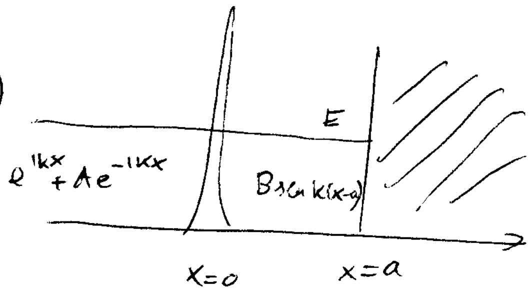
Soluzioni Gennaio 2009

I)



$$\frac{\hbar^2 k^2}{2m} = E$$

Condizioni raccordo:

$x=a \quad \varphi(a)=0 \quad \rightarrow$ già imposta scegliendo la funzione seno in $0<x<a$

$$x=0 \quad \begin{cases} \varphi(0^+) = \varphi(0^-) \\ \varphi'(0^+) - \varphi'(0^-) = \gamma\varphi(0) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 1+A = -B\,\text{sen}\,ka \\ kB\cos ka - ik(1-A) = -\gamma B\,\text{sen}\,ka \end{cases}$$

$$\Downarrow$$

- $$-k(1+A)\,\text{cotg}\,ka - ik(1-A) = \gamma(1+A)$$

$$A(ik - k\,\text{cotg}\,ka - \gamma) = ik + k\,\text{cotg}\,ka + \gamma$$

$$A = \frac{ik + k\,\text{cotg}\,ak + \gamma}{ik - k\,\text{cotg}\,ak - \gamma} = e^{i\delta(k)}$$

$$|A| = \frac{\sqrt{k^2 + (k\,\text{cotg}\,ka + \gamma)^2}}{\sqrt{k^2 + (k\,\text{cotg}\,ka + \gamma)^2}} = 1$$

- Per $\gamma \to \infty \qquad A \to -1 \qquad \delta \to \pi$

$$-B\,\text{sen}\,ka = 1+A \to 0 \quad \Rightarrow \quad B=0$$

$$\varphi(x) = \begin{cases} e^{ikx} - e^{-ikx} = 2i\,\text{sen}\,kx & x<0 \\ 0 & x>0 \end{cases}$$

come per una barriera infinita a $x=0$

Il coefficiente di riflessione $R = \dfrac{|J_{rifl}|}{|J_{inc}|} = |A| = 1$;

per curiosità calcoliamo anche

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right) \qquad \frac{\partial J}{\partial x} = 0$$

per eq. continuità in uno stato stazionario

$$J_{x<0} = \frac{\hbar k}{m}\left(1-|A|^2\right) \equiv 0$$

mentre $$J_{x>0} = \frac{\hbar |B|^2}{2mi}\left[\operatorname{sen} k'(x-a)\frac{d \operatorname{sen} k'(x-a)}{dx} - \operatorname{sen} k'(x-a)\frac{d \operatorname{sen} k'(x-a)}{dx}\right] \equiv 0$$

$J(x) = 0 \quad \forall x$. È continua in $x=0$ anche se $\nabla\psi$ ha un salto.

- Costruendo un pacchetto $x<0$:
$$\int dk\, g(k)\, e^{ikx - i\frac{\hbar k^2 t}{2m}} + \int dk\, g(k) A(k)\, e^{-ikx - i\frac{\hbar k^2}{2m}t}$$
$$= \int dk\, g(k)\, e^{ikx - i\frac{\hbar k^2 t}{2m}} + \int dk\, g(k)\, e^{-ikx - i\frac{\hbar k^2}{2m}t + i\delta(k)}$$

fase stazionaria $\Rightarrow$

$\begin{cases} k_0 \text{ centro del pacchetto} \\ \frac{\hbar^2 k_0^2}{2m} = E_0 \end{cases}$

$x - \frac{\hbar k_0 t}{m} = 0$ $\qquad$ $-x - \frac{\hbar k_0 t}{m} + \frac{d\delta}{dk}(k_0) = 0$

$x = \frac{\hbar k_0}{m} t$ $\qquad$ $x = -\frac{\hbar k_0}{m} t + \frac{d\delta}{dk}$

arriva a $x=0$ a $t=0$ $\qquad$ emerge a $x=0$ per $t = +\frac{m}{\hbar k_0}\frac{d\delta}{dk}(k_0)$ $\qquad$ || ritardo

II)
$$S_x = \frac{\hbar}{\sqrt2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt2}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \qquad S_x^2 - S_y^2 = \frac{\hbar^2}{2}\left[\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix} - \begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}\right]$$

$$H = \mu\hbar\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$ che ha autovettori $|\pm\rangle \equiv \frac{1}{\sqrt2}\begin{pmatrix}1\\0\\\pm1\end{pmatrix}$, $|0\rangle \equiv \begin{pmatrix}0\\1\\0\end{pmatrix}$

e autovalori $H = \pm\mu\hbar$ $\qquad$ $H = 0$

Autovettori di S_x sono $\frac{1}{2}\begin{pmatrix}1\\\pm\sqrt2\\1\end{pmatrix}$, $\frac{1}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$S_x: \ \pm\hbar \ , \ 0$

a $t=0$ $$|\psi(0)\rangle = \frac12\begin{pmatrix}1\\\sqrt2\\1\end{pmatrix} = \frac12\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{\sqrt2}\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{\sqrt2}|H=\mu\hbar\rangle + \frac{1}{\sqrt2}|H=0\rangle$$

quindi $$|\psi(t)\rangle = \frac{e^{-i\mu t}}{\sqrt2}|H=\mu\hbar\rangle + \frac{1}{\sqrt2}|H=0\rangle = \frac12\begin{pmatrix}e^{-i\mu t}\\\sqrt2\\e^{-i\mu t}\end{pmatrix}$$

$$P(S_x=-\hbar) = \left|\langle S_x=-\hbar|\psi(t)\rangle\right|^2 = \left|\frac12(1,-\sqrt2,1)\cdot\frac12\begin{pmatrix}e^{-i\mu t}\\\sqrt2\\e^{-i\mu t}\end{pmatrix}\right|^2 =$$

$$\left| \frac{e^{-i\mu t}}{2} - \frac{1}{2} \right|^2 = \left| e^{-\frac{i\mu t}{2}} \left(\frac{e^{-\frac{i\mu t}{2}} - e^{\frac{i\mu t}{2}}}{2} \right) \right|^2 = \operatorname{sen}^2 \frac{\mu t}{2}$$

$$\langle S_z \rangle = \hbar \, \frac{1}{2} (e^{i\mu t}, \sqrt{2}, e^{+i\mu t}) \cdot \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} e^{-i\mu t} \\ \sqrt{2} \\ e^{-i\mu t} \end{pmatrix} = \frac{\hbar}{4} (1-1) = 0$$

$$\langle S_x \rangle = \frac{\hbar}{\sqrt{2}} \, \psi^* \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \psi = \frac{\hbar}{\sqrt{2}} \, \frac{1}{2} (e^{i\mu t}, \sqrt{2}, e^{i\mu t}) \cdot \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 2e^{-i\mu t} \\ \sqrt{2} \end{pmatrix} = \frac{\hbar}{4\sqrt{2}} \left(2\sqrt{2} e^{i\mu t} + 2\sqrt{2} e^{-i\mu t} \right)$$

$$= \hbar \cos \mu t$$

$$\langle S_y \rangle = \frac{\hbar}{\sqrt{2}} \, \psi^* \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \psi = \frac{\hbar}{\sqrt{2}} \, \frac{1}{2} (e^{i\mu t}, \sqrt{2}, e^{i\mu t}) \cdot \frac{1}{2} \begin{pmatrix} -i\sqrt{2} \\ 0 \\ i\sqrt{2} \end{pmatrix} = 0$$

III) $$H = \sum_{i=1}^{2} \frac{p_i^2}{2m} + \sum_{ij}^{2} \frac{m\omega^2}{2} x_i Q_{ij} x_j \qquad \text{con} \qquad Q_{ij} = \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

Q_{ij} che è simmetrica può essere diagonalizzato con un cambiamento di base ortogonale $\quad \tilde{x}_i = O_{ij} x_j \qquad OO^T = I$

L'energia potenziale $\sum_{ij}^{2} x_i Q_{ij} x_j = \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 \qquad \lambda_i$ autovalori di Q

L'energia cinetica rimane invariante:

$$p_i = -i\hbar \frac{d}{dx_i} = -i\hbar \frac{d\tilde{x}_j}{dx_i} \frac{d}{d\tilde{x}_j} = -i\hbar \, O_{ji} \frac{d}{d\tilde{x}_j} = O_{ji} \tilde{p}_j \Rightarrow$$ trasformazione ortogonale sui p_i

$$\tilde{p}_1^2 + \tilde{p}_2^2 = p_1^2 + p_2^2$$

Autovalori e autovettori di Q

$$\det \begin{pmatrix} \frac{5}{2} - \lambda & -3/2 \\ -3/2 & \frac{5}{2} - \lambda \end{pmatrix} = 0$$

$$\left(\lambda - \frac{5}{2}\right)^2 = \frac{9}{4} \Rightarrow \lambda = \frac{5}{2} \pm \frac{3}{2} = 1, 4$$

$$\begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (1,4) \begin{pmatrix} a \\ b \end{pmatrix}$$

$\lambda = 1 \to a = b \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 4 \to a = -b \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Quindi $$H = \frac{\tilde{p}_1^2}{2m} + \frac{\tilde{p}_2^2}{2m} + \frac{m\omega^2}{2} \underbrace{\left(\tilde{x}_1^2 + 4 \tilde{x}_2^2 \right)}_{\frac{5x_1^2 + 5x_2^2 - 6x_1 x_2}{2}}$$

$$\boxed{\text{un segno overall in } \tilde{x}_1 \text{ e } \tilde{x}_i \text{ è irrilevante, il segno in } \tilde{x}_2 \text{ è scelto per comodità (rotazione)}}$$

$$\tilde{x}_1 = \frac{x_1 + x_2}{\sqrt{2}}$$

$$\tilde{x}_2 = \frac{-x_1 + x_2}{\sqrt{2}}$$

$$V(\tilde{x}_1,\tilde{x}_2) = \frac{m\omega^2\tilde{x}_1^2}{2} + \frac{m}{2}(2\omega)^2\tilde{x}_2^2$$

- Le due frequenze sono $\omega, 2\omega$ e lo spettro

$$E = \hbar\omega\left(n_1+\frac{1}{2}\right) + \hbar(2\omega)\left(n_2+\frac{1}{2}\right) = \hbar\left(n_1+2n_2+\frac{3}{2}\right)$$

- $x_1 = \dfrac{\tilde{x}_1-\tilde{x}_2}{\sqrt{2}} \qquad x_2 = \dfrac{\tilde{x}_1+\tilde{x}_2}{\sqrt{2}}$

$$\langle 00|x_1|00\rangle = \frac{1}{\sqrt{2}}\langle 00|\tilde{x}_1|00\rangle + \frac{1}{\sqrt{2}}\langle 00|\tilde{x}_2|00\rangle$$
$$= \frac{1}{\sqrt{2}}\langle 0|\tilde{x}_1|0\rangle\langle 0|0\rangle + \dots \qquad = 0$$

$$\langle 00|x_1^2|00\rangle = \frac{1}{2}\langle 00|\tilde{x}_1^2 - 2\tilde{x}_1\tilde{x}_2 + \tilde{x}_2^2|00\rangle =$$
$$\frac{1}{2}\langle 00|\tilde{x}_1^2|00\rangle + \frac{1}{2}\langle 00|\tilde{x}_2^2|00\rangle =$$
$$= \frac{1}{2}\frac{\hbar}{m\omega}\left(0+\frac{1}{2}\right) + \frac{1}{2}\frac{\hbar}{2m\omega}\left(0+\frac{1}{2}\right)$$

(notare 2ω)

$$= \frac{3\hbar}{8m\omega}$$

Per un oscillatore:

$\langle n|x|n\rangle = 0$

$\langle n|x^2|n\rangle = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$

- Se $E = \frac{7}{2}\hbar\omega$ con probabilità 1 $\qquad |\psi\rangle = a_1|20\rangle + a_2|01\rangle$

$\qquad |a_1|^2+|a_2|^2 = 1$

$$\begin{cases} E = \hbar\omega\frac{3}{2} & |00\rangle \\ E = \frac{5}{2}\hbar\omega & |10\rangle \\ E = \frac{7}{2}\hbar\omega & |20\rangle, |01\rangle \quad \text{dopp. degenere} \\ \text{etc} \dots \end{cases}$$

$$\langle x_1^2\rangle = \frac{1}{2}\langle \tilde{x}_1^2 - 2\tilde{x}_1\tilde{x}_2 + \tilde{x}_2^2\rangle = \frac{|a_1|^2}{2}\langle 20|\tilde{x}_1^2+\tilde{x}_2^2|20\rangle + \frac{|a_2|^2}{2}\langle 01|\tilde{x}_1^2+\tilde{x}_2^2|01\rangle$$

altri termini zero

$$= \frac{|a_1|^2}{2}\left(\frac{\hbar}{m\omega}\left(2+\frac{1}{2}\right) + \frac{\hbar}{2m\omega}\left(0+\frac{1}{2}\right)\right) + \frac{|a_2|^2}{2}\left(\frac{\hbar}{m\omega}\left(0+\frac{1}{2}\right) + \frac{\hbar}{2m\omega}\left(1+\frac{1}{2}\right)\right)$$

$$|a_1|^2+|a_2|^2 = 1 \Longrightarrow \qquad = \frac{\hbar}{2m\omega}\left(\frac{5}{4} - \frac{3}{2}|a_1|^2\right) \equiv \frac{5\hbar}{8m\omega}$$

$$\Longrightarrow |a_1|^2 = 0 \Longrightarrow |\psi\rangle = |01\rangle$$