over the paths of the oscillator can be performed when calculating the transition amplitude of the system from the initial time 0 to the final time T. With $x(0) = x$ and $x(T) = x'$, Feynman calls the function so obtained $G_\gamma(x, x'; T)$, obtaining finally the formula for the transition amplitude

$$\langle \chi_T | 1 | \psi_0 \rangle_S = \int \chi_T(Q_m, x) e^{\frac{i}{\hbar} S_0[\ldots Q_i \ldots]} G_\gamma(x, x'; T) \psi_0(Q_0, x') \times dx dx' \frac{\sqrt{g} dQ_m \cdots \sqrt{g} dQ_0}{N_m \cdots N_1},$$

where the Q's are the coordinates of the system other than the oscillator.

By using the last expression in the problem of particles interacting through an intermediate oscillator having $x(0) = \alpha$ and $x(T) = \beta$, Feynman shows that the expected value of a functional of the coordinates of the particles alone (such as a transition amplitude) can be obtained with a certain action that does not involve the oscillator coordinates, but only the constants α and β.[16] This eliminates the oscillator from the dynamics of the problem. Various other initial and/or final conditions on the oscillator are shown to lead to a similar result. A brief section labeled "Conclusion" completes the thesis.

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[16] In the abstract at the end of the thesis this conclusion concerning the interaction of two systems is summarized as follows: "It is shown that in quantum mechanics, just as in classical mechanics, under certain circumstances the oscillator can be completely eliminated, its place being taken by a direct, but, in general, not instantaneous, interaction between the two systems."

THE PRINCIPLE OF LEAST ACTION IN QUANTUM MECHANICS

RICHARD P. FEYNMAN

Abstract

A generalization of quantum mechanics is given in which the central mathematical concept is the analogue of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some form of least action principle be available.

It is shown that if the action is the time integral of a function of velocity and position (that is, if a Lagrangian exists), the generalization reduces to the usual form of quantum mechanics. In the classical limit, the quantum equations go over into the corresponding classical ones, with the same action function.

As a special problem, because of its application to electrodynamics, and because the results serve as a confirmation of the proposed generalization, the interaction of two systems through the agency of an intermediate harmonic oscillator is discussed in detail. It is shown that in quantum mechanics, just as in classical mechanics, under certain circumstances the oscillator can be completely eliminated, its place being taken by a direct, but, in general, not instantaneous, interaction between the two systems.

The work is non-relativistic throughout.

I. Introduction

Planck's discovery in 1900 of the quantum properties of light led to an enormously deeper understanding of the attributes and behaviour of matter, through the advent of the methods of quantum mechanics. When, however, these same methods are turned to the problem of light and the electromagnetic field great difficulties arise which have not been surmounted satisfactorily, so that Planck's observations still

It has been the purpose of this introduction to indicate the motivation for the problems which are discussed herein. It is to be emphasized again that the work described here is complete in itself without regard to its application to electrodynamics, and it is this circumstance which makes it appear advisable to publish these results as an independent paper. One should therefore take the viewpoint that the present paper is concerned with the problem of finding a quantum mechanical description applicable to systems which in their classical analogue are expressible by a principle of least action, and not necessarily by Hamiltonian equations of motion.

The thesis is divided into two main parts. The first deals with the properties of classical systems satisfying a principle of least action, while the second part contains the method of quantum mechanical description applicable to these systems. In the first part are also included some mathematical remarks about functionals. All of the analysis will apply to non-relativistic systems. The generalization to the relativistic case is not at present known.

II. Least Action in Classical Mechanics

1. The Concept of a Functional

The mathematical concept of a functional will play a rather predominant role in what is to follow so that it seems advisable to begin at once by describing a few of the properties of functionals and the notation used in this paper in connection with them. No attempt is made at mathematical rigor.

To say F is a functional of the function $q(\sigma)$ means that F is a number whose value depends on the *form* of the function $q(\sigma)$ (where σ is just a parameter used to specify the form of $q(\sigma)$). Thus,

$$F = \int_{-\infty}^{\infty} q(\sigma)^2 e^{-\sigma^2} d\sigma \qquad (1)$$

is a functional of $q(\sigma)$ since it associates with every choice of the function $q(\sigma)$ a number, namely the integral. Also, the area under

a curve is a functional of the function representing the curve, since to each such function a number, the area is associated. The expected value of the energy in quantum mechanics is a functional of the wave function. Again,

$$F = q(0) \qquad (2)$$

is a functional, which is especially simple because its value depends only on the value of the function $q(\sigma)$ at the one point $\sigma = 0$.

We shall write, if F is a functional of $q(\sigma)$, $F[q(\sigma)]$. A functional may have its argument more than one function, or functions of more than one parameter, as

$$F[x(t,s),\, y(t,s)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t,s)y(t,s)\sin\omega(t-s)dtds\,.$$

A functional $F[q(\sigma)]$ may be looked upon as a function of an infinite number of variables, the variables being the value of the function $q(\sigma)$ at each point σ. If the interval of the range of σ is divided up into a large number of points σ_i, and the value of the function at these points is $q(\sigma_i) = q_i$, say, then approximately our functional may be written as a function of the variables q_i. Thus, in the case of equation (1) we could write, approximately,

$$F(\cdots q_i \cdots) = \sum_{i=-\infty}^{\infty} q_i^2 e^{-\sigma_i^2}(\sigma_{i+1} - \sigma_i)\,.$$

We may define a process analogous to differentiation for our functionals. Suppose the function $q(\sigma)$ is altered slightly to $q(\sigma) + \lambda(\sigma)$ by the addition of a small function $\lambda(\sigma)$. From our approximate viewpoint we can say that each of the variables is changed from q_i to $q_i + \lambda_i$. The function is thereby changed by an amount

$$\sum_i \frac{\partial F(\cdots q_i \cdots)}{\partial q_i}\lambda_i\,.$$

In the case of a continuous number of variables, the sum becomes an integral and we may write, to the first order in λ,

$$F[q(\sigma) + \lambda(\sigma)] - F[q(\sigma)] = \int K(t)\lambda(t)dt\,, \qquad (3)$$

where $K(t)$ depends on F, and is what we shall call the functional derivative of F with respect to q at t, and shall symbolize, with Eddington,[5] by $\frac{\delta F[q(\sigma)]}{\delta q(t)}$. It is not simply $\frac{\partial F(\cdots q_i \cdots)}{\partial q_i}$ as this is in general infinitesimal, but is rather the sum of these $\frac{\partial F}{\partial q_i}$ over a short range of i, say from $i+k$ to $i-k$, divided by the interval of the parameter, $\sigma_{i+k} - \sigma_{i-k}$.

Thus we write,

$$F[q(\sigma)+\lambda(\sigma)] = F[q(\sigma)] + \int \frac{\delta F[q(\sigma)]}{\delta q(t)} \lambda(t) dt$$
$$+ \text{higher order terms in } \lambda . \tag{4}$$

For example, in equation (1) if we substitute $q+\lambda$ for q, we obtain

$$\begin{aligned} F[q+\lambda] &= \int [q(\sigma)^2 + 2q(\sigma)\lambda(\sigma) + \lambda(\sigma)^2] e^{-\sigma^2} d\sigma \\ &= \int q(\sigma)^2 e^{-\sigma^2} d\sigma + 2\int q(\sigma)\lambda(\sigma) e^{-\sigma^2} d\sigma \\ &\quad + \text{higher terms in } \lambda . \end{aligned}$$

Therefore, in this case, we have $\frac{\delta F[q]}{\delta q(t)} = 2q(t)e^{-t^2}$. In a similar way, if $F[q(\sigma)] = q(0)$, then $\frac{\delta F}{\delta q(t)} = \delta(t)$, where $\delta(t)$ is Dirac's delta symbol, defined by $\int \delta(t) f(t) dt = f(0)$ for any continuous function f.

The function $q(\sigma)$ for which $\frac{\delta F}{\delta q(t)}$ is zero for all t is that function for which F is an extremum. For example, in classical mechanics the action,

$$\mathscr{A} = \int L(\dot{q}(\sigma), q(\sigma)) d\sigma \tag{5}$$

is a functional of $q(\sigma)$. Its functional derivative is,

$$\frac{\delta \mathscr{A}}{\delta q(t)} = -\frac{d}{dt}\left\{ \frac{\partial L(\dot{q}(t), q(t))}{\partial \dot{q}} \right\} + \frac{\partial L(\dot{q}(t), q(t))}{\partial q} . \tag{6}$$

If $\mathscr{A}$ is an extremum the right hand side is zero.

[5] A. S. Eddington, "The Mathematical Theory of Relativity" (1923) p. 139.
Editor's note: We have changed Eddington's symbol for the functional derivative to that now commonly in use.

2. The Principle of Least Action

For most mechanical systems it is possible to find a functional, $\mathscr{A}$, called the action, which assigns a number to each possible mechanical path, $q_1(\sigma), q_2(\sigma) \ldots q_N(\sigma)$, (we suppose N degrees of freedom, each with a coordinate $q_n(\sigma)$, a function of a parameter (time) σ) in such a manner that this number is an extremum for an actual path $\bar{q}(\sigma)$ which could arise in accordance with the laws of motion. Since this extremum often is a minimum this is called the principle of least action. It is often convenient to use the principle itself, rather than the Newtonian equations of motion as the fundamental mechanical law. The form of the functional $\mathscr{A}[q_1(\sigma) \ldots q_N(\sigma)]$ depends on the mechanical problem in question.

According to the principle of least action, then, if $\mathscr{A}[q_1(\sigma) \ldots q_N(\sigma)]$ is the action functional, the equations of motion are N in number and are given by,

$$\frac{\delta \mathscr{A}}{\delta q_1(t)} = 0, \frac{\delta \mathscr{A}}{\delta q_2(t)} = 0, \ldots, \frac{\delta \mathscr{A}}{\delta q_N(t)} = 0 \tag{7}$$

(We shall often simply write $\frac{\delta \mathscr{A}}{\delta q(t)} = 0$, as if there were only one variable). That is to say if all the derivatives of $\mathscr{A}$, with respect to $q_n(t)$, computed for the functions $\bar{q}_m(\sigma)$ are zero for all t and all n, then $\bar{q}_m(\sigma)$ describes a possible mechanical motion for the systems.

We have given an example, in equation (5), for the usual one dimensional problem when the action is the time integral of a Lagrangian (a function of position and velocity, only). As another example consider an action function arising in connection with the theory of action at a distance:

$$\mathscr{A} = \int_{-\infty}^{\infty} \left\{ \frac{m(\dot{x}(t))^2}{2} - V(x(t)) + k^2 \dot{x}(t)\dot{x}(t+T_0) \right\} dt . \tag{8}$$

It is approximately the action for a particle in a potential $V(x)$, and interacting with itself in a distant mirror by means of retarded and advanced waves. The time it takes for light to reach the mirror from the particle is assumed constant, and equal to $T_0/2$. The quantity

k^2 depends on the charge on the particle and its distance from the mirror. If we vary $x(t)$ by a small amount, $\lambda(t)$, the consequent variation in $\mathscr{A}$ is,

$$\delta\mathscr{A} = \int_{-\infty}^{\infty} \{m\dot{x}(t)\dot{\lambda}(t) - V'(x(t))\lambda(t) + k^2\dot{\lambda}(t)\dot{x}(t+T_0) + k^2\dot{\lambda}(t+T_0)\dot{x}(t)\}dt$$

$$= \int_{-\infty}^{\infty} \{-m\ddot{x}(t) - V'(x(t)) - k^2\ddot{x}(t+T_0) - k^2\ddot{x}(t-T_0)\}\lambda(t)dt\,, \quad \text{by integrating by part}$$

so that, according to our definition (4), we may write,

$$\frac{\delta\mathscr{A}}{\delta x(t)} = -m\ddot{x}(t) - V'(x(t)) - k^2\ddot{x}(t+T_0) - k^2\ddot{x}(t-T_0)\,. \tag{9}$$

The equation of motion of this system is obtained, according to (7) by setting $\frac{\delta\mathscr{A}}{\delta x(t)}$ equal to zero. It will be seen that the force acting at time t depends on the motion of the particle at other time than t. The equations of motion cannot be described directly in Hamiltonian form.

3. Conservation of Energy. Constants of the Motion[6]

The problem we shall study in this section is that of determining to what extent the concepts of conservation of energy, momentum, etc., may be carried over to mechanical problems with a general form of action function. The usual principle of conservation of energy asserts that there is a function of positions at the time t, say, and of velocities of the particles whose value, for the actual motion of the particles, does not change with time. In our more general case however, the forces do not involve the positions of the particles only at one particular time, but usually a calculation of the forces requires

[6] This section is not essential to an understanding of the remainder of the paper.

a knowledge of the paths of the particles over some considerable range of time (see for example, Eq. (9)). It is not possible in this case generally to find a constant of the motion which only involves the positions and velocities at one time.

For example, in the theory of action at a distance, the kinetic energy of the particles is not conserved. To find a conserved quantity one must add a term corresponding to the "energy in the field". The field, however, is a functional of the motion of the particles, so that it is possible to express this "field energy" in terms of the motion of the particles. For our simple example (8), account of the equations of motion (9), the quantity,

$$E(t) = \frac{m(\dot{x}(t))^2}{2} + V(x(t)) - k^2 \int_t^{t+T_0} \ddot{x}(\sigma - T_0)\dot{x}(\sigma)d\sigma + k^2\dot{x}(t)\dot{x}(t+T_0)\,, \tag{10}$$

has, indeed, a zero derivative with respect to time. The first two terms represent the ordinary energy of the particles. The additional terms, representing the energy of interaction with the mirror (or rather, with itself) require a knowledge of the motion of the particle from the time $t - T_0$ to $t + T_0$.

Can we really talk about conservation, when the quantity conserved depends on the path of the particles over considerable ranges of time? If the force acting on a particle be $F(t)$ say, so that the particle satisfies the equation of motion $m\ddot{x}(t) = F(t)$, then it is perfectly clear that the integral,

$$I(t) = \int_{-\infty}^{t} [m\ddot{x}(t) - F(t)]\dot{x}(t)dt \tag{11}$$

has zero derivative with respect to t, when the path of the particle satisfies the equation of motion. Many such quantities having the same properties could easily be devised. We should not be inclined to say (11) actually represents a quantity of interest, in spite of its constancy.