Soluzioni covarianti delle eq. di Maxwell

$\partial_\mu F^{\mu\nu} = 4\pi J^\nu \quad , \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$

potenziali: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \Rightarrow 2^a$ eq. soddisfatta

Lorenz gauge: $\partial_\mu A^\mu = 0 \quad \Rightarrow \boxed{\Box A^\mu = 4\pi J^\mu}$

(è gauge fixing covariante)

- cerco funzione di Green per operatore $\Box$:

$\Box_x D(x,x') = \delta^{(4)}(x-x')$ dove $\delta^{(4)}(x-x') = \delta(x^0 - x'^0)\, \delta(\vec{x} - \vec{x}')$

x'^μ non è $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

- invarianza traslazionale di $\Box$ e di $\delta^{(4)}$ $\Rightarrow D = D(x-x')$

- $\Box D(x) = \delta^{(4)}(x)$

Se conosco $D(x)$ $\Rightarrow A^\mu(x) = A_0^\mu(x) + \int d^4x'\, D(x-x')\, 4\pi J^\mu(x')$

dove $A_0^\mu(x)$ è soluzione di $\Box A_0^\mu = 0$

dim: ovvio: $\Box A^\mu(x) = \Box A_0^\mu(x) + \int d^4x' \underbrace{\Box D(x-x')}_{\delta^{(4)}(x-x')} 4\pi J^\mu(x')$

$= \emptyset \quad + 4\pi J^\mu(x)$

- Per trovare D, Fourier transform (in 4 dimensioni): $\tilde{D}(k) = \int d^4x\, D(x)\, e^{ikx}$

$D(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{D}(k)\, e^{-ik\cdot x} \qquad k\cdot x = k^\mu x_\mu = k^0 x^0 - \vec{k}\cdot\vec{x}$

$\delta^{(4)}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \qquad$ (ovvio da $\delta(x) = \int \frac{dk}{2\pi} e^{ikx}$)

$\Box e^{-ikx} = -k^2 e^{-ikx}$

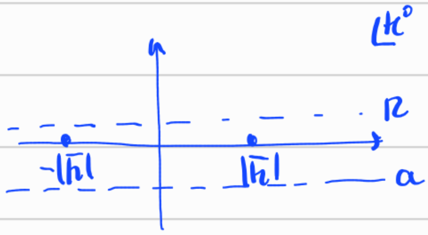
$$\Box D(x) = \delta^{(4)}(x) \quad \xmapsto{F.T.} \quad -K^2 \tilde{D}(K) = 1$$

$$\Rightarrow \boxed{\tilde{D}(K) = -\frac{1}{K^2}} \qquad K^2 = K^\mu K_\mu$$

Tornando in x^μ space: $D(x) = \int \frac{d^4k}{(2\pi)^4} \frac{-1}{K^2} e^{-iKx} = -\int \frac{d^3k \, dk^0}{(2\pi)^4} \frac{e^{-iKx}}{(k^0)^2 - |\vec{k}|^2}$

• Piano complesso:

ci sono 2 poli $k^0 = \pm |\vec{k}|$, non integrabili $\Rightarrow$ PROLUNGAMENTO ANALITICO



$k^0 \mapsto k^0 = Re(k^0) + i\, Im(k^0)$

Voglio usare Cauchy

$$\Rightarrow e^{-iKx} = e^{-ik^0x^0} e^{+i\vec{k}\cdot\vec{x}} =$$

$$= e^{-ix^0 Re(k^0)} \underbrace{e^{x^0 Im(k^0)}}$$

Se $x^0 > 0 \Rightarrow Im(k^0) < 0 \Rightarrow$

Se $x^0 < 0 \Rightarrow Im(k^0) > 0 \Rightarrow$

* cammino R (ritardato)

$x^0 < 0 \rightarrow$ chiudo sopra $\rightarrow$ NO POLI $\rightarrow$ risultato nullo $\Rightarrow \sim \theta(x^0)$

$x^0 > 0 \rightarrow$ chiudo sotto $\rightarrow$ poli

$$\int \frac{dk^0}{2\pi} \frac{e^{-ix^0k^0}}{(k^0)^2 - |\vec{k}|^2} = \frac{1}{2\pi}(-2\pi i) \sum_{residues} = -i \left[\lim_{k^0 \to |\vec{k}|} \frac{e^{-ix^0k^0}}{k^0 + |\vec{k}|} + \lim_{k^0 \to -|\vec{k}|} \frac{e^{-ix^0k^0}}{k^0 - |\vec{k}|} \right]$$

$$= \frac{-i}{2|\vec{k}|}\left[e^{-ix^0|\vec{k}|} - e^{+ix^0|\vec{k}|}\right] = -\frac{\sin(x^0|\vec{k}|)}{|\vec{k}|}$$

Quindi, riassumendo

$$D_r(x) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{-k^2} = \int \frac{d^3k}{(2\pi)^3}\frac{\sin(x^0|\vec{k}|)}{|\vec{k}|}e^{i\vec{k}\cdot\vec{x}}\,\theta(x^0)$$

Coord. sferiche:

$$\int \frac{d^3k}{(2\pi)^3}\frac{\sin(x^0|\vec{k}|)}{|\vec{k}|}e^{i\vec{k}\cdot\vec{x}} = \left\{\text{coord. sferiche, qui } k = |\vec{k}|\right\}$$

$$= \frac{1}{(2\pi)^3}\int k^2\,dk\,d\cos\vartheta\,d\varphi\,\frac{1}{k}\sin(x^0k)\,e^{-ik|\vec{x}|\cos\vartheta}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty k\,dk\,\sin(x^0k)\int_{-1}^{1}dc\,e^{-ik|\vec{x}|c}$$

$$= \frac{1}{(2\pi)^2}\int\left[\quad\right]\frac{e^{-ik|\vec{x}|c}}{-ik|\vec{x}|}\Bigg|_{c=-1}^{c=1}$$

$$= \frac{1}{(2\pi)^2}\frac{i}{|\vec{x}|}\int_0^\infty dk\,\sin(x^0k)\left[e^{-ik|\vec{x}|} - e^{ik|\vec{x}|}\right]$$

$$= \frac{1}{2\pi^2}\frac{1}{|\vec{x}|}\int_0^\infty dk\,\sin(x^0k)\sin(k|\vec{x}|)$$

$$= \left\{\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}\right\}$$

$$= -\frac{1}{8\pi^2}\frac{1}{|\vec{x}|}\int_0^\infty\left[e^{ix^0k} - e^{-ix^0k}\right]\left[e^{ik|\vec{x}|} - e^{-ik|\vec{x}|}\right]$$

$$= \frac{-1}{8\pi^2}\frac{1}{|\vec{x}|}\int_0^\infty dk\left[e^{ik(x^0+|\vec{x}|)} + e^{-ik(x^0+|\vec{x}|)} - e^{-ik(x^0-|\vec{x}|)} - e^{ik(x^0-|\vec{x}|)}\right]$$

$$= \frac{-1}{8\pi^2}\frac{1}{|\vec{x}|}\int_{-\infty}^{+\infty} dk\left[e^{ik(x^0+|\vec{x}|)} - e^{ik(x^0-|\vec{x}|)}\right]$$

$$= \frac{-1}{8\pi^2}\frac{1}{|\bar{x}|}\left[\delta(x^0+|\bar{x}|) - \delta(x^0-|\bar{x}|)\right]\cdot(2\pi)$$

$$D_r(x) = \frac{-1}{4\pi}\frac{1}{|\bar{x}|}\left[\delta(x^0+|\bar{x}|) - \delta(x^0-|\bar{x}|)\right]\theta(x^0)$$

$$\Rightarrow D_r(x) = \frac{1}{4\pi|\bar{x}|}\delta(x^0-|\bar{x}|)\,\theta(x^0)$$

$$\boxed{D_r(x-x') = \frac{\theta(x^0-x^{0\prime})\,\delta(x^0-x^{0\prime}-|\bar{x}-\bar{x}'|)}{4\pi|\bar{x}-\bar{x}'|}}$$

FUNZIONE DI GREEN RITARDATA (causale)

* cammino a (advanced, avanzata)

$$\boxed{D_a(x-x') = \frac{\theta(-(x^0-x^{0\prime}))\,\delta(x^0-x^{0\prime}+|\bar{x}-\bar{x}'|)}{4\pi|\bar{x}-\bar{x}'|}}$$

ADVANCED GREEN FUNCTION

- scrittura covariante: compaiono tempi e spazi separatamente, ma in realtà è un'espressione covariante

$$\delta((x-x')^2) = \delta((x^0-x^{0\prime})^2 - |\bar{x}-\bar{x}'|^2) = \{(x^0-x^{0\prime}) = \pm|\bar{x}-\bar{x}'|\}$$

$$= \frac{\delta(x^0-x^{0\prime}-|\bar{x}-\bar{x}'|) + \delta(x^0-x^{0\prime}+|\bar{x}-\bar{x}'|)}{2|\bar{x}-\bar{x}'|}$$

$$\Rightarrow \boxed{\begin{aligned} D_r(x-x') &= \frac{1}{2\pi}\theta(x^0-x^{0\prime})\,\delta((x-x')^2) \\ D_a(x-x') &= \frac{1}{2\pi}\theta(x^{0\prime}-x^0)\,\delta((x-x')^2) \end{aligned}}$$

questo δ è invariante di Lorentz.