Risoluzione MQ - Settembre 2009

(I)

Lo stato è una combinazione dei primi tre livelli e le funzioni d'onda sono proporzionali a

$\frac{H_n}{\sqrt{2^n n!}}$

$$\phi_0 \sim e^{-\frac{\alpha}{2}x^2}$$

$$\phi_1 \sim \frac{H_1(\sqrt{\alpha}x)}{\sqrt{2}} e^{-\frac{\alpha}{2}x^2} \sim \sqrt{2}\sqrt{\alpha}x\, e^{-\frac{\alpha}{2}x^2}$$

$$\phi_2 \sim \frac{H_2(\sqrt{\alpha}x)}{2\sqrt{2}} e^{-\frac{\alpha}{2}x^2} \sim \left(\sqrt{2}\,\alpha x^2 - \frac{1}{\sqrt{2}}\right) e^{-\frac{\alpha}{2}x^2}$$

Quindi
$$|\phi\rangle = N'\left(|0\rangle + \sqrt{2}|1\rangle + \sqrt{2}\left(|2\rangle + \frac{|0\rangle}{\sqrt{2}}\right)\right)$$
$$= N'\left(2|0\rangle + \sqrt{2}|1\rangle + \sqrt{2}|2\rangle\right)$$

normalizzando $\longrightarrow$
$$= \frac{\sqrt{2}|0\rangle + |1\rangle + |2\rangle}{2}$$

- Le energie sono $\hbar\omega(n+\frac{1}{2})$

$$\langle E\rangle = \sum E_n P(E_n) = \frac{1}{2}\cdot\frac{\hbar\omega}{2} + \frac{1}{4}\,\frac{3\hbar\omega}{2} + \frac{1}{4}\,\frac{5\hbar\omega}{2}$$
$$= \frac{\hbar\omega}{8}10 = \frac{5\hbar\omega}{4}$$

- $x = \frac{1}{\sqrt{2\alpha}}(a + a^{+})$ $\qquad a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$\langle n|x|n\rangle = 0$$

$$\langle 1|x|0\rangle \left(= \langle 0|x|1\rangle\right) = \langle 1|\frac{a^{+}}{\sqrt{2\alpha}}|0\rangle = \frac{1}{\sqrt{2\alpha}}\langle 1|1\rangle = \frac{1}{\sqrt{2\alpha}}$$

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$$\langle 2|x|1\rangle \left(= \langle 1|x|2\rangle\right) = \langle 2|\frac{a^{+}}{\sqrt{2\alpha}}|1\rangle = \frac{1}{\sqrt{\alpha}}$$

$$\langle\phi|x|\phi\rangle = 2\left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\langle 1|x|0\rangle + \frac{1}{2}\cdot\frac{1}{2}\langle 1|x|2\rangle\right) = \frac{1}{\sqrt{\alpha}}$$

[2 punti] • $\langle n|x^2|n\rangle = \frac{1}{2\alpha}\langle n|\, a^2 + \underset{\sqrt{n+1}\sqrt{n+1}}{aa^\dagger} + \underset{\sqrt{n}\sqrt{n}}{a^\dagger a} + a^{\dagger 2}\,|n\rangle = \frac{2n+1}{2\alpha} = \frac{1}{\alpha}\left(n+\frac{1}{2}\right)$

$$\langle 0|x^2|2\rangle = \langle 2|x^2|0\rangle = \frac{1}{2\alpha}\langle 2|a^{\dagger 2}|0\rangle = \frac{1}{2\alpha}\langle 2|a^\dagger|1\rangle = \frac{1}{\sqrt{2}\alpha}$$

$$\langle \phi|x^2|\phi\rangle = \frac{1}{2}\langle 0|x^2|0\rangle + \frac{1}{4}\langle 1|x^2|1\rangle + \frac{1}{4}\langle 2|x^2|2\rangle + 2\frac{1}{2}\frac{1}{\sqrt{2}}\langle 2|x^2|0\rangle$$

$$= \frac{1}{2}\frac{1}{2\alpha} + \frac{1}{4}\frac{3}{2\alpha} + \frac{1}{4}\frac{5}{2\alpha} + \frac{1}{2\alpha}$$

$$= \frac{1}{8\alpha}(2+3+5+4) = \frac{7}{4\alpha}$$

II

$$E_n = -\frac{E_0}{n^2}$$

• $$|\psi\rangle = N\left(|100\rangle + a|200\rangle + |210\rangle\right)$$

$$N = \frac{1}{\sqrt{2+a^2}}$$

$$\langle\psi|H|\psi\rangle = \frac{1}{2+a^2}\left(-E_0 - a^2\frac{E_0}{4} - \frac{E_0}{4}\right) = -\frac{5+a^2}{4(2+a^2)}E_0$$

poiché $\langle H\rangle$ deve essere $-\frac{E_0}{2}$ e a positivo $\Rightarrow$ $\boxed{a=1}$

• $$H = \mu\vec{L}\cdot\vec{B} = \mu L_x B$$

Nella base di L_x $\qquad n=2$

$$\begin{cases} |n11\rangle_x = \dfrac{|n11\rangle + \sqrt{2}|n10\rangle + |n1-1\rangle}{2} & \text{con energia} \quad -\dfrac{E_0}{4} + \mu\hbar B \\ |n10\rangle_x = \dfrac{|n11\rangle - |n1-1\rangle}{\sqrt{2}} & \qquad -\dfrac{E_0}{4} \\ |n1-1\rangle_x = \dfrac{|n11\rangle - \sqrt{2}|n10\rangle + |n1-1\rangle}{2} & \qquad -\dfrac{E_0}{4} - \mu\hbar B \end{cases}$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle_x + |200\rangle_x + \frac{|211\rangle_x - |21-1\rangle_x}{\sqrt{2}}\right) \Rightarrow |\psi(t)\rangle = |100\rangle_x \frac{e^{+i\frac{E_0 t}{\hbar}}}{\sqrt{3}} + |200\rangle_x \frac{e^{+i\frac{E_0 t}{4\hbar}}}{\sqrt{3}} + \frac{|211\rangle_x}{\sqrt{6}} e^{+i\frac{E_0 t}{4\hbar} - i\mu Bt} - \frac{|21-1\rangle_x}{\sqrt{6}} e^{+i\frac{E_0 t}{4\hbar} + i\mu Bt}$$

$$\Rightarrow \begin{cases} P(L_z=0) = |\langle 100|\psi(t)\rangle|^2 + |\langle 200|\psi(t)\rangle|^2 + |\langle 210|\psi(t)\rangle|^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\cos^2\mu Bt \\ P(L_z=\pm\hbar) = |\langle 21\pm1|\psi(t)\rangle|^2 = \frac{1}{6}\,\mathrm{sen}^2\mu Bt \end{cases}$$

III Dai libri: gli autostati di $\vec{n}\cdot\vec{S}$, $\vec{n} = (\text{sen}\theta\cos\varphi, \text{sen}\theta\,\text{sen}\varphi, \cos\theta)$

sono
$$|+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \text{sen}\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$$
$$|-\rangle = \begin{pmatrix} -\text{sen}\frac{\theta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$$
con autovalori $\pm\hbar/2$

$H = \mu\vec{B}\cdot\vec{S}$: gli autovalori sono $\pm\mu B\frac{\hbar}{2}$

e $\vec{B} = B\vec{n}$ con $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{4}$

quindi gli autostati sono
$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/8} \\ e^{i\pi/8} \end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{-i\pi/8} \\ e^{i\pi/8} \end{pmatrix}$$

Si chiedono informazioni su $\vec{S}\cdot\vec{n}$ dove $\vec{n}$ è ruotato di 90 gradi rispetto a $\vec{B}$. I suoi autovettori sono $\left(\theta = \frac{\pi}{2}, \varphi = -\frac{\pi}{4}\right)$
$$|+\rangle_{\vec{n}} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/8} \\ e^{-i\pi/8} \end{pmatrix} \qquad |-\rangle_{\vec{n}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{i\pi/8} \\ e^{-i\pi/8} \end{pmatrix}$$

$$|\psi(0)\rangle = |+\rangle_{\vec{n}} = \underbrace{\langle +|+\rangle_{\vec{n}}}_{\cos\frac{\pi}{4}}|+\rangle + \underbrace{\langle -|+\rangle_{\vec{n}}}_{\left(-i\,\text{sen}\frac{\pi}{4}\right)}|-\rangle = \frac{|+\rangle - i|-\rangle}{\sqrt{2}}$$

quindi
$$|\psi(t)\rangle = \frac{|+\rangle}{\sqrt{2}}e^{-\frac{i\mu Bt}{2}} - \frac{i}{\sqrt{2}}|-\rangle e^{\frac{i\mu Bt}{2}}$$

$$P\left(\vec{S}\cdot\vec{n} = -\frac{\hbar}{2}; t\right) = \left|{}_{\vec{n}}\langle -|\psi(t)\rangle\right|^2 = \frac{1}{\ } \left| \frac{i\,\text{sen}\frac{\pi}{4}}{\sqrt{2}} e^{-\frac{i\mu Bt}{2}} - \frac{i}{\sqrt{2}}\cos\frac{\pi}{4}\, e^{\frac{i\mu Bt}{2}} \right|^2 = \text{sen}^2\frac{\mu Bt}{2}$$

[Per simmetria si poteva più semplicemente cambiare assi e portare $\vec{B}$ lungo l'asse z e $\vec{S}\cdot\vec{n}$ lungo l'asse x]