4 FEB 09 SOLUZIONI

I Le autofunzioni normalizzate della buca sono

$$\psi_n(x) = \sqrt{\frac{2}{a}}\,\mathrm{sen}\,\frac{n\pi x}{a} \quad \text{con} \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

quindi $\psi_1(x) = \sqrt{\frac{2}{a}}\,\mathrm{sen}\,\frac{\pi x}{a}$, $\psi_2 = \sqrt{\frac{2}{a}}\,\mathrm{sen}\,\frac{2\pi x}{a} = \sqrt{\frac{2}{a}}\,2\,\mathrm{sen}\,\frac{\pi x}{a}\cos\frac{\pi x}{a}$

$$\psi(x,0) = C\left(\psi_1(x) + \frac{\psi_2(x)}{2}\right) \xrightarrow[\text{normalizzato}]{} \psi(x,0) = \frac{2\psi_1(x) + \psi_2(x)}{\sqrt{5}}$$

- $$\langle E\rangle = \sum E_n P(E_n) = \underbrace{\frac{4}{5}}_{P(E_1)}\cdot\frac{\pi^2\hbar^2}{2ma^2} + \underbrace{\frac{1}{5}}_{P(E_2)}\,\frac{4\pi^2\hbar^2}{2ma^2} = \frac{4\pi^2\hbar^2}{5ma^2}$$

- $$\psi(x,t) = \frac{2e^{-i\frac{\pi^2\hbar}{2ma^2}t}\psi_1(x) + e^{-\frac{4\pi^2\hbar}{2ma^2}t}\psi_2(x)}{\sqrt{5}}$$

lo stato finale è $\psi_f(x) = \dfrac{\psi_1(x) - 2\psi_2(x)}{\sqrt{5}}$

e la probabilità

$$P = |\langle\psi_f|\psi(t)\rangle|^2 = \left|\frac{2}{5}e^{-\frac{i\pi^2\hbar}{2ma^2}t} - \frac{2}{5}e^{-\frac{i4\pi^2\hbar}{2ma^2}t}\right|^2$$

$$= \frac{16}{25}\,\mathrm{sen}^2\,\frac{3\pi^2\hbar}{4ma^2}t$$

- la probabilità che la particella sia in $a\le x\le 2a$ sarà zero se la buca è raddoppiata istantaneamente

$$\tilde\psi(x,0) = \begin{cases}\dfrac{2\psi_1(x)+\psi_2(x)}{\sqrt{5}} & 0\le x\le a\\ 0 & a\le x\le 2a\end{cases}$$

Le autofunzioni del sistema finale sono $\psi_n^{FIN}(x) = \sqrt{\frac{2}{2a}}\,\mathrm{sen}\,\frac{n\pi x}{2a}$ e il primo stato eccitato è $\psi_2^{FIN}(x) = \sqrt{\frac{1}{a}}\,\mathrm{sen}\,\frac{\pi x}{a}$ che coincide con $\frac{1}{\sqrt{2}}\psi_1(x)$ in $0\le x\le a$

$$P = \left|\langle \psi_2^{FIN} | \tilde{\psi} \rangle\right|^2 = \left|\int_0^{2a} \psi_2^{FIN\,*}(x)\, \tilde{\psi}(x)\, dx\right|^2 \equiv \left|\int_0^{a} \psi_2^{FIN}(x)\, \tilde{\psi}(x)\, dx\right|^2$$

$$= \left|\int_0^{a} \frac{1}{\sqrt{2}} \psi_1^*(x)\, \frac{2\psi_1(x) + \psi_2(x)}{\sqrt{5}}\right|^2 = \left|\sqrt{\frac{2}{5}}\right|^2 = \frac{2}{5}$$

(II) $H = \hbar I + \hbar \varepsilon \sigma_1$ che ha autovalori ed autovettori (basta diagonalizzare σ_1) $E_\pm = \hbar \pm \hbar\varepsilon$ $\quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

- $\psi(0) = \begin{pmatrix}1\\0\end{pmatrix} = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \rightarrow \psi(t) = e^{-it}\, \frac{e^{-i\varepsilon t}|+\rangle + e^{i\varepsilon t}|-\rangle}{\sqrt{2}} = e^{-it}\begin{pmatrix}\cos\varepsilon t\\ -i\,\mathrm{sen}\,\varepsilon t\end{pmatrix}$

$$P(|-\rangle) = \left|(0,1)\cdot e^{-it}\begin{pmatrix}\cos\varepsilon t\\ -i\,\mathrm{sen}\,\varepsilon t\end{pmatrix}\right|^2 = \mathrm{sen}^2\varepsilon t$$

- $\langle C \rangle = \langle \psi(t) | C | \psi(t) \rangle = e^{it}(\cos\varepsilon t, i\,\mathrm{sen}\,\varepsilon t)\; i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\varepsilon t\\ -i\,\mathrm{sen}\,\varepsilon t\end{pmatrix} e^{-it}$

$$= (\cos\varepsilon t, i\,\mathrm{sen}\,\varepsilon t)\begin{pmatrix}-\mathrm{sen}\,\varepsilon t\\ i\cos\varepsilon t\end{pmatrix} = -2\,\mathrm{sen}\,\varepsilon t\cos\varepsilon t$$
$$= -\mathrm{sen}\,2\varepsilon t$$

- ~~$[H,\sigma_1] = 0$ σ_1 commuta con H e il valore~~

* $C = \sigma_2$ ha autovettori $|c=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$, $|c=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$

$$P(c=1) = |\langle c=1|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1,-i)\cdot e^{-it}\begin{pmatrix}\cos\varepsilon t\\ -i\,\mathrm{sen}\,\varepsilon t\end{pmatrix}\right|^2$$
$$= \left|\frac{\cos\varepsilon t - \mathrm{sen}\,\varepsilon t}{\sqrt{2}}\right|^2 = \frac{1 - \mathrm{sen}\,2\varepsilon t}{2}$$

$$P(c=-1) = |\langle c=-1|\psi(t)\rangle|^2 = \left|\frac{\cos\varepsilon t + \mathrm{sen}\,\varepsilon t}{\sqrt{2}}\right|^2 = \frac{1 + \mathrm{sen}\,2\varepsilon t}{2}$$

- le teorema di Ehrenfest: $\frac{d}{dt}\langle C\rangle = \left\langle \frac{\partial}{\partial t} C\right\rangle + \frac{1}{i\hbar}\langle [C,H]\rangle$

$$\begin{cases} [C,H] = [\sigma_2, \hbar I + \hbar\varepsilon\sigma_1] = -2i\hbar\varepsilon\sigma_3 \\ \frac{\partial C}{\partial t} = 0 \end{cases}$$

$$\langle [C,H]\rangle = -2i\hbar\varepsilon\,(\cos\varepsilon t, +i\,\mathrm{sen}\,\varepsilon t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varepsilon t\\ -i\,\mathrm{sen}\,\varepsilon t\end{pmatrix}$$
$$= -2i\hbar\varepsilon(\cos^2\varepsilon t - \mathrm{sen}^2\varepsilon t)$$
$$= -2i\hbar\varepsilon\cos 2\varepsilon t$$

e quindi $\frac{d\langle C\rangle}{dt} = -2\cos 2\varepsilon t = \frac{1}{i\hbar}\langle [C,H]\rangle$

III $\psi = f(r)\, r\, \underbrace{\left(1 + \frac{iy}{r}\right)}_{}$

solo e parte angolare
serve per studiare L^2, L_x, L_z

Ora $Y_{00} = \frac{1}{\sqrt{4\pi}}$, $Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\, \frac{x \pm iy}{r} \longrightarrow \frac{iy}{r} = -\frac{Y_{11} + Y_{1-1}}{2}\sqrt{\frac{8\pi}{3}}$

$$\psi(x) \cong c\left(\sqrt{4\pi}Y_{00} - \sqrt{4\pi}\,\frac{Y_{11} + Y_{1-1}}{\sqrt{6}}\right) \xrightarrow[\text{normalizzazione}]{} \psi \sim \frac{\sqrt{6}\,Y_{00} - Y_{11} - Y_{1-1}}{\sqrt{8}}$$

$$\psi = \sqrt{\frac{6}{8}}|00\rangle - \frac{1}{\sqrt{8}}|11\rangle - \frac{1}{\sqrt{8}}|1-1\rangle$$

- $\vec{L}^2 = \hbar\ell(\ell+1)$ e $\ell=0$ oppure $\ell=1$

$P(\ell=0) = 6/8$
$P(\ell=1) = 1/8 + 1/8 = 2/8$

$L_z = \hbar m$ e $m=0,\ m=1,\ m=-1$

$P(m=0) = 6/8 \quad P(m=1) = 1/8 \quad P(m=-1) = 1/8$

$L_x = 0, \pm\hbar$ con autofunzioni $|00\rangle_x = |00\rangle$

$$|1\pm1\rangle_x = \frac{|11\rangle \pm \sqrt{2}|10\rangle + |1-1\rangle}{2} \rightarrow \frac{1}{2}\begin{pmatrix}1\\ \pm\sqrt{2}\\ 1\end{pmatrix}$$

$$|10\rangle_x = \frac{|11\rangle - |1-1\rangle}{\sqrt{2}} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$$

$$\psi = \sqrt{\frac{6}{8}}|00\rangle_x - \frac{1}{\sqrt{8}}\left(|1-1\rangle_x + |11\rangle_x\right)$$

$P(L_x=0) = 6/8$
$P(L_x=\hbar) = 1/8$
$P(L_x=-\hbar) = 1/8$

- $\langle L_z\rangle = \sum P(L_z)\hbar m = \frac{6}{8}\cdot 0 + \frac{1}{8}\hbar + \frac{1}{8}(-\hbar) = 0$

$\langle L_x\rangle = \sum P(L_x)\ell_x = \frac{6}{8}\cdot 0 + \frac{1}{8}\hbar + \frac{1}{8}(-\hbar) = 0$

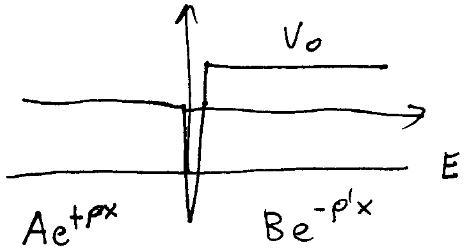
- $$\psi(t) = \sqrt{\frac{6}{8}}|00\rangle_x - \frac{1}{\sqrt{8}}\left(|1-1\rangle_x e^{i\mu Bt} + |11\rangle_x e^{-i\mu Bt}\right)$$

oppure, se uno vuole riscriverla:

$$\psi(t) = \sqrt{\frac{6}{8}}\,|00\rangle - \frac{1}{\sqrt{8}}\left(|11\rangle\cos\mu Bt + |1{-}1\rangle\cos\mu Bt - i\sqrt{2}\sin\mu Bt\,|10\rangle\right)$$

oppure: $\psi(x,t) = f(r)\left(x + iy\cos\mu Bt + iz\sin\mu Bt\right)$

IV



- $Ae^{+\rho x}$ $\quad$ $Be^{-\rho' x}$

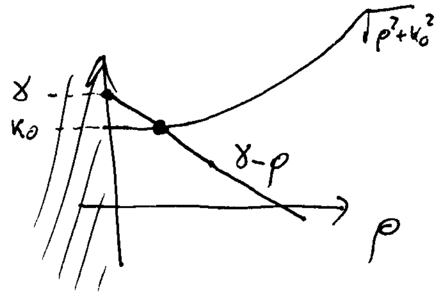
$$\frac{\hbar^2\rho^2}{2m} = -E \qquad \frac{\hbar^2\rho'^2}{2m} = V_0 - E \quad \Rightarrow \quad \rho'^2 = \rho^2 + \frac{2mV_0}{\hbar^2} \equiv \rho^2 + k_0^2$$

Condizioni di raccordo $\begin{cases} \varphi(0+) = \varphi(0-) \\ \varphi'(0+) - \varphi'(0-) = -\gamma\,\varphi(0) \end{cases}$ $\quad\begin{cases} A = B \\ -\rho' B - \rho A = -\gamma A \end{cases}$

$$\Downarrow$$

$$\rho + \rho' = \gamma$$

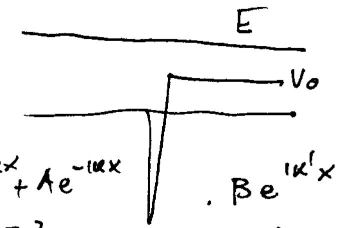
$$\sqrt{\rho^2 + k_0^2} = \gamma - \rho$$



stato legato se $\gamma \geq k_0 \rightarrow k_0^2 \leq \gamma^2 \rightarrow V_0 \leq \frac{\hbar^2\gamma^2}{2m}$

- $$\int_{-\infty}^{+\infty} |\varphi(x)|^2 dx = \frac{|A|^2}{2\rho} + \frac{|B|^2}{2\rho'} = |A|^2\frac{\gamma}{2\rho\rho'} \equiv 1 \quad \rightarrow \quad A = \sqrt{\frac{2\rho\rho'}{\gamma}}$$

$$P(x<0) = \int_{-\infty}^{0} |\varphi(x)|^2 dx = \frac{2\rho\rho'}{\gamma}\int_{-\infty}^{0} e^{2\rho x}\,dx = \frac{\rho'}{\gamma}$$

- 

$e^{ikx} + Ae^{-ikx}$ $\quad$ $Be^{ik'x}$

$$\frac{\hbar^2 k^2}{2m} = E \qquad \frac{\hbar^2 k'^2}{2m} = E - V_0$$

Raccordo: $\quad 1 + A = B$

$$ik'B - ik(1-A) = -\gamma B$$

$$\Downarrow$$

$$(\gamma + ik')(1+A) - ik(1-A) = $$

$$A = \frac{-\gamma + i(k-k')}{\gamma + i(k+k')}$$

$$\Rightarrow \quad R = |A|^2 \qquad T = 1 - |A|^2 \quad \Rightarrow \quad R = \frac{\gamma^2 + (k-k')^2}{\gamma^2 + (k+k')^2} \qquad T = \frac{4kk'}{\gamma^2 + (k+k')^2}$$