Special relativity

Problem 52

In a certain inertial frame two light pulses are emitted, a distance 5 km apart and separated by 5 μs. An observer who is travelling, parallel to the line joining the points where the pulses are emitted, at a velocity v with respect to this frame notes that the pulses are simultaneous. Find v.

Solution

The 'standard configuration' in special relativity problems involves two inertial frames S and S' such that, according to observers stationary with respect to the frame S, the frame S' has a velocity v in the x-direction. If $\Delta x, \Delta y, \Delta z, \Delta t$ are the intervals measured in S between two events, and $\Delta x', \Delta y', \Delta z', \Delta t'$ are the intervals between the same events measured in S', the relations between the intervals are given by the Lorentz transformations:

$$\Delta x' = \gamma(\Delta x - v\Delta t), \qquad \Delta x = \gamma(\Delta x' + v\Delta t'),$$
$$\Delta y' = \Delta y, \qquad \Delta y = \Delta y',$$
$$\Delta z' = \Delta z, \qquad \Delta z = \Delta z',$$
$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2), \qquad \Delta t = \gamma(\Delta t' + v\Delta x'/c^2),$$

where

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}}.$$

We will assume that S is the frame in which the pulses are emitted with a time separation of 5 μs, so that $\Delta x = 5$ km, $\Delta t = 5\ \mu$s. We require to find the frame S' in which $\Delta t' = 0$. From the Lorentz transformation, we can see that this is so if

$$v = c^2\Delta t/\Delta x.$$

▶ Inserting the values of Δt and Δx gives $v = 9 \times 10^7\ \text{m s}^{-1}$ (= 0.3 c).

Problem 53

Observer A sees two events at the same place ($\Delta x = \Delta y = \Delta z = 0$) and separated in time by $\Delta t = 10^{-6}$ s. A second observer B sees them to be separated by $\Delta t' = 2 \times 10^{-6}$ s. What is the separation in space of the two events according to B? What is the speed of B relative to A?

Solution

Observer A is at rest in frame S, and observer B is at rest in the frame S'. The Lorentz transformation for $\Delta t'$ gives

$$\Delta t' = \gamma\Delta t$$

(since $\Delta x = 0$), so we must have $\gamma = 2$. Now since

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}}$$

it follows that

$$v = c\sqrt{\left(1 - \frac{1}{\gamma^2}\right)},$$

▶ so $v = c(1 - 1/4)^{1/2} = \sqrt{3}\,c/2$. The Lorentz transformation for $\Delta x'$ gives

$$\Delta x' = -\gamma v\Delta t$$

(again using the fact that $\Delta x = 0$), so

$$\Delta x' = -2\frac{\sqrt{3}}{2}3 \times 10^8 \times 10^{-6}\ \text{m} = -520\ \text{m}.$$

Thus according to observer B, the spatial separation of the two events is
▶ $\Delta x' = -520$ m, $\Delta y' = \Delta z' = 0$.

[We could have calculated the magnitude of $\Delta x'$ directly, without first calculating v, by using the *Lorentz invariant interval*. This is defined as

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2,$$

and it can be shown that $\Delta s^2 = \Delta s'^2$, i.e. that the interval between two events is the same in any inertial frame. In this problem, $\Delta y' = \Delta z' = 0$ so that

$$0 - c^2\Delta t^2 = \Delta x'^2 - c^2\Delta t'^2,$$

which can be rearranged to give

$$\Delta x'^2 = c^2(\Delta t'^2 - \Delta t^2).$$

Putting $\Delta t' = 2\ \mu$s and $\Delta t = 1\ \mu$s gives $\Delta x' = \pm 520$ m.]

Problem 54

Two inertial frames of reference S and S' are in the standard configuration, frame S' having velocity v with respect to frame S. At the instant when their spatial origins O and O' coincide, a light beam is emitted from O and O' along the positive x- and x'-axis. The beam is reflected by a mirror M fixed in S at a distance d from O and with its plane perpendicular to the x-axis. Consider the following three events:

(1) light beam reaches M,
(2) reflected beam returns to O',
(3) reflected beam returns to O.

Calculate the times of these events as measured by observers in frame S. Use the Lorentz transformation to determine the times of these events as measured by observers in frame S'. Show how observers in frame S' would explain their measurements without reference to frame S.

Solution

It will be helpful to draw a space–time diagram of the events in S, as shown in figure 53.

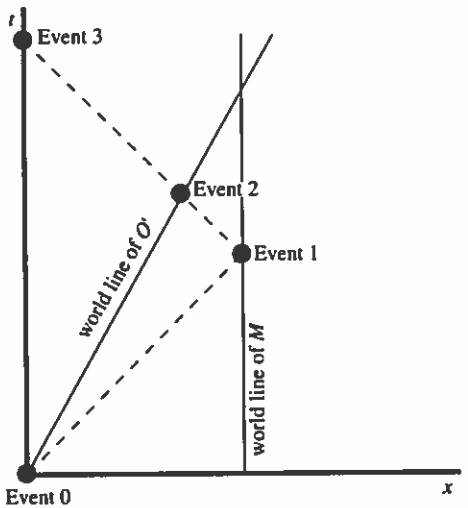

Figure 53

The dashed lines show the world-line of the light beam. We will assume that clocks in both frames are synchronised to $t = t' = 0$ when the origins of the frames coincide at event 0 (emission of the light beam).

The mirror M is a distance d from the origin O, so light will take a time d/c to reach it. Thus

$$t_1 = d/c.$$

When we perform the Lorentz transformations later we will also need the x-coordinate of this event, which is obviously

$$x_1 = d.$$

At time t_1, the light beam is at $x = d$. After this time, the light beam travels in the negative x-direction, so at time $t > t_1$ its x-coordinate is

$$\begin{aligned} x &= d - c(t - t_1) \\ &= d - c(t - d/c) \\ &= 2d - ct. \end{aligned}$$

At time t, the x-coordinate of O' is vt, so at time t_2, when O' and the light beam meet, we must have

$$2d - ct_2 = vt_2.$$

Thus

$$t_2 = \frac{2d}{c + v}.$$

The x-coordinate of this event can be found by substituting into either $x = 2d - ct_2$ or $x = vt_2$, to give

$$x_2 = \frac{2dv}{c + v}.$$

The time coordinate of event 3 is easy to calculate, since it is just the time required for the beam of light to travel from O to M and back again. Thus

$$t_3 = 2d/c.$$

Clearly

$$x_3 = 0.$$

Now let us calculate the time-coordinates of these events in the frame S', using the Lorentz transformation

$$t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Thus

$$t'_1 = \gamma\left(\frac{d}{c} - \frac{vd}{c^2}\right) = \frac{\gamma d}{c}(1 - v/c),$$

$$t'_2 = \gamma\left(\frac{2d}{c+v} - \frac{2v^2 d}{c^2(c+v)}\right) = \frac{2\gamma d}{c+v}(1 - v^2/c^2)$$

(which is equal to $2t'_1$),

$$t'_3 = \gamma\left(\frac{2d}{c} - 0\right) = \frac{2\gamma d}{c}.$$

In order to see how observers in S' would interpret these measurements, it is helpful to draw the space-time diagram for the frame S', as shown in figure 54.

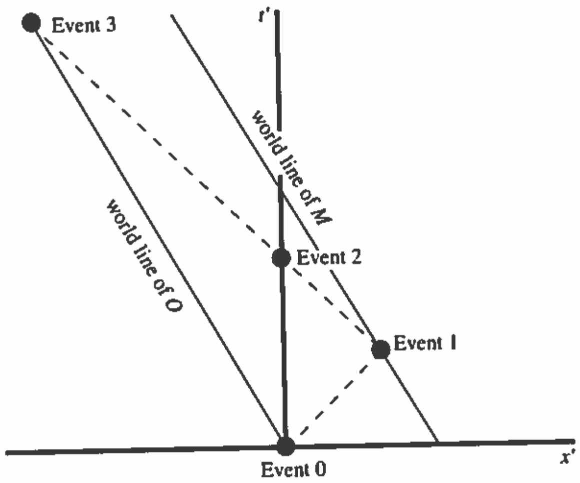

Figure 54

According to the observers in S', O is moving in the negative x'-direction at some speed u. We can use their measurements of t'_1 and t'_3 to calculate u as follows:

At time t', the x'-coordinate of O is

$$x' = -ut'.$$

For times less than t'_1, the light beam moves in the positive x'-direction, and for times greater than t'_1 it moves in the negative x'-direction. Thus at time $2t'_1$ the light beam has an x'-coordinate of zero, and at time $t' > t'_1$ the x'-coordinate is

$$x' = -c(t' - 2t'_1).$$

The x'-coordinates of O and of the light beam must be equal at $t' = t'_3$, so

$$ut'_3 = c(t'_3 - 2t'_1)$$

and hence

$$u = c\left(1 - \frac{2t'_1}{t'_3}\right).$$

Substituting the observed values

$$t'_1 = \frac{\gamma d}{c}\left(1 - \frac{v}{c}\right) \quad \text{and} \quad t'_3 = \frac{2\gamma d}{c}$$

gives $u = v$, so the observers in S' see the frame S moving backwards at speed v. [We could have written this down straight away, since it follows directly from the postulates of special relativity.]

The observers in S' can also use their measurements to calculate the distance d' between O and M: At time t', the x'-coordinate of M is

$$x' = d' - ut' = d' - vt'.$$

At time t' the x'-coordinate of the light beam is

$$x' = ct',$$

and when $t' = t'_1$ these x'-coordinates must be equal. Thus

$$ct'_1 = d' - vt'_1,$$

hence

$$d' = (c + v)t'_1.$$

Substituting the observed value of t'_1 gives

$$d' = \frac{\gamma d}{c}\left(1 - \frac{v}{c}\right)(c + v) = \gamma d\left(1 - \frac{v^2}{c^2}\right) = d\left(1 - \frac{v^2}{c^2}\right)^{1/2} = \frac{d}{\gamma}.$$

The observers in S' thus measure the distance from O to M to be d/γ, which is in agreement with the length-contraction formula.

Problem 55

A member of a colony on a moon of Jupiter is required to salute the UN flag at the same time as it is being done on Earth, at noon in New York. If observers in all inertial frames are to agree that he has performed his duty, for how long must he salute? (The distance from Earth to Jupiter is 8×10^8 km. The relative motion of the Earth and Jupiter's moon may be ignored.)

Solution

Write down the Lorentz transformation for t in differential form,

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2),$$

and let S be the frame of reference in which the Earth (and the moon of Jupiter) is at rest, and S' the rest frame of an arbitrary observer.

Identify the salute on Earth and the salute on the moon of Jupiter as two events. In the frame S, the spatial displacement Δx between these two events is D (i.e. the distance from Earth to Jupiter) and the temporal separation Δt is T.

We require that in $S'\Delta t'$ is zero, hence

$$T = vD/c^2.$$

Thus the member of the colony must be saluting at a time vD/c^2 after the salute on Earth, as measured in the frame S, and since v can vary between $\pm c$ he must salute for a total time of

$$\begin{aligned} &2D/c \\ &= 2 \times 8 \times 10^{11}/(3 \times 10^8)\ \text{s} \\ &= 5.3 \times 10^3\ \text{s} \\ &\approx 1.5\ \text{hours.} \end{aligned}$$

Problem 56

Two rockets A and B depart from Earth at steady speeds of $0.6c$ in opposite directions, having synchronised clocks with each other and with Earth at departure. After one year as measured in Earth's reference frame, rocket B emits a light signal. At what times, in the reference frames of the Earth and of rockets A and B, does rocket A receive the signal?

Solution

In this problem we have three inertial frames to consider, so we will use coordinates x and t to denote quantities measured in the Earth's frame, x_A and t_A to denote quantities measured in A's frame, and x_B and t_B to

denote quantities measured in B's frame. We will assume that, according to an observer in the Earth's frame, rocket A is travelling at speed v in the positive x-direction and rocket B in the negative x-direction. With these assumptions, we can write down the Lorentz transformations:

$$\begin{aligned} x &= \gamma(x_A + vt_A), & x_A &= \gamma(x - vt), \\ t &= \gamma(t_A + vx_A/c^2), & t_A &= \gamma(t - vx/c^2), \\ x &= \gamma(x_B - vt_B), & x_B &= \gamma(x + vt), \\ t &= \gamma(t_B - vx_B/c^2), & t_B &= \gamma(t + vx/c^2), \end{aligned}$$

where

$$v = 0.6c$$

and

$$\gamma = (1 - v^2/c^2)^{-1/2} = 5/4.$$

It will be convenient to use a system of units in which time is measured in years and distance in light-years, in which case c has a value of 1. Again, it is helpful to draw a space-time diagram in the Earth's frame of reference, as shown in figure 55.

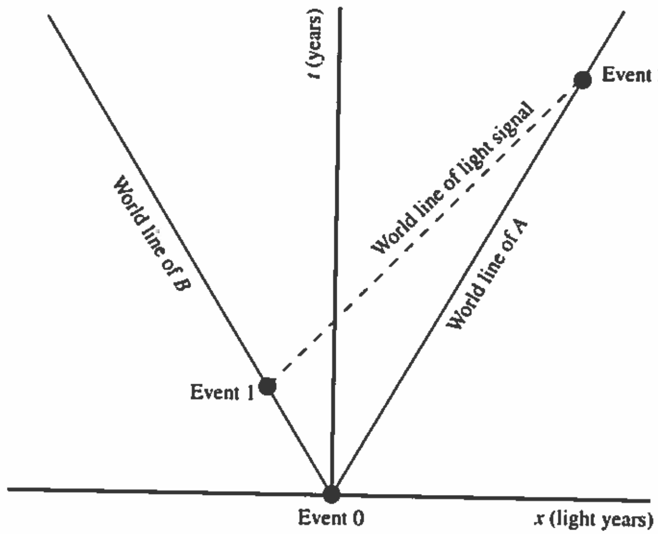

Figure 55

In the Earth's frame of reference, event 1 (B emits the signal) has coordinates

$$\begin{aligned} x &= -0.6, \\ t &= +1, \end{aligned}$$

so at time $t > 1$ the x-coordinate of the light signal is

$$x = -0.6 + (t - 1) = t - 1.6.$$

At time t, the x-coordinate of rocket A is

$$x = +0.6t.$$

The x-coordinates must be equal at event 2, so for this event we must have

$$t - 1.6 = 0.6t,$$

hence $t = 4$. Thus according to an observer in the Earth frame, event 2 occurs after 4 years. The x-coordinate of this event is therefore $0.6 \times 4 = 2.4$.

We can now substitute these coordinates into the Lorentz transformation formulae to find the time coordinates of event 2 in A's and B's frame of reference:

$$t_A = \gamma(t - vx)$$

(remember that $c = 1$)

$$= \frac{5}{4}(4 - 0.6 \times 2.4) = 3.2,$$

so that, according to A, event 2 occurs after 3.2 years.

$$t_B = \gamma(t + vx)$$
$$= \frac{5}{4}(4 + 0.6 \times 2.4) = 6.8,$$

so according to B, event 2 occurs after 6.8 years.

Problem 57

A very fast train of proper length L_0 rushes through a station which has a platform of length L ($< L_0$). What must be its speed v such that the back of the train is opposite one end of the platform at exactly the same instant as the front of the train is opposite the other end, according to an observer on the platform?

According to this observer, two porters standing at either end of the platform (distance L apart) kick the train simultaneously, thereby making dents in it. When the train stops, the dents are at a distance L_0 apart. How is the difference between L and L_0 explained by (a) the observer on the platform, and (b) an observer travelling in the train?

Solution

According to the observer on the platform, the train undergoes a Lorentz contraction by a factor of γ, so that its length is L_0/γ. This must clearly be equal to L if the two ends of the train are to align with the two ends of the platform, so

$$\gamma = (1 - v^2/c^2)^{-1/2} = L_0/L.$$

Rearranging,

$$v/c = (1 - L^2/L_0^2)^{1/2},$$

so

$$v = c(1 - L^2/L_0^2)^{1/2}.$$

(a) According to the observer on the platform, the kicks are administered at either end of the train so the dents will be at the two ends of the train. When the train stops this is found to be the case, although the train is no longer undergoing a Lorentz contraction so the separation of the dents is greater than it was when the train was in motion.

(b) According to an observer on the train, the train is of length L_0 but the platform is moving at velocity $-v$ so it undergoes Lorentz contraction from its proper length L to a length $L/\gamma = L^2/L_0$. The fact that two porters standing this distance apart nevertheless manage to make dents in the train separated by L_0 is explained by the fact that the kicks are not administered simultaneously.

We can show this using the Lorentz transformations. Let us identify frame S as the frame in which the platform is stationary, and S' as the frame in which the train is stationary. The train thus has a velocity $+v$ in the x-direction relative to the platform. We will call the two porters A and B, and assume that their x-coordinates are 0 and L respectively in the frame S. We will also assume that, in frame S, the kicks occur at time $t = 0$. Thus we have, using the Lorentz transformations, the data shown in Table 5.

Table 5

	In frame S	In frame S'
A kicks the train	$x = 0$, $t = 0$	$x' = 0$, $t' = 0$
B kicks the train	$x = L$, $t = 0$	$x' = \gamma L$, $t' = -\gamma vL/c^2$

In order for an observer at rest in the frame S' to measure the distance between the kicks as L_0, we must have $\gamma = L_0/L$ as before. However, we

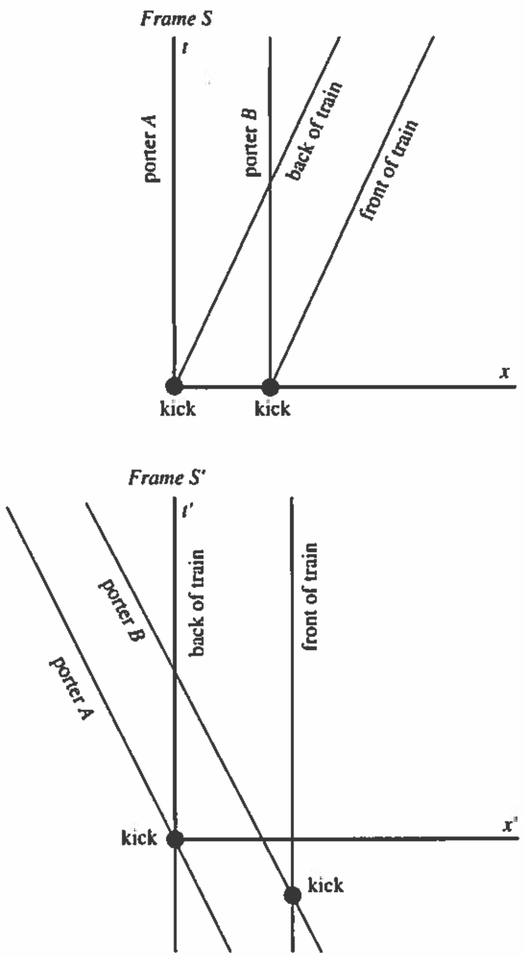

Figure 56

see that in this frame porter B's kick is administered earlier than porter A's kick, by an interval of time equal to $\gamma v L/c^2$. To show that this is consistent with the Lorentz contraction of the platform observed by the train, we can use this time interval to calculate the length of the platform according to an observer in S'.

According to observers in S', the porters are moving at speed v in the negative x'-direction, so at $t' = 0$, porter B is at $x' = L_0 - \gamma v^2 L/c^2$ and porter A is at $x' = 0$. The length of the platform, according to an observer in S', is thus $L_0 - \gamma v^2 L/c^2$. Using the fact that $\gamma = L_0/L$, this can be rewritten as $L_0(1 - v^2/c^2)$, and using the fact that $(1 - v^2/c^2) = 1/\gamma^2$, it can be simplified to L_0/γ^2. Thus according to an observer moving

with the train, the length of the platform is $L_0/\gamma^2 = L/\gamma$, so it has been Lorentz-contracted as we expect.

This can be summarised on space-time diagrams in S and S', shown in figure 56.

Problem 58

Given two observers O and O', with O' moving at uniform velocity v in the positive x-direction relative to O, use the appropriate Lorentz transformations to show that if an object is moving with velocity component $u_{x'}$ in the frame of reference of O', then

$$u_x = \frac{u_{x'} + v}{1 + \dfrac{v u_{x'}}{c^2}},$$

where u_x is the corresponding velocity component according to O.

(a) A space ship is launched from Earth and maintains a uniform velocity of $0.900c$. The space ship subsequently launches a small rocket in the forward direction with a speed of $0.900c$ relative to the ship. What is the speed of the small rocket relative to the Earth?

(b) According to observations on the Earth, the nearest star to the solar system is 4.25 light years away. A space ship which leaves the Earth and travels at uniform velocity takes 4.25 years, according to ship-borne clocks, to reach the star. What is the speed of the space ship, expressed as a fraction of the speed of light c?

Solution

The frames of reference of O and O' have the standard relationship, so that the Lorentz transformations are given, in differential form, by

$$dx' = \gamma(dx - v\,dt),$$

$$dt' = \gamma\left(dt - \frac{v\,dx}{c^2}\right).$$

[We could also write down the reverse transformations, but in fact we don't need them.] The ratio of these two expressions gives the component $u_{x'}$:

$$u_{x'} = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \dfrac{v\,dx}{c^2}}.$$

Multiplying through by the denominator of the right-hand side and rearranging, we obtain

$$dx\left(1+\frac{vu_{x'}}{c^2}\right)=dt(u_{x'}+v),$$

from which we can write down the required result:

▶ $$u_x=\frac{dx}{dt}=\frac{u_{x'}+v}{1+\dfrac{vu_{x'}}{c^2}}.$$

(a) This result can be applied directly if we identify O with an observer on Earth and O' with an observer on the space ship. We have $v = 0.900c$ and $u_{x'} = 0.900c$, so

$$u_x=\frac{0.900c+0.900c}{1+\dfrac{(0.900c)(0.900c)}{c^2}}=\frac{1.800c}{1.810}.$$

▶ The speed of the rocket relative to Earth is thus $0.994c$.

(b) We can solve this using the Lorentz transformations. Let us identify S as the frame in which the Earth and the star are at rest, and S' as the frame in which the space ship is at rest, and synchronise clocks to $t = t' = 0$ when $x = x' = 0$. If we put D for the distance to the star in the frame S, the coordinates in S of the event of the space ship reaching the star are

$$x = D,$$
$$t = D/v.$$

Thus the time coordinate of this event in the space ship's frame S' is

$$\begin{aligned} t' &= \gamma(t - vx/c^2)\\ &= \gamma(D/v - Dv/c^2)\\ &= (\gamma D/v)(1 - v^2/c^2)\\ &= D/\gamma v. \end{aligned}$$

Now we are given that $t' = 4.25$ years and $D/c = 4.25$ years, so it follows that

$$\gamma v/c = 1.$$

Putting

$$\beta = v/c$$

for convenience, we have

$$\gamma\beta = 1,$$

and since

$$\gamma=\frac{1}{\sqrt{(1-\beta^2)}},$$

this gives

$$\frac{\beta^2}{1-\beta^2}=1,$$

so that

$$\beta^2 = 1/2,$$

▶ hence $v = c/\sqrt{2}$.

Problem 59

To an observer, two bodies of equal rest mass collide head on with equal but opposite velocities $4c/5$ and cohere. To a second observer, one body is initially at rest. Find the apparent velocity of the other, moving mass before the collision and compare its initial energy in the two frames of reference.

Solution

It is clear that the apparent velocity of the second mass must be equal to the relative velocity of the two masses, which is given by the relativistic addition of $4c/5$ and $4c/5$:

$$v=\frac{4c/5+4c/5}{1+(4c/5)(4c/5)(1/c^2)}=\frac{8c/5}{41/25}$$

▶ $$= 40c/41.$$

The total initial energy of either particle in the first frame is γmc^2, where m is its rest mass and γ is the Lorentz factor for a speed of $4c/5$, thus

$$E=\frac{mc^2}{\sqrt{\left(1-\dfrac{4^2}{5^2}\right)}}$$
$$= (5/3)mc^2.$$

In the second frame, the same formula applies for the moving particle but γ is now the value appropriate to a speed of $40c/41$, thus

$$E = \frac{mc^2}{\sqrt{\left(1 - \frac{40^2}{41^2}\right)}} = (41/9)mc^2.$$

Thus the initial energy of the moving particle is greater by a factor of 41/15 in the second frame.

Problem 60

A beam of monochromatic light, whose wavelength in free space is λ, is split into two separate beams and each is then passed through identical troughs of water. Show that if the water in one trough is stationary and the water in the other trough is moving with speed v ($\ll c$) in the direction of the light, the phase difference between the emerging beams is

$$\Delta\phi = (2\pi L/\lambda)(n^2 - 1)(v/c),$$

where L is the length of the troughs and n is the refractive index of the stationary water. Suggest suitable values for L and v in an experimental arrangement for verifying this result.

Solution

Figure 57 shows the arrangement.

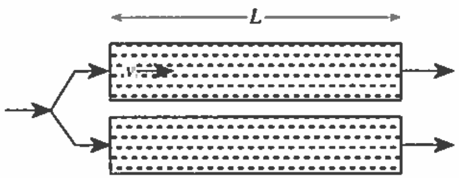

Figure 57

The refractive index n defines the speed at which light propagates with respect to the medium as c/n, and since the moving water moves forward (i.e. in the same direction as the light) in the laboratory frame at speed v, the speed of the light relative to the laboratory frame must be the relativistic sum of v and c/n, i.e.

$$\frac{v + \frac{c}{n}}{1 + \frac{v}{nc}}.$$

Thus the time taken for light to travel from one end of the trough of moving water to the other, measured in the laboratory frame, is

$$T_{moving} = L(1 + v/nc)/(c/n + v)$$
$$= (Ln/c)(1 + v/nc)(1 + vn/c)^{-1}.$$

Since $v \ll c$, we can use the binomial theorem to expand this expression to the first order in v, giving

$$T_{moving} \approx (Ln/c)(1 + v/nc - vn/c).$$

The time required for light to travel through the trough of stationary water can be deduced from this expression by substituting into it $v = 0$, to give

$$T_{stationary} = Ln/c$$

(which is obviously correct), so the difference in travel times through the two troughs is

$$\Delta T = (Ln/c)(v/c)(n - 1/n)$$
$$= (Lv/c^2)(n^2 - 1).$$

The phase difference $\Delta\phi$ is given by $2\pi\nu\Delta T$ where ν is the frequency, and substituting $\nu = c/\lambda$ finally gives

$$\Delta\phi = (2\pi L/\lambda)(n^2 - 1)(v/c)$$

as required.

Water has a refractive index n of approximately 1.33 at optical wavelengths (say 500 nm), and if we assume that $\Delta\phi$ must be at least $\pi/2$ to produce a measurable effect, substitution into this expression shows that Lv must exceed about 50 m^2 s^{-1}. Possible values for a demonstration might be $L = 5$ m and $v = 10$ m s^{-1}.

Problem 61

In its rest frame, a source emits light in a conical beam of width ±45°. In a frame moving towards the source at speed v, the beam width is ±30°. What is v?

Solution

There are several ways of solving this. The longest, but most basic, is to use the Lorentz transformations directly. It is also possible to use the formulae for relative velocity, or (the shortest method) to use the aberration formula.

(1) Using the Lorentz transformation directly.

First we need to set up two inertial frames S and S' in which to describe the problem. If we adopt the standard configuration in which frame S' has a positive velocity v in the x-direction when observed in frame S, figure 58 shows that we can consider the light to be emitted at up to $\pm 45°$ from the x'-axis in the frame S'.

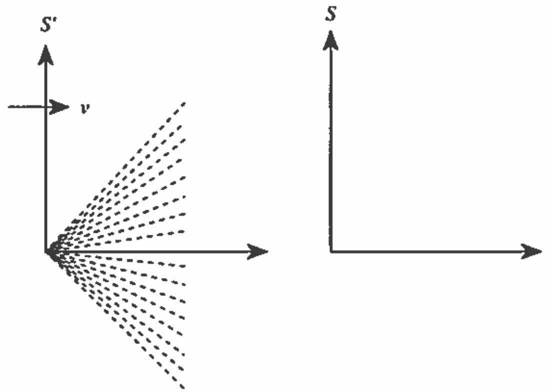

Figure 58

We can write the spatial parts of the Lorentz transformations in differential form:

$$\Delta x = \gamma(\Delta x' + v\Delta t'),$$
$$\Delta y = \Delta y'.$$

If we consider a photon travelling at the very edge of the cone, in S' it will make an angle of 45° with the x'-axis. If the photon connects two events which have a separation along the x'-axis of $\Delta x'$, the separation along the y'-axis must be numerically equal to $\Delta x'$ (because the photon is travelling at an angle of 45°), and the time interval $\Delta t'$ must be $\sqrt{2}\Delta x'/c$ (because the photon travels a distance $\sqrt{2}\,\Delta x'$ at the speed of light).

Thus we have

$$\Delta x',$$
$$\Delta y' = \Delta x',$$
$$\Delta t' = \sqrt{2}\,\Delta x'/c.$$

Transforming into the frame S, we find

$$\Delta x = \gamma\left(\Delta x' + \frac{\sqrt{2}\,v\Delta x'}{c}\right) = \gamma\Delta x'\left(1 + \frac{\sqrt{2}\,v}{c}\right),$$
$$\Delta y = \Delta x'.$$

The photon must make an angle of $\arctan(\Delta y/\Delta x)$ with the x-axis (which we are told is 30°), so

$$\gamma\left(1 + \frac{\sqrt{2}\,v}{c}\right) = \frac{1}{\tan 30°} = \sqrt{3}.$$

Substituting $\gamma = (1 - v^2/c^2)^{-1/2}$ gives

$$\frac{1 + \sqrt{2}(v/c)}{\sqrt{(1 - (v/c)^2)}} = \sqrt{3}.$$

If we put $\beta = v/c$ for convenience, and multiply throughout by $(1 - \beta^2)^{1/2}$, we obtain

$$1 + \sqrt{2}\,\beta = (3 - 3\beta^2)^{1/2}.$$

Squaring:

$$1 + 2\sqrt{2}\,\beta + 2\beta^2 = 3 - 3\beta^2.$$

This can be rearranged as a quadratic equation in β:

$$5\beta^2 + 2\sqrt{2}\,\beta - 2 = 0.$$

Solving the quadratic gives $v/c = +0.410$ or -0.976. Clearly we require the positive solution, so our result is $v = 0.41\,c$. [The negative solution corresponds to the transformed cone making an angle of −30° with the axis. It was introduced when we squared our expression for β.]

(2) Using the velocity transformation formulae.

A photon travelling along the edge of the cone has velocity components, in S', of

$$v'_x = c/\sqrt{2}$$

and

$$v'_y = c/\sqrt{2}.$$

In the frame S, these will transform to v_x and v_y where, in general,

$$v_x = \frac{v + v'_x}{1 + vv'_x/c^2}$$

and

$$v_y = \frac{v'_y(1 - v^2/c^2)^{1/2}}{1 + vv'_x/c^2}.$$

In this particular case, we thus obtain

$$v_x = \frac{v + \frac{c}{\sqrt{2}}}{1 + \frac{v}{\sqrt{2}c}},$$

$$v_y = \frac{c}{\sqrt{2}} \frac{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{v}{\sqrt{2}c}}.$$

Since $v_y/v_x = \tan 30° = 1/\sqrt{3}$ we have

$$\frac{1 + \sqrt{2}(v/c)}{\sqrt{(1 - (v/c)^2)}} = \sqrt{3}$$

as before.

[If we were unable to remember the transformation for v_y, we could solve the problem using the velocity transformation for v_x alone, using the fact that the speed of light is the same in all inertial frames, as shown in figure 59.

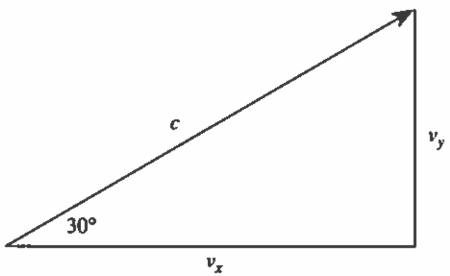

Figure 59

Thus $v_x/c = \cos 30° = \sqrt{3}/2$, so

$$\frac{v + \frac{c}{\sqrt{2}}}{1 + \frac{v}{\sqrt{2}c}} = \frac{\sqrt{3}c}{2}.$$

Cross-multiplying this expression gives

$$2v + \sqrt{2}c = \sqrt{3}c + \sqrt{(3/2)}\,v,$$

which can be solved to give

$$v = c\frac{\sqrt{3} - \sqrt{2}}{2 - \sqrt{\frac{3}{2}}}$$

$= 0.410c$ as before.]

(3) Using the aberration formula for light making an angle ϕ with the x-axis and ϕ' with the x'-axis.

$$\tan(\phi/2) = \sqrt{\frac{1 - v/c}{1 + v/c}}\tan(\phi'/2).$$

Substitution of $\phi = 30°$ and $\phi' = 45°$ gives

$$\frac{1 - v/c}{1 + v/c} = \frac{\tan^2 15°}{\tan^2 22.5°} = 0.4185,$$

therefore

▶ $$\frac{v}{c} = \frac{1 - 0.4185}{1 + 0.4185} = 0.41.$$

Problem 62

Estimate the minimum frequency of a γ-ray that causes a deuteron to disintegrate into a proton and a neutron, commenting on any assumptions you make. The masses of the particles are

$$m_d = 2.0141m_u,$$
$$m_p = 1.0078m_u,$$
$$m_n = 1.0087m_u.$$

Solution

We will assume that the deuteron is at rest, and that the proton and neutron are created at rest. This cannot be quite correct, since it violates the principle of conservation of momentum, but the masses of the particles involved are large so the associated velocities will be small.

The total mass of the products is $2.0165m_u$, which is greater than the mass of the deuteron by $0.0024m_u$. The extra mass must be provided by the energy of the photon, so the minimum possible frequency must be given by

$$h\nu = 0.0024m_u c^2.$$

▶ Substituting $m_u = 1.66 \times 10^{-27}$ kg gives $\nu = 5.4 \times 10^{20}$ Hz.

[We can roughly check the reliability of our assumption as follows. The momentum of this photon is given by $h\nu/c = p = 1.2 \times 10^{-21}$ kg m s^{-1}. If all of this momentum were transferred to (say) the proton, it would acquire a velocity of $p/m = 7.2 \times 10^5$ m s^{-1}. This is much less than the speed of light, so we are justified in ignoring changes in mass caused by the velocities of the particles. In fact, the error in our calculation can be shown to be less than 0.1%.]

Problem 63

What is the speed of an electron which has a total energy of 1 MeV?

Solution

The total energy E is given by

$$E = \gamma m_0 c^2,$$

so

$$\begin{aligned}\gamma &= E/(m_0c^2) \\ &= 10^6 \times 1.60 \times 10^{-19}/(9.11 \times 10^{-31})/(3.00 \times 10^8)^2 \\ &= 1.95.\end{aligned}$$

Now

$$\gamma = (1 - v^2/c^2)^{-1/2},$$

so

$$\begin{aligned}v/c &= (1 - 1/\gamma^2)^{1/2} \\ &= 0.86.\end{aligned}$$

Therefore

▶ $$v = 0.86c.$$

Problem 64

A particle of rest mass m_0 is travelling so that its total energy is just twice its rest mass energy. It collides with a stationary particle of rest mass m_0 to form a new particle. What is the rest mass of the new particle?

Solution

Before the collision, particle 1 (the moving particle) has a total energy of $2m_0c^2$ and a non-zero momentum which we will call p, as shown in figure 60. Particle 2 has a total energy of m_0c^2 and a momentum of zero. Thus by conservation of energy and momentum, the new particle has a total energy of $3m_0c^2$ and a total momentum of p.

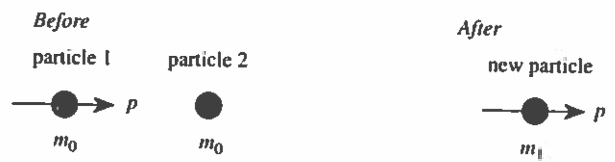

Figure 60

We can find p by using the energy–momentum invariant for particle 1:

$$E^2 - p^2c^2 = m_0^2c^4;$$

rearranging,

$$\begin{aligned}p^2 &= E^2/c^2 - m_0^2c^2 \\ &= (2m_0c^2)^2/c^2 - m_0^2c^2 \\ &= 3m_0^2c^2.\end{aligned}$$

Applying the same invariant to the new particle (whose rest mass we will call m_1), we have

$$E^2 - p^2c^2 = m_1^2c^4,$$

so

$$\begin{aligned}m_1^2c^4 &= (3m_0c^2)^2 - 3m_0^2c^2 \cdot c^2 \\ &= 6m_0^2c^4.\end{aligned}$$

Thus

$$m_1 = \sqrt{6}\, m_0.$$

Problem 65

Explain carefully why an uncharged pi-meson (mass 134 MeV/c^2) can always decay into two photons whereas a photon of sufficient energy can decay into an electron–positron pair only in the presence of matter.

Solution

(1) Decaying pi-meson.

However the pi-meson is moving, we can define its zero momentum frame (ZMF) in which it is at rest. In this frame it can clearly decay into two photons of equal and opposite momentum, each of which carries a total energy of 67 MeV. If the process is possible in one inertial frame, it must be possible in all inertial frames even though the details (energy and momentum of the photons) will differ.

(2) Decaying photon.

If we assume this to be possible in the absence of matter, the electron–positron pair produced by the decay will have a ZMF. However, a photon cannot have zero momentum, so the decay process is impossible in the ZMF and hence in all frames.

In the presence of another particle, however, we can allocate energy and momentum between the photon and the particle as required, and the process becomes possible. For example, let us consider a photon of energy $h\nu$ decaying in the presence of a particle of rest mass M to produce an electron–positron pair (each of rest mass m), and the original particle of mass M, all at rest (see figure 61).

Since all the particles after the decay are at rest, the total momentum of the system is zero. Since the photon's momentum is $+h\nu/c$, the initial momentum of the particle of mass M must be $-h\nu/c$.

The total energy of the system after the decay is $(M + 2m)c^2$, which must be equal to the total energy before the decay. Since the energy of the photon is $h\nu$, the initial energy of the particle of mass M must be $(M + 2m)c^2 - h\nu$.

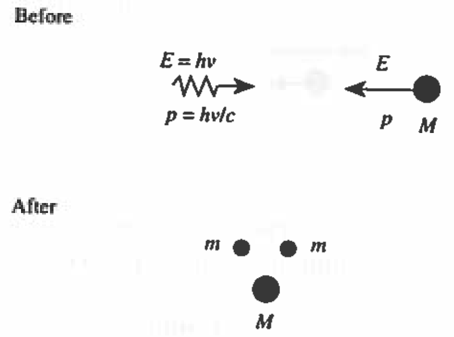

Figure 61

We know that for a single particle of rest mass M, the energy E and momentum p are related by

$$E^2 - p^2c^2 = M^2c^4,$$

so on substituting for E and p we obtain

$$(M + 2m)^2c^4 + h^2\nu^2 - 2(M + 2m)c^2h\nu - h^2\nu^2 = M^2c^4.$$

Therefore

$$h\nu = \frac{(M + 2m)^2c^4 - M^2c^4}{2(M + 2m)c^2} = \frac{2mc^2(1 + m/M)}{(1 + 2m/M)}.$$

Thus there is a solution for any value of M, and we deduce that the decay is possible if the photon has sufficient energy.

Problem 66

A particle of rest mass m moving along the x-axis with velocity v collides with a particle of rest mass $m/2$ moving along the x-axis with velocity $-v$. If the two particles coalesce, find the rest mass of the resulting particle.

Solution

Figure 62 shows the situation before and after the collision. Particle 1 (the one with mass m) has a total energy of γmc^2, where $\gamma = (1 - v^2/c^2)^{-1/2}$, and particle 2 has a total energy of $0.5\gamma mc^2$, so by conservation of energy the resulting particle must have a total energy of $1.5\gamma mc^2$.

Figure 62

Particle 1 has a momentum of $\gamma m v$, and particle 2 has a momentum of $-0.5\gamma m v$, so by conservation of momentum the resulting particle must have a momentum of $0.5\gamma m v$.

We now know the energy and momentum of the resulting particle, so we can use the energy–momentum invariant to find its rest mass m_1:

$$E^2 - p^2c^2 = m_1^2c^4,$$

so

$$\begin{aligned} m_1^2 &= E^2/c^4 - p^2/c^2 \\ &= (9/4)\gamma^2 m^2 - (1/4)\gamma^2 m^2 v^2/c^2 \\ &= (1/4)\gamma^2 m^2 (9 - v^2/c^2). \end{aligned}$$

Substituting the expression for $\gamma = (1 - v^2/c^2)^{-1/2}$, we finally obtain

$$m_1 = \frac{m}{2}\sqrt{\frac{9 - v^2/c^2}{1 - v^2/c^2}}.$$

As a simple check, we can see that when $(v/c)^2 \ll 1$ this tends to $3m/2$, which is clearly the correct classical limit.

Problem 67

The proton collider at CERN in Geneva makes use of proton and antiproton beams travelling in opposite directions. Explain the advantages of this technique over that of using an antiproton beam hitting stationary protons by calculating the minimum energy of the antiprotons ($\bar{\text{p}}$) needed to give the following reaction in which $\Omega\bar{\Omega}$ particle–antiparticle pairs are produced:

$$\text{p} + \bar{\text{p}} \rightarrow \text{p} + \bar{\text{p}} + \Omega + \bar{\Omega}:$$

(a) for colliding antiproton and proton beams;
(b) for antiprotons hitting stationary protons.

Express your answers in terms of the proton rest-mass energy.

(The Ω has a mass of $1.78m_\text{p}$.)

Solution

(a) Figure 63 shows the situation before and after the collision. If the proton and antiproton collide with equal and opposite velocities, the laboratory frame is the zero momentum frame (ZMF). The resulting system of particles thus has no net momentum, so the configuration of minimum energy is when all four particles are at rest. The total energy after the collision is thus

$$2m_\text{p}c^2 + 2m_\Omega c^2 = 5.56m_\text{p}c^2.$$

By conservation of energy, this must be equal to the total energy of the two particles before the collision, so by symmetry the energy of the antiproton must be half of this value, i.e. $2.78m_\text{p}c^2$.

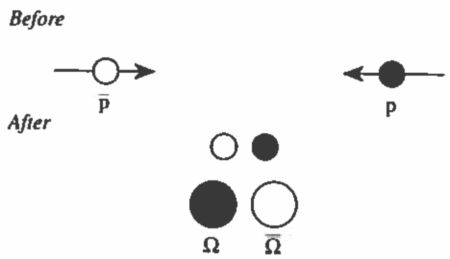

Figure 63

(b) If the proton is initially at rest, the description in part (a) is still valid except that we need to transform it into a different inertial frame, as shown in figure 64.

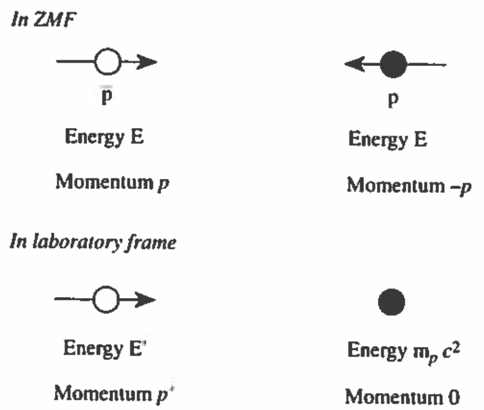

Figure 64

In the zero momentum frame, the system has a total energy of $2E$ and a total momentum of zero. In the laboratory frame, it has a total energy of $E' + m_p c^2$ and a total momentum of p'. Now for any system, the quantity

$$E^2 - p^2c^2$$

is a Lorentz invariant, i.e. it is the same in any inertial frame, so we have the relationship

$$(2E)^2 = (E' + m_p c^2)^2 - p'^2c^2,$$

which can be expanded as

$$4E^2 = E'^2 + m_p^2c^4 + 2E'm_pc^2 - p'^2c^2.$$

However, the energy E' and the momentum p' of the antiproton are related by

$$E'^2 - p'^2c^2 = m_p^2c^4,$$

and we can substitute this result to eliminate p':

$$2E^2 = m_p^2c^4 + E'm_pc^2.$$

Therefore

$$E' = (2E^2 - m_p^2c^4)/m_pc^2.$$

Now E is the value that we calculated in part (a), and we want to express E' in terms of the proton rest-mass energy, so if we divide this expression by m_pc^2 we obtain

$$\begin{aligned}\frac{E'}{m_pc^2} &= 2\left(\frac{E}{m_pc^2}\right)^2 - 1 \\ &= 2(2.78)^2 - 1 \\ &= 14.5.\end{aligned}$$

Thus the minimum antiproton energy if the proton is stationary is

▶ $14.5m_pc^2$.

Problem 68

A proton of total energy E collides elastically with a second proton at rest in the laboratory. After the collision the two protons follow trajectories which are disposed symmetrically at angles $\pm\phi/2$ to the direction of the

incident particle. By considering the motion in the laboratory frame, or otherwise, show that

$$\cos\phi = \frac{E - E_0}{E + 3E_0},$$

where E_0 is the rest mass energy of the proton.

What is the value of ϕ when the first proton is accelerated from rest through a potential difference of 1.5×10^9 V before colliding with the second proton?

Solution

Let us call the momentum of the initially moving particle p_1, and the energy and (modulus) momentum of each particle after the collision E_2 and p_2 respectively, as shown in figure 65.

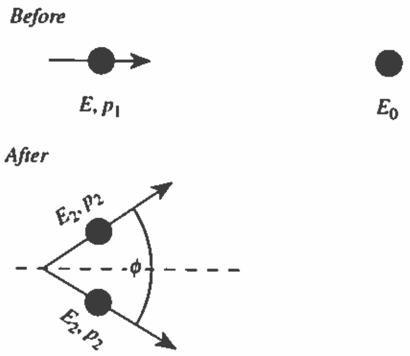

Figure 65

Conservation of momentum gives

$$p_1 = 2p_2\cos(\phi/2).$$

The relationship between E and p_1 is

$$E^2 - p_1^2c^2 = E_0^2,$$

which we can substitute into our expression to obtain

$$(E^2 - E_0^2) = 4p_2^2c^2\cos^2(\phi/2).$$

We can eliminate p_2 from this by using the energy–momentum relation again:

$$E_2^2 - p_2^2c^2 = E_0^2,$$

which gives

$$(E^2 - E_0^2) = 4(E_2^2 - E_0^2)\cos^2(\phi/2).$$

Finally we can eliminate E_2 using the principle of conservation of energy:

$$E + E_0 = 2E_2,$$

which gives

$$(E^2 - E_0^2) = (E^2 - 3E_0^2 + 2EE_0)\cos^2(\phi/2),$$

so

$$\cos^2\left(\frac{\phi}{2}\right) = \frac{E^2 - E_0^2}{E^2 + 2EE_0 - 3E_0^2}.$$

Recalling that $\cos\phi = 2\cos^2(\phi/2) - 1$, we can rearrange this to give

$$\cos\phi = \frac{E^2 + E_0^2 - 2EE_0}{E^2 + 2EE_0 - 3E_0^2} = \frac{(E - E_0)^2}{(E - E_0)(E + 3E_0)}.$$

Hence

$$\cos\phi = \frac{E - E_0}{E + 3E_0}$$

as required.

If the first proton is accelerated through a potential V, it acquires a kinetic energy of eV, so

$$E = E_0 + eV;$$

therefore

$$\begin{aligned} E/E_0 &= 1 + eV/E_0 \\ &= 1 + eV/mc^2 \\ &= 1 + 1.60 \times 10^{-19} \times 1.5 \times 10^9/(1.67 \times 10^{-27})/(3.00 \times 10^8)^2 \\ &= 2.60, \end{aligned}$$

so

$$\begin{aligned} \cos\phi &= (2.60 - 1)/(2.60 + 3) \\ &= 0.286; \end{aligned}$$

therefore $\phi = 73°$.

Problem 69

Consider the elastic scattering of a photon of frequency ν by a stationary electron (the Compton effect). Find an expression for the wavelength change of a photon scattered through 180°. What is the energy of a photon of initial energy 1 MeV after a single 180° scattering?

Solution

Let us assume that the photon has a frequency ν' after being scattered, and that the electron acquires a momentum p as a result of the collision, as shown in figure 66.

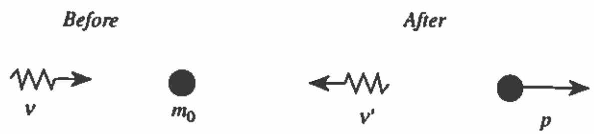

Figure 66

Since the momentum of a photon of frequency ν is given by $h\nu/c$, conservation of momentum gives

$$\frac{h\nu}{c} = p - \frac{h\nu'}{c}, \tag{1}$$

and since the energy of a particle of rest mass m_0 and momentum p is given by $(m_0^2c^4 + p^2c^2)^{1/2}$, conservation of energy gives

$$h\nu + m_0c^2 = h\nu' + (m_0^2c^4 + p^2c^2)^{1/2}. \tag{2}$$

Rearranging (2) to separate the square root gives

$$(m_0^2c^4 + p^2c^2)^{1/2} = h\nu - h\nu' + m_0c^2,$$

and squaring this gives

$$\begin{aligned} m_0^2c^4 + p^2c^2 = h^2\nu^2 + h^2\nu'^2 + m_0^2c^4 - 2h^2\nu\nu' + 2h\nu m_0c^2 \\ - 2h\nu' m_0c^2. \end{aligned} \tag{3}$$

We can eliminate p from this by rearranging (1),

$$pc = h\nu + h\nu',$$

and squaring,

$$p^2c^2 = h^2\nu^2 + h^2\nu'^2 + 2h^2\nu\nu'.$$

Substituting this expression into (3) and simplifying gives

$$2h\nu\nu' = (\nu - \nu')m_0c^2.$$

If we divide throughout by $\nu\nu' m_0 c$ we obtain

$$\frac{2h}{m_0c} = \frac{\nu - \nu'}{\nu\nu'}c = \left(\frac{1}{\nu'} - \frac{1}{\nu}\right)c = \lambda' - \lambda.$$

Thus the change in the wavelength on being scattered through 180° is
▶ $2h/m_0c$. [The quantity h/m_0c is the *Compton wavelength* of a particle of rest mass m_0.]

A photon of energy E has a wavelength ch/E, so taking $E = 1$ MeV $= 1.602 \times 10^{-13}$ J gives $\lambda = 1.240 \times 10^{-12}$ m. $2h/m_0c$ has a value of 4.853×10^{-12} m if m_0 is taken as the rest mass of the electron, so the final wavelength λ' is 6.093×10^{-12} m. The energy of a photon of this
▶ wavelength is 3.260×10^{-14} J or 0.20 MeV.

Quantum, atomic and nuclear physics

Problem 70

The Andromeda Nebula, at a distance of 2×10^{22} m from the Earth, radiates 8×10^{27} W in the spectral line of frequency 1420 MHz. Estimate the number of photons received per second when the nebula is observed by a radio telescope of collecting area 100 m^2.

Solution

The energy of a photon of frequency ν is $h\nu$, so if the nebula radiates a power P this must correspond to

$$\frac{P}{h\nu}$$

photons per unit time.

At a distance D, all of these photons will be spread uniformly (assuming the nebula radiates uniformly and there is no absorption) over an area $4\pi D^2$, so the number of photons received in unit time by an area A will be

$$\frac{PA}{4\pi D^2 h\nu}.$$

Substituting $P = 8 \times 10^{27}$ W, $A = 10^2$ m^2, $D = 2 \times 10^{22}$ m and
▶ $\nu = 1.42 \times 10^9$ Hz yields 1.7×10^8 photons per second.

Problem 71

What is the force experienced by a mirror when it reflects all the light from a laser with a power of 10 mW?

Solution

Write P for the laser power, and assume that the photons have frequency ν. Since the energy of the photons is $h\nu$, the number incident per unit