Some details on coordinate transformations, the invariance of the action and conservation laws.

For simplicity in the following I suppress the x^i index.

$$S = \int dt\, L(t, x(t), \dot{x}(t)) = \int dt\, L(t, x(t), v(t))$$

Now I write the more general coordinate transformations:

$$\begin{cases} t' = t'(t) = t + \delta t(t) = t + \varepsilon_t\, \bar{\delta} t(t) & \text{(i)} \\ x'(t) = x'(x(t)) = x(t) + \delta x(t) = x(t) + \varepsilon_x\, \bar{\delta} x(t) & \text{(ii)} \end{cases}$$

The 2nd equation represents how x' changes, as a function of x, keeping everything at fixed time t

First equation: functional form of $t \to t'(t)$
Second equation: functional form of $x \to x'(x)$

For future convenience, ε_t and ε_x are infinitesimal coefficients, introduced for bookkeeping. In particular, we will be interested in understanding how S changes under the above transformation, keeping only the linear terms.

The variation of S is:

$$\delta' S = \int dt'\, L(t', x'(t'), v'(t')) - \int dt\, L(t, x(t), v(t))$$

where $v(t) = dx/dt$ $\quad v'(t') = dx'/dt'$

I denote the variation as $\delta' S$ to distinguish this variation wrt the one where t does not change, i.e., in general:

$$\delta f(t,x,\dot{x}) = f(t, x+\delta x, \dot{x}+\delta\dot{x}) - f(t,x,\dot{x})$$

$$\delta' f(t,x,\dot{x}) = f(t+\delta t, x(t+\delta t) + \delta x(t+\delta t), \ldots) - f(t,x,\dot{x})$$

In order to compute $\delta' S$, I need to determine $x'(t')$ and $v'(t')$ as functions of δt and δx, keeping only the linear terms:

$$x'(t') = x'(t+\delta t) = x'(t) + \delta t \frac{d}{dt} x'(t) + \ldots$$

$$\frac{d}{dt} x'(t) = \frac{d}{dt}\left(x(t) + \varepsilon_x \bar{\delta} x(t)\right) = \frac{d}{dt} x(t) + \varepsilon_x \frac{d}{dt} \bar{\delta} x(t)$$

$$\frac{d}{dt'} x'(t') = \frac{dt}{dt'} \frac{d}{dt}\left[x'(t) + \delta t \frac{dx'}{dt} + \ldots\right]$$

$$\frac{dt}{dt'} = \frac{1}{dt'/dt} = \left(1 + \frac{d}{dt}\delta t\right)^{-1} = 1 - \frac{d}{dt}(\delta t)$$

$$\Rightarrow \frac{dx'(t')}{dt'} = \left[1 - \frac{d}{dt}(\delta t)\right] \frac{d}{dt}\left[x(t) + \varepsilon_x \bar{\delta} x + \varepsilon_t \bar{\delta} t \frac{dx'}{dt}\right]$$

$$= \left[1 - \varepsilon_t \frac{d}{dt} \bar{\delta} t\right]\left[v + \varepsilon_x \frac{d}{dt} \bar{\delta} x + \varepsilon_t \left(\frac{d\bar{\delta} t}{dt} v + \bar{\delta} t\, a\right) + \ldots\right]$$

$$= \frac{dx}{dt} + \frac{d}{dt}\delta x + (\delta t)\, a + v\varepsilon_t \cancel{\frac{d(\bar{\delta} t)}{dt}} - v\varepsilon_t \cancel{\frac{d(\bar{\delta} t)}{dt}}$$

$$\frac{d}{dt}x'(t) = \frac{d}{dt}x(t) + \varepsilon_x \frac{d}{dt}\bar{\delta}x$$

Hence: $\frac{dx'(t')}{dt'} = \frac{dx}{dt} + O(\varepsilon)$, $\frac{dx'}{dt} = \frac{dx}{dt} + O(\varepsilon)$. In particular:

$$\frac{dx'}{dt'} = \frac{dx}{dt} + \frac{d}{dt}\delta x + \delta t\, a \qquad (v'-v)$$

I denoted, and will denote, with $O(\varepsilon)$, a generic term linear in ε_t or ε_x.

We obtained: $x'(t') = x'(t) + \delta t \frac{d}{dt}x'(t) =$

$$= x(t) + \delta x(t) + \delta t\, v(t) + O(\varepsilon^2)$$

where $v(t) = dx/dt$.

Summarizing:

$$\begin{cases} t' = t + \delta t = t + \varepsilon_t \bar{\delta}t & \text{(iii)} \\ x'(t') = x(t) + \delta x(t) + \delta t\, v(t) = \\ \quad = x(t) + (\varepsilon_x)\bar{\delta}x + (\varepsilon_t \bar{\delta}t)\frac{dx}{dt} & \text{(iv)} \end{cases}$$

We have now all the ingredients. The above equations, in particular the 2nd one, can be written as

$$\delta' = \delta + \delta t \frac{d}{dt} \qquad (v)$$

where δ' = variation in t and x, δ = variation in x.

If applied to x, the identity above yields

$$\delta' x = \delta x + \delta t \frac{dx}{dt} \quad , \text{ that is}$$

$$x'(t') - x(t) = \delta x + \delta t \frac{d}{dt} x , \text{ that is equation (iv).}$$

Equation (v) is valid more in general, if applied to an arbitrary function. Let's see this on L:

$$\delta' L = L(t', x'(t'), v'(t')) - L(t, x(t), v(t))$$

$$= \frac{\partial L}{\partial t}(t'-t) + \frac{\partial L}{\partial x}(x'(t') - x(t)) + \frac{\partial L}{\partial v}(v'(t') - v(t)) + \dots$$

$$= \underline{\delta t}\, \partial_t L + (\delta x + \underline{\delta t}\, v)\partial_x L + \left(\frac{d}{dt}\delta x + \underline{\delta t}\, a\right)\partial_v L$$

Since $\frac{dL}{dt} = \partial_t L + (\partial_x L)\dot{x} + (\partial_v L)\dot{v}$, we have

$$\delta' L = \delta t \frac{dL}{dt} + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial v}\frac{d}{dt}\delta x$$

Moreover:

$$\delta L = L(t, x+\delta x, v+\delta v) - L(t, x, v)$$

$$= (\partial_x L)\,\delta x + (\partial_v L)\,\delta v$$

where $\delta v = \frac{d}{dt}(x+\delta x) - \frac{d}{dt}x = \frac{d}{dt}\delta x$

hence:

$$\delta' L = \delta L + \delta t \frac{dL}{dt} .$$

(which is the same as (v), this time applied to L).

Now we can compute $\delta' S$:

$$\delta' S = \int dt' \, L(t', x'(t'), v'(t')) - \int dt \, L =$$

$$= \int dt \frac{dt'}{dt} L(t', \dots) - \int dt \, L$$

$$= \int dt \left[\left(1 + \frac{d}{dt}(\delta t)\right) L(t', \dots) - L(t, \dots) \right]$$

$$= \int dt \left[L(t', \dots) - L(t, \dots) + \frac{d}{dt}(\delta t) L(t', \dots) \right]$$

$$= \int dt \left[\delta' L + L \frac{d}{dt}(\delta t) \right] \qquad (\delta' S)$$

where in the last term we have used, obviously, that

$$\frac{d}{dt}(\delta t) L(t', \dots) = \frac{d}{dt}(\delta t) \left[L(t, \dots) + O(\varepsilon) \right]$$

$$= \frac{d}{dt}(\delta t) L(t, \dots) + O(\varepsilon^2)$$

The $(\delta' S)$ equation can also be obtained using (v) applied on S (using Leibnitz, after all these are differentials..)

$$\delta' S = \delta'\left[\int dt\, L\right] = \int \delta'(dt)\, L + dt\,(\delta' L)$$

and $\delta'(dt) = dt' - dt = dt\left(\frac{dt'}{dt} - 1\right) = dt\left(\frac{d}{dt}(t+\delta t) - 1\right)$

$$= dt\ \frac{d}{dt}(\delta t)$$