

An attempt to a β rays theory

NOTA DI ENRICO FERMI

(ricevuto 1933)

Summary. — A quantitative theory of β -rays emission is proposed in which «neutrino» is admitted; electrons and neutrinos emission from a nucleus at a β decay is treated with a procedure similar to the one followed for radiation theory to describe a light quantum emission by an excited atom. Formulae are derived for the mean life and for the distribution of the β -rays continuum spectrum, which are compared with experimental data.

Basic assumptions of the theory ⁽¹⁾

1. —

In attempting to construct a theory of nuclear electrons and of the β rays emission, two well known difficulties are encountered. The first is that the primary β rays are emitted by the nuclei with a continuous velocity distribution. If the energy conservation principle is not abandoned, we must therefore admit that a fraction of the energy made available in the β decay escapes our present observation possibilities. According to a proposal by PAULI existence can be supposed, for instance, of a new particle, the so called «neutrino» with a zero electric charge and a mass of the order of magnitude of the electron mass or less. One admits, furthermore, that in every β process are simultaneously emitted an electron, observed as a β ray, and a neutrino escaping observation and carrying away part of the energy.

A second difficulty for the nuclear electrons theory arises because the current relativistic theories of light particles (electrons or neutrinos) do not satisfactorily deal with the possibility that such particles could be bound into orbits of nuclear size.

As a consequence it seems more appropriate to admit with HEISENBERG⁽²⁾ that all nuclei only consist of heavy particles, protons and neutrons. Nevertheless to understand the possibility of β rays emission, we shall attempt the construction of a theory for the emission of light particles from a nucleus in analogy with the emission theory of a light

⁽¹⁾ Cfr. preliminary note in «La Ricerca Scientifica», **2**, n. 12, 1933.

⁽²⁾ W. HEISENBERG, «Zs. für Phys.», **77**, 1, 1932; E. MAJORANA, «Zs. für Phys.», **82**, 137, 1933.

quantum by an excited atom in the ordinary radiation process. In radiation theory the total number of light quanta is not constant; the quanta are created at the moment an excited atom emits them, while they disappear when absorbed. In analogy we shall try to build the theory of the β rays upon the following hypotheses:

- a) The total number of electrons and neutrinos is not necessarily constant.
- b) Electrons (or neutrinos) can be created or destroyed. Such a possibility does not have any analogy with the possibility of creation or annihilation of an electron-positron pair; if, indeed, a positron is interpreted as a DIRAC «hole», we can simply view the latter process as a quantum transition of an electron from a state of negative energy to one with positive energy, with conservation of the total (infinite) number of electrons.
- b) Heavy particles, neutron and proton, can be considered, according to HEISENBERG, as two different quantum states of a heavy particle. We shall formulate this by introducing an internal coordinate ρ for the heavy particle, which can take only two values: $\rho = +1$, if the particle is a neutron; $\rho = -1$, if the particle is a proton.
- c) The Hamiltonian of the entire system, consisting in heavy and light particles, has to be chosen so that every transition of a proton into a neutron occurs together with the creation of an electron and a neutrino. Notice that in this way electric charge conservation is guaranteed.

The operators of the theory

2. —

A mathematical formalism allowing the construction of a theory in agreement with the above items can be constructed most easily via DIRAC-JORDAN-KLEIN's method⁽³⁾ known as «second quantization». We shall therefore consider the probability amplitudes ψ and φ of electrons and neutrinos, in ordinary space, as well as their complex conjugates ψ^* and φ^* as operators; while for the heavy particles we shall employ the usual representation in configuration space, in which, of course, also ρ will have to be considered among the coordinates.

Introduce first two operators Q and Q^* acting over functions of the two values variable ρ as the linear substitutions

$$(1) \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Q^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It is immediately recognized that Q describes transitions from proton to neutron, and Q^* the opposite transitions from neutron to proton.

As it is known, the meaning of the probability amplitudes ψ and φ interpreted as operators is the following. Let

$$\psi_1, \psi_2, \dots, \psi_n, \dots$$

⁽³⁾ Cfr. for instance P. JORDAN and O. KLEIN, «ZS. für Phys.», **45**, 751. 1927; W. HEISENBERG, «Ann. d. Phys.», **10**, 888, 1931.

be a system of individual quantum states for the electrons. Let then

$$(2) \quad \psi = \sum_s \psi_s a_s, \quad \psi^* = \sum_s \psi_s^* a_s^*,$$

The amplitudes a_s and their conjugated a_s^* are operators acting on functions of the occupation numbers N_1, N_2, \dots of the individual states. If PAULI's principle holds each of the N_s can only take values 0, 1; and the operators a_s, a_s^* are defined as follows:

$$(3) \quad \begin{aligned} a_s \Psi(N_1, N_2, \dots, N_s, \dots) &= (-1)^{N_1+N_2+\dots+N_{s-1}} (1 - N_s) \Psi(N_1, N_2, \dots, 1 - N_s, \dots) \\ a_s^* \Psi(N_1, N_2, \dots, N_s, \dots) &= (-1)^{N_1+N_2+\dots+N_{s-1}} N_s \Psi(N_1, N_2, \dots, 1 - N_s, \dots) \end{aligned}$$

The operator a_s^* induces creation of an electron in the quantum state s while the operator a_s induces its destruction.

Correspondingly to (2) we shall set, for the neutrinos:

$$(4) \quad \varphi = \sum_\sigma \varphi_\sigma b_\sigma; \quad \varphi^* = \sum_\sigma \varphi_\sigma^* b_\sigma^*;$$

The complex conjugated b_σ and b_σ^* operate on the functions of the occupation numbers $M_1, M_2, \dots, M_\sigma, \dots$ of the individual quantum states $\varphi_1, \varphi_2, \dots, \varphi_\sigma, \dots$ of neutrinos. Admitting that also such particles follow PAULI's principle, the numbers M_σ can take only the two values 0, 1; and it is

$$(5) \quad \begin{aligned} b_\sigma \Psi(M_1, M_2, \dots, M_\sigma, \dots) &= (-1)^{M_1+M_2+\dots+M_{\sigma-1}} (1 - M_\sigma) \Psi(M_1, M_2, \dots, 1 - M_\sigma, \dots) \\ b_\sigma^* \Psi(M_1, M_2, \dots, M_\sigma, \dots) &= (-1)^{M_1+M_2+\dots+M_{\sigma-1}} M_\sigma \Psi(M_1, M_2, \dots, 1 - M_\sigma, \dots) \end{aligned}$$

The operators b_σ and b_σ^* determine respectively disappearance and creation of a neutrino in the state σ .

The Hamiltonian function

3. –

Energy of the total system, consisting in the heavy and light particles, is the sum of the energy H_{heavy} of heavy particles + the energy H_{light} of the light particles + the interaction energy between the light and heavy particles.

Restricting, for simplicity, consideration to only one heavy particle, we shall write the first term in the form

$$(6) \quad H_{heavy} = \frac{1+\rho}{2} \mathcal{N} + \frac{1-\rho}{2} \mathcal{P}$$

where \mathcal{N} and \mathcal{P} are the operators representing the energy of the neutron and the proton. Notice indeed that for $\rho = +1$ (neutron) Eq.(6) reduces to \mathcal{N} ; while for $\rho = -1$ (proton) it reduces to \mathcal{P} .

To write the energy H_{light} in the simplest way, we shall take stationary states for the quantum states ψ_s, φ_s of electrons and neutrinos. For the electrons we shall take the eigenfunctions in the coulombian field of the nucleus (suitably screened to take into account the action of the atomic electrons); for neutrinos we shall simply take DE BROGLIE plane waves, because forces possibly acting on neutrinos are certainly extremely weak. Let $H_1, H_2, \dots, H_s, \dots$ and $K_1, K_2, \dots, K_\sigma, \dots$ the energies of the stationary states of the electrons and of the neutrinos: we shall then have

$$(7) \quad H_{light} = \sum_s H_s N_s + \sum_\sigma K_\sigma M_\sigma$$

We are left with writing the interaction energy. It consists first in the coulombian energy between proton and electron; in the case of heavy nuclei, however, the attraction exercised by a single proton is not very important⁽⁴⁾ and in any event it does not contribute to the β decay process. To avoid complicating needlessly the matter we shall not take this term into account. We must instead add to the Hamiltonian a term to satisfy condition c) of Sec.1.

A term that necessarily couples the transformation of a proton into a neutron with an electron and neutrino creation has, according to the results of Sec.2, the form

$$(8) \quad Q^* a_s^* b_\sigma^*$$

while the conjugate operator

$$(8') \quad Q a_s b_\sigma$$

couples the inverse processes (transmutation of a proton into a neutron and disappearance of an electron and a neutrino).

A term satisfying condition c) will then have the form

$$(9) \quad \mathcal{H} = Q \sum_{s,\sigma} c_{s\sigma} a_s b_\sigma + Q^* \sum_{s,\sigma} c_{s\sigma}^* a_s^* b_\sigma^*$$

where $c_{s\sigma}$ and $c_{s\sigma}^*$ are quantities that can depend upon coordinates, momenta, etc... of the heavy particle.

Further determination of \mathcal{H} cannot be done other than by a maximum simplicity criterion; naturally, however, the possibilities of choices of \mathcal{H} are constrained by the need that \mathcal{H} be invariant under change of coordinates and that it should satisfy linear momentum conservation.

If at first we neglect relativistic effects the simplest choice for Eq.(9) is the following

$$(10) \quad \mathcal{H} = g [Q \psi(x) \varphi(x) + Q^* [\psi^*(x) \varphi^*(x)]]$$

Eq.(10) by no means represents the only possible choice of \mathcal{H} ; equally admissible would be any scalar expression like like

$$L(p) \psi(x) M(p) \varphi(x) N(p) + c.c.$$

⁽⁴⁾ Coulombian attraction due to the many other protons must, naturally, be taken into account as a static field.

where $L(p), M(p), N(p)$ are suitable functions of the heavy particle. Since, on the other hand, consequences of Eq.(10) have shown to be in agreement with experimental data it seems useless, for the time being, to have recourse to more complicated expressions.

It is however essential to extend Eq.(10) to treat relativistically at least light particles. Of course, even such a generalization does not seem possible without some degree of arbitrariness. The most general solution to the problem appears to be the following: relativistically, instead of ψ and φ , there are two four-ples $\psi_1, \psi_2, \psi_3, \psi_4$ and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ of DIRAC's functions. Consider the 16 bilinear independent combinations of $\psi_1, \psi_2, \psi_3, \psi_4$ and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. When the system of reference undergoes a LORENTZ transformation, the 16 bilinear combinations undergo a linear substitution, which yields a representation of the LORENTZ group. In particular the four bilinear combinations

$$(11) \quad \begin{aligned} A_0 &= -\psi_1\varphi_2 + \psi_2\varphi_1 + \psi_3\varphi_4 - \psi_4\varphi_3 \\ A_1 &= \psi_1\varphi_3 - \psi_2\varphi_4 - \psi_3\varphi_1 + \psi_4\varphi_2 \\ A_2 &= i\psi_1\varphi_3 + i\psi_2\varphi_4 - i\psi_3\varphi_1 - i\psi_4\varphi_2 \\ A_3 &= -\psi_1\varphi_4 - \psi_2\varphi_3 + \psi_3\varphi_2 + \psi_4\varphi_1 \end{aligned}$$

are transformed as the components of a quadrivector, *i.e.* as the components of the electromagnetic quadripotential.⁽⁵⁾ It is therefore natural to introduce in the Hamiltonian of the heavy particle the four quantities

$$g(QA_i + Q^*A_i^*)$$

as in the corresponding quadripotential. Here is met a difficulty depending on the lack of knowledge of a relativistic wave equation for the heavy particles. Nevertheless, if the heavy particle speed is small compared to c , it will be possible to consider only the term corresponding to eV (V being the scalar potential) and write

$$(12) \quad \begin{aligned} \mathcal{H} = & g \left[Q(-\psi_1\varphi_2 + \psi_2\varphi_1 + \psi_3\varphi_4 - \psi_4\varphi_3) + \right. \\ & \left. + Q^*(-\psi_1^*\varphi_2^* + \psi_2^*\varphi_1^* + \psi_3^*\varphi_4^* - \psi_4^*\varphi_3^*) \right] \end{aligned}$$

To the latter term others must be added of the order v/c . However, for the time being,

⁽⁵⁾ **NdT**: This means using $A_\mu = \psi^T g_\mu \varphi$ with

$$g_0 = \begin{pmatrix} -i\sigma_y & 0 \\ 0 & i\sigma_y \end{pmatrix}, \quad g_1 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

hence $g_0 \equiv \delta$, see Eq.(14), and $g_j = -\delta \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$, $j = 1, 2, 3$, with the usual Pauli matrices σ_j .

The matrices are determined by requiring that the spinor transformation $\varphi'(x) = S(\Lambda)\varphi(\Lambda^{-1}x)$ defines a covariant vector: *i.e.* $S^T(\Lambda)g_\nu S(\Lambda) = \Lambda_\nu^\mu g_\mu$ for any Lorentz transformation Λ (therefore the g_μ are not the usual γ -matrices). This is obtained from the infinitesimal generator of Lorentz transformations $T = \sum_{\mu, \nu} \frac{1}{8} \lambda^{\mu\nu} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \equiv \sum_{\mu < \nu} \frac{1}{2} \lambda^{\mu\nu} \gamma_\mu \gamma_\nu$ (see S. Schweber, *An introduction to relativistic quantum field theory*, Row-Peterson & Co., 1961, p.77, Eq.(78)), with $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ by successively choosing $\lambda^{\mu\nu} = 0$ except $\lambda^{0j} = -\lambda^{j0} = 1$, $j = 1, 2, 3$: but I find that g_2 should have the opposite sign; however this changes the *r.h.s.* of A_2 in Eq.(11) which, of course, is never used in the paper, see Eq.(12).

we shall neglect such terms, because the neutrons and protons speeds in the nuclei are usually small compared to c (Cfr. §9).

In a symbolic matrix representation Eq.(12) can be written

$$(13) \quad \mathcal{H} = g [Q\tilde{\psi}^* \delta \varphi + Q^* \tilde{\psi} \delta \varphi^*]$$

where ψ and φ are understood to be written as single column matrices, and the tilde sign transforms a matrix into the transposed conjugated; furthermore it is

$$(14) \quad \delta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

With the above notations one finds, comparing Eq.(12) and (9)

$$(15) \quad c_{s\sigma} = g\tilde{\psi}_s^* \delta \varphi_\sigma; \quad c_{s\sigma}^* = g\tilde{\psi}_s \delta \varphi_\sigma^*$$

where ψ_s and φ_σ represent the normalized eigenfunctions with four components of the electron states s and of the neutrino states σ , considered as functions of the position (x, y, z) occupied by the heavy particle.

The perturbation matrix

4. –

With the just established Hamiltonian it is possible to develop the β decay theory in complete analogy with radiation theory. In the latter, the HAMILTON function consists, as is known, of the sum: atom Energy + Radiation field Energy + Interaction between atom and radiation; the last term is considered as a perturbation of the other two. Analogously we shall take as unperturbed Hamiltonian

$$(16) \quad H_{heavy} + H_{light}.$$

Perturbation is represented by the interaction term Eq.(13).

Quantum states of the unperturbed system can be enumerated as follows:

$$(17) \quad (\rho, n, N_1, N_2, \dots, N_s, \dots, M_1, M_2, \dots, M_\sigma, \dots)$$

where the first number ρ assumes one of the values ± 1 , indicating whether the heavy particle is a neutron or a proton. The second number indicates the neutron or the proton quantum state. For $\rho = +1$ (neutron) let the corresponding eigenfunction be

$$(18) \quad u_n(x)$$

while for $\rho = -1$ (proton) let the eigenfunction be

$$(19) \quad v_n(x)$$

The remaining numbers $N_1, N_2, \dots, N_s, \dots, M_1, M_2, \dots, M_\sigma, \dots$ can only take values 0, 1 and indicate which electron and neutrino states are occupied.

Examining the general form Eq.(9) of the perturbation energy, it is immediately realized that it has non zero matrix elements only for transitions in which either a heavy particle is changed from proton to neutron while at the same time an electron and a neutrino are created, or viceversa.

Through Eqs. (1),(3),(5),(9),(18),(19) it is easily found that the corresponding matrix element is

$$(20) \quad \mathcal{H}_{-1mN_1N_2\dots1_s\dots M_1M_2\dots1_\sigma\dots}^{1nN_1N_2\dots0_s\dots M_1M_2\dots0_\sigma\dots} = \pm \int v_m^* c_{s\sigma}^* u_n d\tau$$

where the integration has to be extended to the whole configurations space of the heavy particle (except over the coordinate ρ); the sign \pm means more precisely

$$(-1)^{N_1+N_2+\dots+N_{s-1}+M_1+M_2+\dots+M_{\sigma-1}}$$

and disappears from the calculations that will follow. To the opposite transition corresponds a matrix element complex conjugate of Eq.(20).

Taking Eq.(15), Eq.(20) becomes

$$(21) \quad \mathcal{H}_{-1m1_s1_\sigma}^{1n0_s0_\sigma} = \pm g \int v_m^* u_n \tilde{\psi}_s \delta\varphi_\sigma^* d\tau$$

where, for brevity, we omitted writing in the *r.h.s.* all labels which remain unchanged.

Theory of β decays

5. –

A β decay is a process in which a nuclear neutron is transformed into a proton, while at the same time, with the above described mechanism, an electron is emitted, which is observed as a β particle, and a neutrino. To compute the probability of such a process we shall admit that, at time $t = 0$, there is a neutron in a nuclear state with eigenfunction $u_n(x)$, and that furthermore the electronic state s and the neutrino state σ are not occupied, *i.e.* $N_s = M_\sigma = 0$. At $t = 0$ it will therefore be possible to suppose equal to 1 the probability amplitude of the state $(1, n, 0_s, 0_\sigma)$, *i.e.*

$$(22) \quad a_{1,n,0_s,0_\sigma} = 1$$

while initially the probability amplitude of the state $(-1, m, 1_s, 1_\sigma)$, in which the neutron is transformed into a proton with eigenfunction $v_m(x)$ emitting an electron and a neutrino in the states s and σ vanishes.

Applying the usual formulae of perturbation theory we find, for a time short enough so that Eq.(22) is still approximately valid,

$$(23) \quad a_{-1,m,1_s,1_\sigma} = -\frac{2\pi i}{h} \mathcal{H}_{-1m1_s1_\sigma}^{1n0_s0_\sigma} e^{\frac{2\pi i}{h}(-W+H_s+K_\sigma)t}$$

where W indicates the energy difference between the neutron state and the proton state.

Integrating Eq.(23) we find (since for $t = 0$ it is $a_{-1,m,1_s,1_\sigma} = 0$)

$$(24) \quad a_{-1,m,1_s,1_\sigma} = -\mathcal{H} \begin{matrix} 1n & 0_s & 0_\sigma \\ -1m & 1_s & 1_\sigma \end{matrix} \frac{e^{\frac{2\pi i}{h}(-W+H_s+K_G)t} - 1}{-W+H_s+K_\sigma}$$

The probability of the considered transition is then

$$(25) \quad |a_{-1,m,1_s,1_\sigma}|^2 = 4 \left| \mathcal{H} \begin{matrix} 1n & 0_s & 0_\sigma \\ -1m & 1_s & 1_\sigma \end{matrix} \right|^2 \frac{\sin^2 \frac{\pi t}{h}(-W+H_s+K_\sigma)}{-W+H_s+K_\sigma}$$

To compute the mean life of the neutron state u_n it is necessary to sum Eq.(25) over all unoccupied electron and neutrino states. An important simplification of such sum is obtained after remarking that the DE BROGLIE wavelength for electron or neutrinos with energies of a few million Volts is considerably larger than the nucleus dimensions. It is therefore possible, at first approximation, to consider the eigenfunctions ψ_s and φ_σ as constants within the nucleus size. Eq. (21) thus becomes

$$(26) \quad \mathcal{H} \begin{matrix} 1n & 0_s & 0_\sigma \\ -1m & 1_s & 1_\sigma \end{matrix} = \pm g \tilde{\psi}_s \delta \varphi_\sigma^* \int v_m^* u_n d\tau$$

where, here and in the following, ψ_s and φ_σ must be intended to be evaluated in the nucleus (Cfr. §8). From Eq.(26) we deduce

$$(27) \quad \left| \mathcal{H} \begin{matrix} 1n & 0_s & 0_\sigma \\ -1m & 1_s & 1_\sigma \end{matrix} \right|^2 = g^2 \left| \int v_m^* u_n d\tau \right|^2 \tilde{\psi} \delta \varphi_\sigma^* \tilde{\varphi}_\sigma^* \delta \psi_\sigma.$$

The neutrino states σ are determined by their momentum p_σ and by the spin direction. If, for ease of normalization, quantization is performed inside a volume Ω , with dimensions which will be eventually let to infinity, the normalized neutrino eigenfunctions are DIRAC's plane waves with density Ω . Easy algebraic considerations allow us to perform in Eq(26) the average with respect to all orientations of p_σ and of the spin. (In so doing only the positive energy states must be considered; the negative energy states must be eliminated via an artifice similar to DIRAC's holes theory). One gets

$$(28) \quad \overline{\left| \mathcal{H} \begin{matrix} 1n & 0_s & 0_\sigma \\ -1m & 1_s & 1_\sigma \end{matrix} \right|^2} = \frac{g^2}{4\Omega} \left| \int v_m^* u_n d\tau \right|^2 \left(\tilde{\psi}_s \psi_s - \frac{\mu_\sigma^2}{K_\sigma} \tilde{\psi}_s \beta \psi_s \right)$$

where μ is the neutrino mass at rest and β id DIRAC'S matrix

$$(29) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Remarking that the number of neutrino states with positive energy and momentum between p_σ and $p_\sigma + dp_\sigma$ is $8\pi\Omega p_\sigma^2 dp_\sigma / h^3$ and that $\partial K_\sigma / \partial p_\sigma = v_\sigma$, with v_σ the neutrino speed in the state σ and finally that Eq.(25) has a strong peak for the value p_σ for which there is no variation of the unperturbed energy, *i.e.* for which

$$(30) \quad -W + H_s + K_\sigma = 0$$

it is possible to perform in the usual way⁽⁶⁾ the sum of Eq.(25) with respect to σ and to find

$$(31) \quad t \cdot \frac{8\pi^2 g^2}{h^4} \left| \int v_m^* n u_n d\tau \right|^2 \frac{p_\sigma^2}{v_\sigma} \left(\tilde{\psi}_s \psi_s - \frac{\mu c^2}{K_\sigma} \tilde{\psi}_s \beta \psi_s \right)$$

p_σ being the neutrino momentum value for which Eq.(30) holds.

The elements that determine the transition probability

6. –

Eq.(31) gives the probability that within time t a β -decay occurs in which the electron is emitted in the state s . As it should be, such probability is proportional to time (t has been supposed small in comparison with the mean life); the coefficient of t gives the transition probability for the considered process; it is

$$(32) \quad P_s = \frac{8\pi^3 g^2}{h^4} \left| \int v_m^* n u_n d\tau \right|^2 \frac{p_\sigma^2}{v_\sigma} \left(\tilde{\psi}_s \psi_s - \frac{\mu c^2}{K_\sigma} \tilde{\psi}_s \beta \psi_s \right)$$

Remark that:

a) The free neutrino states have always $K_\sigma \geq \mu c^2$. It is therefore necessary, so that Eq.(30) can possibly be satisfied, that

$$(33) \quad H_s \leq W - \mu c^2$$

Equality corresponds to the upper limit of the β -rays spectrum.

b) furthermore for the non occupied electronic states $H_s \geq m c^2$ and one finds, to make the β decay possible, the following condition

$$(34) \quad W \geq (m + \mu) c^2$$

Hence in the nucleus there has to be an occupied neutronic state higher enough above a free protonic state in order that the β emission can occur.

c) According to Eq.(32) P_s depends on the eigenfunctions u_n and v_m of the heavy particle in the nucleus, through the matrix element

$$(35) \quad Q_{mn}^* = \int v_m^* u_n^* d\tau$$

⁽⁶⁾ For a description of the methods that are used to perform such sums cfr. and review article about the radiation theory. For instance E. FERMI, «Rev. of Mod. Phys.», **4**, 87, 1932.

Such matrix element plays, in the β -rays theory, a role analogous to the one of the electric moment matrix element in radiation theory. The matrix element in Eq.(35) has normally size of order 1; however it often happens that, due to particular symmetries of the eigenfunctions u_n and v_m^* , Q_{mn}^* vanishes exactly. In such cases we shall call the β -transitions *forbidden*. But one should not expect that the forbidden transitions are really impossible, because Eq.(32) is only an approximate formula. We shall come back on this point in Sec.9.

The neutrino mass

7. —

The transition probability Eq.(32) determines, among other properties, the form of the β rays continuum spectrum. We shall discuss here how the form of such spectrum depends on the neutrino rest mass, so that this mass can be determined by comparison with the experimentally observed form of the spectrum. The mass μ appears in Eq.(32) in the factor p_σ^2/v_σ in particular. The μ dependence of the form of the energy distribution curve is specially strong near the β rays maximum energy E_0 . It is easily recognized that the distribution curve for energies E near the maximum value E_0 , behaves, aside from a E independent factor, as

$$(36) \quad \frac{p_\sigma^2}{v_\sigma} = \frac{1}{c^2}(\mu c^2 + E_0 - E) \sqrt{(E_0 - E)^2 + 2\mu c^2(E_0 - E)}$$

In Fig.1 the end of the distribution curve is represented for $\mu = 0$, and for a small value of μ or for a large one. The closer similarity with the experimental curves is achieved for $\mu = 0$. Therefore we reach the conclusion the the neutrino mass is zero or, in any case, is small compared to the electron mass⁽⁷⁾. In the following calculations we shall set, for simplicity, $\mu = 0$.

Then, taking also into account Eq.(30),

$$(37) \quad v_\sigma = c; \quad K_\sigma = cp_\sigma; \quad p_\sigma = \frac{K_\sigma}{c} = \frac{W - H_s}{c}$$

and inequalities eq.(33) and (34) become:

$$(38) \quad H_s \leq W; \quad W \geq mc^2$$

Finally the transition probability Eq.(32) takes the form

$$(39) \quad P_s = \frac{8\pi^3 g^2}{c^3 h^4} \left| \int v_m^* u_n d\tau \right|^2 \tilde{\psi}_s \psi_s (W - H_s)^2.$$

⁽⁷⁾ In a recent note F. PERRIN, «C.R.», **197**, 1625, 1933, reaches via quantitative arguments the same conclusion.

Lifetime and form of the energy distribution curve for allowed transitions.

8. —

From Eq.(39) it is possible to derive a formula expressing how many β transitions, in which the β particle receives a momentum between $mc\eta$ and $mc(\eta + d\eta)$, take place per unit time. For this purpose it is necessary to compute the sum of the values of $\tilde{\psi}_s\psi_s$ in the nucleus, over all states (in the continuum) which are within the indicated momentum interval. Note in this respect that the relativistic eigenfunctions

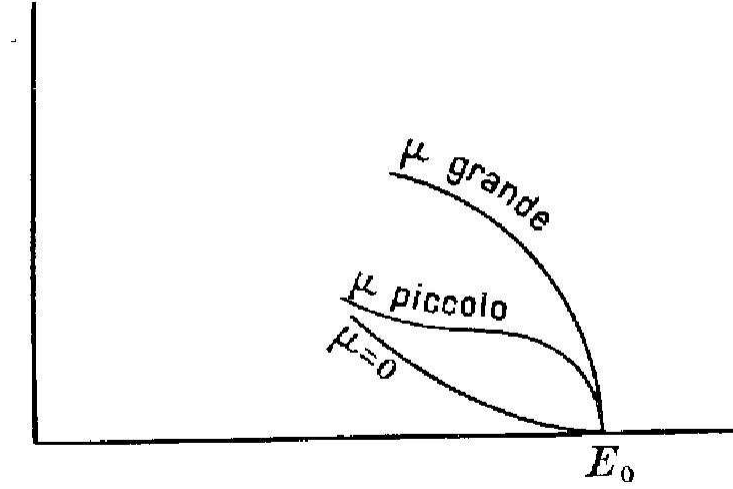


Fig. 1

in Coulomb field for the states with $j = \frac{1}{2}(^2s_{1/2}$ and $^2p_{1/2})$ become infinite in the center. On the other hand Coulomb's law is not valid up to the center of the nucleus, but only up to a distance from it larger than R , where R is the nuclear radius. A heuristic calculation shows that, under plausible hypotheses on the behavior of the electric potential in the nucleus interior, the value of $\psi_s\psi_s$ in the center of the nucleus is very close to the value that $\tilde{\psi}_s\psi_s$ would take, in the case of Coulomb's law, at distance R from the center. Applying the known formulae⁽⁸⁾ for the relativistic continuum spectrum eigenfunctions in Coulomb's field one finds via a somewhat lengthy, but without difficulties, calculations

$$(40) \quad \sum_{d\eta} \tilde{\psi}_s\psi_s = d\delta \cdot \frac{32\pi m^3 c^3}{h^3 [\Gamma(3+2S)]^2} \left(\frac{4\pi mcR}{h} \right)^{2S} \eta^{2+2S} e^{\pi\gamma \frac{\sqrt{1+\eta^2}}{\eta}} \left| \Gamma\left(1+S+i\gamma \frac{\sqrt{1+\eta^2}}{\eta}\right) \right|^2$$

where we set

$$(41) \quad \gamma = Z/137; \quad S = \sqrt{1-\gamma^2} - 1$$

⁽⁸⁾ R.H. HULME, «Proc. Roy. Soc.», **188**, 381, 1931.

The transition probability in an electronic state with momentum in the interval $mc d\eta$ becomes therefore, see Eq.(39),

$$(42) \quad P(\eta)d\eta = d\eta \cdot g^2 \frac{256 \pi^4}{[\Gamma(3+2S)]^2} \frac{m^5 c^4}{h^7} \left(\frac{4\pi mc R}{h} \right)^{2S} \left| \int v_m^* u_n d\tau \right|^2 \eta^{2+2S} \cdot e^{\pi\gamma \frac{\sqrt{1+\eta^2}}{\eta}} \left| \Gamma(1+S+i\gamma \frac{\sqrt{1+\eta^2}}{\eta}) \right|^2 (\sqrt{1+\eta_0^2} - \sqrt{1+\eta^2})^2$$

where η_0 represents the maximum momentum of the emitted β rays, measured in units mc .

For the numerical evaluation of Eq.(42) let us consider the special value $\gamma = 0.6$, which corresponds to $Z = 82.2$, because the atomic numbers of radioactive elements are not far from this value. For $\gamma = 0.6$, from Eq.(41) we get $S = -0.2$. Furthermore we find that for $\eta < 10$ we can set, with sufficient approximation,

$$(43) \quad \eta^{1.6} e^{0.6\pi \frac{\sqrt{1+\eta^2}}{\eta}} \left| \Gamma(0.8 + 0.6i \frac{\sqrt{1+\eta^2}}{\eta}) \right|^2 \simeq 4.5\eta + 1.6\eta^2$$

Eq.(42) becomes, setting in it $R = 9 \cdot 10^{-13}$,

$$(44) \quad P(\eta)d\eta = 1.75 \cdot 10^{95} g^2 \left| \int v_m^* u_n d\tau \right|^2 (\eta + 0.355\eta^2) (\sqrt{1+\eta_0^2} - \sqrt{1+\eta^2})^2$$

The inverse of the mean life is found by integrating Eq.(44) between 0 and $\eta = \eta_0$; one finds

$$(45) \quad \frac{1}{\tau} = 1.75 \cdot 10^{95} g^2 \left| \int v_m^* u_n d\tau \right|^2 F(\eta_0)$$

where we set

$$(46) \quad F(\eta_0) = \frac{2}{3} \sqrt{1+\eta_0^2} - \frac{2}{3} + \frac{\eta_0^4}{12} - \frac{\eta_0^2}{3} + 0.355 \left[-\frac{\eta_0}{4} - \frac{\eta_0^3}{12} + \frac{\eta_0^5}{30} + \frac{\sqrt{1+\eta_0^2}}{4} \log(\eta_0 + \sqrt{1+\eta_0^2}) \right]$$

For small values of the argument, $F(\eta_0)$ behaves as $\eta_0^6/24$: for larger arguments the values of F are collected in the following table

TABELLA I

η_0	$F(\eta_0)$	η_0	$F(\eta_0)$	η_0	$F(\eta_0)$	η_0	$F(\eta_0)$
0	$\eta_0^6/24$	2	1.2	4	29	6	185
1	0.03	3	7.5	5	80	7	380

Forbidden transitions

9. –

Before comparing theory and experiments we still want to illustrate some properties of the forbidden β -transitions.

As already mentioned, a transition is forbidden when the corresponding matrix element Eq.(35) vanishes. If the nucleus model for the individual protons and neutrons quantum states represents a good approximation, the matrix element Q_{mn}^* vanishes, for symmetry reasons, when it is not

$$(47) \quad i = i'$$

i and i' being the angular momentum, which we measure in units $h/2\pi$, of the neutronic state u_n and the proton state v_m . In the case in which the individual quantum states are not a good approximation the selection rule Eq.(47) is replaced by the corresponding

$$(48) \quad I = I'$$

where I and I' are the nucleus angular momentum before and after the β decay.

The selection rules Eq.(47) and (48) are far less rigorous than the selection rules in optics. Exceptions are possible particularly via the following processes:

a) Formula Eq.(26) has been obtained while neglecting the variations of ψ_s and φ_σ within the nucleus extension. If instead such variations are taken into account it becomes possible to obtain β transitions even when Q_{mn}^* vanishes. It is easily recognized that the intensity of such transitions, in order of magnitude, has with respect to that of the allowed transitions, a ratio $(R/\lambda)^2$, λ being the DE BROGLIE wavelength of the light particles. Notice that, at equal neutrino and electron energy, the kinetic energy of the first, when it is near the nucleus, is, because of the electrostatic attraction, much larger and hence the largest effect is due to the variations of ψ_s . An evaluation of the order of magnitude of the intensity of such forbidden processes shows that, at equal energy of the emitted electrons, they must be about hundred times less intense than the normal processes.

Besides the relatively small intensity, a characteristic feature of forbidden transitions has to be looked in the different form of the curve of the β rays energy distribution, which for the forbidden transitions, must yield a smaller number of particles with energy lower than in the normal case.

b) A second possibility for the realization of β transitions forbidden by the selection rule Eq.(48) depends on the feature, already remarked at the end of Sec. §3, that, when the neutrons and protons speed is not negligible compared to the speed of light, we must add to the interaction term Eq.(12) other terms of the order v/c . If, for instance, we admitted also for the heavy particles a DIRAC-like relativistic wave equation, we could add to Eq.(12) terms like

$$(49) \quad g Q (\alpha_x A_1 + \alpha_y A_2 + \alpha_z A_3) + c.c.$$

with $\alpha_x, \alpha_y, \alpha_z$ being the usual DIRAC matrices for the heavy particle and A_1, A_2, A_3 the spatial components of the four-vector defined by the Eq.(11).

A term like Eq.(49) allows also β transitions that do not satisfy the selection rule Eq.(48), and their intensity is, compared to that of the normal processes, of the order of magnitude $(v/c)^2$, *i.e.* of about 1/100. Thus we find a second possibility of forbidden transitions about 100 times less intense than the normal ones.

Comparison with experimental data

10. –

Eq.(45) establishes a relation between the maximum momentum η_0 of the β rays emitted by a substance and its mean lifetime. In such relation intervenes, indeed, an unknown quantity, the integral

$$(50) \quad \int v_m^* u_n d\tau$$

and its evaluation requires knowledge of the nuclear eigenfunctions of the neutron and of the proton. However in the allowed transitions case Eq.(50) is of the order of magnitude of unity. Therefore we expect that the product

$$(51) \quad \tau F(\eta_0)$$

has the same order of magnitude for all allowed transitions. For the forbidden transitions, instead, the mean lifetime will be, as order of magnitude, one hundred times larger, and correspondingly also the product Eq.(51) will be larger. In the following table are collected the products $\tau F(\eta_0)$ for all substances that disintegrate emitting β rays for which we have sufficiently exact data.

TABLE II

Element	τ in hours	η_0	$F(\eta_0)$	$\tau F(\eta_0)$
<i>UX</i> ₂ ...	0.026	5.4	115	3.0
<i>RaB</i> ...	0.64	2.04	1.34	0.9
<i>ThB</i> ...	15.3	1.37	0.176	2.7
<i>ThC</i> ''...	0.076	4.4	44	3.3
<i>AcC</i> ''...	0.115	3.6	17.6	2.0
<i>RaC</i> ...	0.47	7.07	398	190
<i>RaE</i> ...	173	3.23	10.5	1800
<i>ThC</i> ...	2.4	5.2	95	230
<i>MsTh</i> ₂ ...	8.8	6.13	73	640

In the table are, without doubt, recognizable the two groups that we expected; on the other hand such a partition in two groups of the elements that admit primary β rays had been already empirically observed by SARGENT⁽⁹⁾. The values of η_0 have been taken

(⁹) B.W. SARGENT, «Proc. Roy. Soc.», **139**, 659, 1933.

from the quoted work of SARGENT (for a comparison notice that: $\eta_0 = (H\rho)_{\max}/1700$). Besides the data reported in the table, SARGENT gives the data for three other elements, warning that they are not equally reliable as the others. They are: UX_1 for which it is $\tau = 830$; $\eta_0 = 0.76$; $F(\eta_0) = 0.0065$; $\tau F(\eta_0) = 5.4$; hence this element seems to belong to the first group. For AcB it is $\tau = 0.87$; $\eta_0 = 1.24$; $F(\eta_0) = 0.102$; $\tau F(\eta_0) = 0.09$; hence one finds a value for $\tau F(\eta_0)$ about ten times smaller than the values of the first group. Finally for RaD it is: $\tau = 320000$; $\eta_0 = 0.38$ (very uncertain); $F(\eta_0) = 0.00011$; $\tau F(\eta_0) = 35$. This element is therefore to be located at about half way between the two groups. I have been unable to find data about the other elements which emit primary β rays, *i.e.* $MstTh_1$, UY , Ac , AcC , UZ , RaC'' .

All things considered from the above comparison it can be concluded that agreement between theory and experiment is as good as it could have been expected.

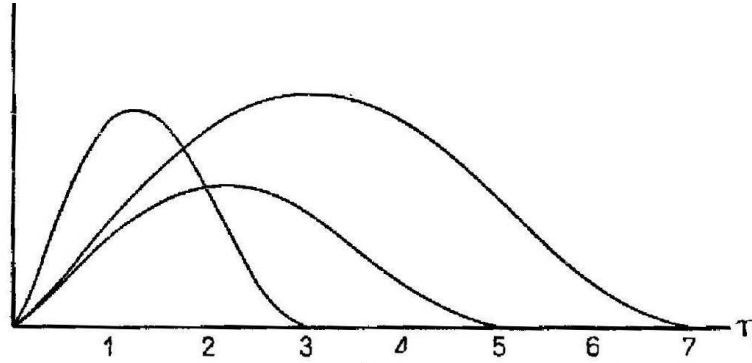


Fig. 2

The observed discrepancies for the elements on which there are uncertainties on the experimental data, RaD and AcB , can be well explained partially by the lack of precise measurements, partially also with the perfectly plausible variations of the matrix element Eq.(50). We must further remark that most β decays come together with γ rays emissions, indicating that most β processes may leave the proton in excited states, which again provides a mechanism that can produce variations in the value of $\tau F(\eta_0)$.

From the data of table II it is possible to deduce a value, although very rough, of the constant g . For instance if we admit that when the matrix element Eq.(50) has value 1, it is $\tau F(\eta_0) = 1 \text{ hour} = 3600 \text{ sec}$, one finds from Eq.(45):

$$g = 4 \cdot 10^{-50} \text{ cm}^3 \cdot \text{erg}$$

a value that, naturally, does not give anything more than the order of magnitude.

We now discuss the form of the distribution curve of the β rays speeds. In the case of the allowed processes, the distribution curve, as a function of η (*i.e.* up to the factor 1700 of $H\rho$) is drawn in Fig. 2, for various values of the maximum momentum η_0 . The curves are sufficiently similar to the experimentally collected ones by SARGENT⁽¹⁰⁾. Only in the low energy portion SARGENT's curves are a little lower than the theoretical ones, a feature which is more visible in the curves in Fig.3 in which energies rather than momenta

⁽¹⁰⁾ B.W. SARGENT, «Proc. Camb. Phil. Soc.», **28**,538, 1932.

are taken as abscissae. It as, however, to be remarked that the low energy portion of the curves is very imprecisely experimentally known ⁽¹¹⁾. Furthermore also theoretically for the forbidden transitions the low energy portion has to be lower than in the case of the curves that pertain to the allowed transitions, drawn in Fig.2 and 3.

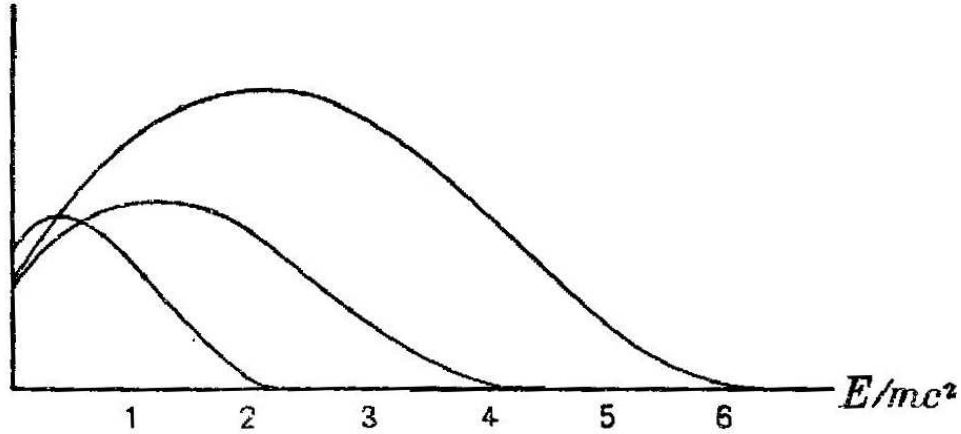


Fig. 3

This must be particularly taken into account in the case of *Ra E*, which is best known for the experimental viewpoint. The β rays emission from this element, as it appears from the anomalously high value of $\tau F(\eta_0)$ (C.f.r. table II) is certainly forbidden, actually it is possible that it becomes allowed only in a second order approximation. I hope to be able, in a forthcoming communication, to make more precise the behavior of the distribution curves for the forbidden transitions.

Summarizing it appears justified to state that the theory, in the form presented here, is found in agreement with the experimental data, which on the other hand are not always very precise. Even if, in a further comparison with experimental data, we should arrive at contradictions, it would always be possible to modify the theory without essentially altering the conceptual foundations. One could precisely save the equation Eq.(9) while choosing differently the $c_{s\sigma}$. This would in particular lead to a different form of the selection rule Eq.(48), and to a different form of the energy distribution curve.

Only a further development of the theory, as well as an increased precision of the experimental data, will possibly indicate whether such a modification will become necessary.

⁽¹¹⁾ *E.g.* c.f.r. RUTHERFORD, ELLIS and CHADWICK, Radiations from active substances, Cambridge, 1930. See, in particular p. 407.

• **Sunto** *Si propone una teoria quantitativa dell'emissione dei raggi β in cui si ammette l'esistenza del «neutrino» e si tratta l'emissione degli elettroni e dei neutrini all'atto della disintegrazione β con un procedimento simile a quello seguito nella teoria dell'irradiazione per descrivere l'emissione di un quanto di luce da un atomo eccitato. Vengono dedotte delle formule per la vita media e per la forma dello spettro continuo dei raggi β , e le si confrontano con i dati sperimentali.*

Translated from the Italian original by GIOVANNI GALLAVOTTI, May 2011

Other English translations exist, possibly based on the almost identical contemporary German version by Fermi himself, and contain useful preliminary comments:

(1) In: “The development of weak interaction theory”, Ed. P.K. Kabir, 'International Science Reviews Series', Vol.5, p.1–21, Gordon-Breach, 1963 (the translation might be based on the Italian version of the collected papers of E. Fermi, printed in 1962, because formula (11) for A_1 contains the same misprint absent in both the original Italian and German versions).

(2) In: F.L. Wilson, *Fermi's theory of beta decay*, American Journal of Physics, **36**, 1150-160, 1968, (from the German version).