Soluzioni 18 Febr. 09

(I) $\frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$ L'oscillatore ha frequenze $\omega_x = \omega_y \equiv \omega \qquad \omega_z = 2\omega$

spettro $$E = \left(n_x+\frac{1}{2}\right)\hbar\omega_x + \left(n_y+\frac{1}{2}\right)\hbar\omega_y + \left(n_z+\frac{1}{2}\right)\hbar\omega_z$$
$$= \hbar\omega\left(n_x+n_y+2n_z+2\right)$$

Poichè $u_n(x) \sim H_n(x)\, e^{-\frac{m\omega}{2\hbar}x^2}$ lo stato è

$$|\psi(0)\rangle \sim |100\rangle + |010\rangle \xrightarrow[\text{normalizzato}]{} |\psi(0)\rangle = \frac{|100\rangle + |010\rangle}{\sqrt{2}}$$

- Per un singolo oscillatore $\langle n_x|x|n_x\rangle = 0$
  $\langle n_x|x^2|n_x\rangle = \frac{\hbar}{\omega_x m}\left(n_x+\frac{1}{2}\right)$

$$\langle r^2\rangle = \langle x^2+y^2+z^2\rangle = \langle 100|x^2|100\rangle + \langle 010|x^2|010\rangle + (\text{stesso per } y^2, z^2)$$

[$\langle 100|x^2|010\rangle$ ‖ 0 ortogonale in y]

$$= \frac{1}{2}\Bigg(\frac{\hbar}{m\omega}\left(1+\frac{1}{2}\right) + \frac{\hbar}{m\omega}\left(0+\frac{1}{2}\right) + \qquad \leftarrow x^2$$
$$\frac{\hbar}{m\omega}\left(0+\frac{1}{2}\right) + \frac{\hbar}{m\omega}\left(1+\frac{1}{2}\right) + \qquad \leftarrow y^2$$
$$\frac{\hbar}{2m\omega}\left(0+\frac{1}{2}\right) + \frac{\hbar}{2m\omega}\left(0+\frac{1}{2}\right)\Bigg) \qquad \leftarrow z^2$$

$$= 2\frac{\hbar}{2m\omega}\left(4+\frac{1}{2}\right) = \frac{9\hbar}{4m\omega}$$

- $L_z = -i\hbar\frac{\partial}{\partial\phi}$ e il problema è a simmetria assiale
  $x = \rho\cos\phi, \quad y = \rho\,\text{sen}\phi, \quad z = z$

$$\psi(\rho,\phi,z) = N(\cos\phi + \text{sen}\phi)\,\rho\, e^{-\frac{m\omega}{2\hbar}\rho^2 - \frac{m\omega}{\hbar}z^2}$$

$$\underset{\text{dipendenza da }\phi:}{\sim} \left[e^{i\phi}\left(\frac{1}{2}+\frac{1}{2i}\right) + e^{-i\phi}\left(\frac{1}{2}-\frac{1}{2i}\right)\right]$$

Autofunzioni di L_z: $L_z \frac{e^{in\phi}}{\sqrt{2\pi}} = n\hbar \frac{e^{in\phi}}{\sqrt{2\pi}}$ $|n\rangle_\phi = \frac{e^{in\phi}}{\sqrt{2\pi}}$

quindi $L_z = \pm\hbar$: $P(L_z=\hbar) = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2}$

$\psi \sim \frac{e^{i\phi}}{\sqrt{2\pi}} \frac{1-i}{2} + \frac{e^{-i\phi}}{\sqrt{2\pi}} \frac{1+i}{2}$ $P(L_z=-\hbar) = \left|\frac{1+i}{2}\right|^2 = \frac{1}{2}$

normalizzato a 1

$\psi = |1\rangle_\phi \frac{1-i}{2} + |-1\rangle \frac{1+i}{2}$

- $H = H_0 + \mu L_z$: il livello originale per $\mu = 0$ era autostato di H_0 con energia $E = 3\hbar\omega$ e può essere scritto come combinazione di autostati comuni a H_0 e L_z:

$|3\hbar\omega; \pm 1\rangle = N(x \pm iy) e^{-\frac{m\omega}{2\hbar}(x^2+y^2+2z^2)}$ $H|3\hbar\omega, \pm 1\rangle = (3\hbar\omega \pm \mu\hbar)|3\hbar\omega; \pm 1\rangle$

$= N e^{\pm i\phi} \rho e^{-\frac{m\omega}{2\hbar}\rho^2 - \frac{m\omega}{\hbar} z^2}$

$\psi(0) = \frac{1-i}{2}|3\hbar\omega; +1\rangle + \frac{1+i}{2}|3\hbar\omega; -1\rangle \Rightarrow \psi(t) = e^{-3i\omega t}\left(\frac{1-i}{2}|3\hbar\omega; 1\rangle e^{-i\mu t} + \frac{1+i}{2}|3\hbar\omega; -1\rangle e^{i\mu t}\right)$

con l'unico effetto di sostituire $\phi \to \phi - \mu t$ e di moltiplicare per una fase

$\psi(\rho, \phi, z) \sim N(\cos(\phi - \mu t) + \sin(\phi - \mu t))\rho\, e^{-\frac{m\omega}{2\hbar}(\rho^2 + 2z^2)}$

II

$H = \frac{\varepsilon}{\hbar}\left((\vec{S}_1 + \vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2\right) - \varepsilon(S_{1z} + S_{2z})$

$\vec{S} = \vec{S}_1 + \vec{S}_2$

$= \frac{\varepsilon}{\hbar}\left(\vec{S}^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right) - S_z\varepsilon = \frac{\varepsilon}{\hbar}\left(\vec{S}^2 - \frac{3}{2}\hbar^2\right) - S_z\varepsilon$

$S = 0, 1$

Tripletto $S^2 = 2\hbar^2$	$\|11\rangle$	$E = -\varepsilon\hbar/2$	$\|++\rangle$
	$\|10\rangle$	$E = \varepsilon\hbar/2$	$\frac{\|+-\rangle + \|-+\rangle}{\sqrt{2}}$
	$\|1-1\rangle$	$E = 3\varepsilon\hbar/2$	$\|--\rangle$
Singoletto	$\|00\rangle$	$E = -3\varepsilon\hbar/2$	$\frac{\|+-\rangle - \|-+\rangle}{\sqrt{2}}$

- $|\psi(0)\rangle = \frac{|+-\rangle + |++\rangle}{\sqrt{2}} = \frac{|11\rangle}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|10\rangle + |00\rangle)\right) = \frac{|11\rangle}{\sqrt{2}} + \frac{|10\rangle + |00\rangle}{2}$

$$|\psi(t)\rangle = \frac{|11\rangle}{\sqrt{2}} e^{\frac{i\varepsilon t}{2}} + \frac{|10\rangle e^{-\frac{i\varepsilon t}{2}} + |00\rangle e^{+\frac{3i\varepsilon t}{2}}}{2} \qquad P(\vec{S}=0) = \left|\frac{e^{\frac{3i\varepsilon t}{2}}}{2}\right|^2 = \frac{1}{4}$$

- $$\langle S_z \rangle = \sum P(S_z = \hbar m)\hbar m = \left|\frac{e^{i\varepsilon t/2}}{\sqrt{2}}\right|^2 \hbar + \left|\frac{e^{-\frac{i\varepsilon t}{2}}}{2}\right|^2 \cdot 0 + \left|\frac{e^{\frac{3i\varepsilon t}{2}}}{2}\right|^2 \cdot 0 = \frac{\hbar}{2}$$

- $$|\psi(t)\rangle = \frac{|++\rangle}{\sqrt{2}} e^{\frac{i\varepsilon t}{2}} + e^{\frac{i\varepsilon t}{2}} \frac{|10\rangle e^{-i\varepsilon t} + |00\rangle e^{i\varepsilon t}}{2} = \frac{|++\rangle}{\sqrt{2}} e^{\frac{i\varepsilon t}{2}} + \frac{e^{\frac{i\varepsilon t}{2}}}{\sqrt{2}}\Big(|+-\rangle \cos\varepsilon t - |-+\rangle i \operatorname{sen}\varepsilon t\Big)$$

$P(S_{z1})$:	$\frac{1}{2}$	$\frac{\cos^2\varepsilon t}{2}$	$\frac{\operatorname{sen}^2\varepsilon t}{2}$
	$S_{z1} = \frac{\hbar}{2}$	$S_{z1} = \frac{\hbar}{2}$	$S_{z1} = -\frac{\hbar}{2}$

$$\langle S_{z1} \rangle = \frac{1}{2}\cdot\frac{\hbar}{2} + \frac{\cos^2\varepsilon t}{2}\frac{\hbar}{2} + \frac{\operatorname{sen}^2\varepsilon t}{2}\left(-\frac{\hbar}{2}\right) =$$

$$= \frac{\hbar}{2}\,\frac{1+\cos 2\varepsilon t}{2} = \frac{\hbar}{2}\cos^2\varepsilon t$$

(III)

base delle armoniche sferiche con $\ell = 1$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \operatorname{sen}\theta\, e^{i\varphi} \qquad Y_{1-1} = \sqrt{\frac{3}{8\pi}} \operatorname{sen}\theta\, e^{-i\varphi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\psi(\theta,\varphi) \sim N\left(Y_{10} + \sqrt{2}\,\frac{Y_{1-1} - Y_{11}}{2}\right) \underset{\text{normalizzando}}{=} \frac{Y_{10}}{\sqrt{2}} + \frac{Y_{1-1} + Y_{11}}{2}$$

$$|\psi\rangle = \frac{|10\rangle}{\sqrt{2}} + \frac{|11\rangle - |1-1\rangle}{2}$$

- $|11\rangle$ $\quad E = \frac{\hbar^2}{I} + \hbar\mu$

  $|10\rangle$ $\quad E = \frac{\hbar^2}{I}$

  $|1-1\rangle$ $\quad E = \frac{\hbar^2}{I} - \hbar\mu$

  $L^2 = \hbar^2\ell(\ell+1) = 2\hbar^2$

  $$|\psi(t)\rangle = e^{-\frac{i\hbar t}{I}}\left(\frac{|10\rangle}{\sqrt{2}} + \frac{|11\rangle e^{-i\mu t} - |1-1\rangle e^{i\mu t}}{2}\right)$$

- $$\langle L_z \rangle = \frac{1}{2}\cdot 0 + \frac{1}{4}\hbar + \frac{1}{4}(-\hbar) = 0$$

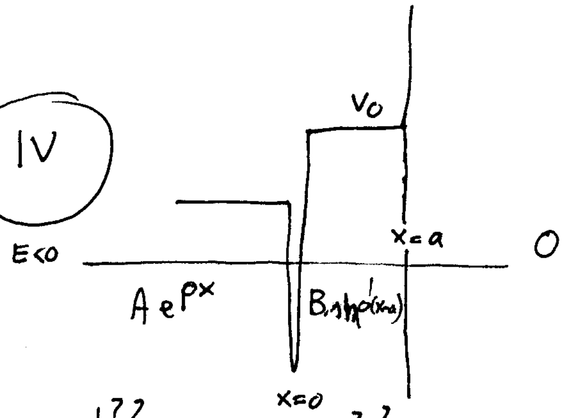
  $$\langle H \rangle = \frac{1}{2}\left(\frac{\hbar^2}{I}\right) + \frac{1}{4}\left(\frac{\hbar^2}{I} + \hbar\mu\right) + \frac{1}{4}\left(\frac{\hbar^2}{I} - \hbar\mu\right) = \frac{\hbar^2}{I}$$

- $$|\psi_f\rangle = -\frac{|10\rangle}{\sqrt{2}} + \frac{|11\rangle - |1-1\rangle}{2}$$

$$P = |\langle \psi_f | \psi(t) \rangle|^2 = \left| -\frac{1}{2} + \frac{1}{4} e^{-i\mu t} + \frac{1}{4} e^{i\mu t} \right|^2 = \left| \frac{\cos \mu t - 1}{2} \right|^2$$

$$= \left| -\mathrm{sen}^2 \frac{\mu t}{2} \right|^2 = \mathrm{sen}^4 \frac{\mu t}{2}$$

IV



- Stati legati:

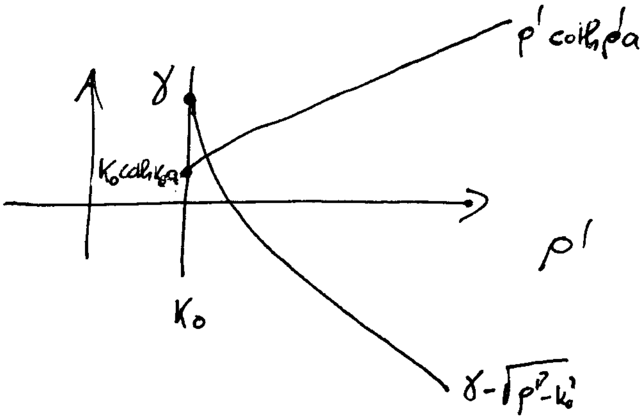
$$\varphi(0+) = \varphi(0-)$$
$$\varphi'(0+) - \varphi'(0-) = -\gamma \varphi(0)$$

$$\frac{\hbar^2 \rho^2}{2m} = -E \qquad \frac{\hbar^2 \rho'^2}{2m} = -E + V_0 \implies \frac{\hbar^2 \rho'^2}{2m} = \frac{\hbar^2 \rho^2}{2m} + V_0 \implies \rho'^2 = \rho^2 + k_0^2 \qquad k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$

$$\begin{cases} A = -B \, sh \rho' a \\ +B\rho' ch \rho' a - A\rho = -\gamma A \end{cases} \implies +\rho' \coth \rho' a + \rho = \gamma$$

$$\begin{cases} \rho' \coth \rho' a = \gamma - \rho = \gamma - \sqrt{\rho'^2 - k_0^2} \\ \rho'^2 = \rho^2 + k_0^2 \end{cases}$$

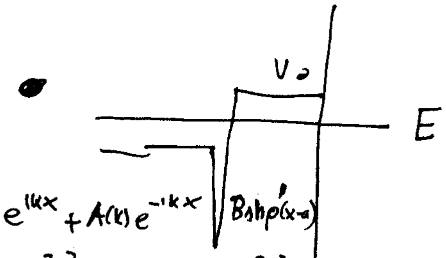
$$\boxed{\rho' \geq k_0}$$



stati legati se $\gamma \geq k_0 \coth k_0 a$

cioè se $V_0 \leq \overline{V_0}$ con $\boxed{\sqrt{\frac{2m\overline{V_0}}{\hbar^2}} \coth \sqrt{\frac{2m\overline{V_0}}{\hbar^2}} a = \gamma}$

(che ha soluzione se $\gamma > \frac{1}{a}$)

- 

$$\frac{\hbar^2 k^2}{2m} = E \qquad \frac{\hbar^2 \rho'^2}{2m} = V_0 - E$$

$$\begin{cases} 1 + A = -B \, sh \rho' a \\ +B\rho' ch \rho' a - ik(1-A) = -\gamma(1+A) \end{cases}$$

$$\Downarrow$$

$$(1+A)(\gamma - \rho' \coth \rho' a) - ik(1-A) = 0$$

$$|A| = 1 \qquad A = \frac{ik - \gamma + \rho' \coth \rho' a}{ik + \gamma - \rho' \coth \rho' a} \implies \begin{array}{l} R = |A| = 1 \\ T = 1 - R = 0 \end{array}$$

- $V_0 \to \infty$: – niente stati legati
  – $\rho' \to \infty$: $A \to -1$ $\quad \varphi_I(x) = e^{ikx} - e^{-ikx} \sim \mathrm{sen}\, kx$