An Introduction to the Standard Model of Electroweak Interactions

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Contents

Chapter 1

Introduction

The aim of these lectures is a description of the construction and the main phenomenological implications of the Glashow-Weinberg-Salam unified theory of weak and electromagnetic interactions (universally referred to as the *standard model*.) Basic knowledge in quantum field theory [1],[2] and elementary group theory [3] is assumed, as well as familiarity with the fundamental phenomenology of weak interactions [4].

No attempt will be made to give a full list of references. Such a list can be found in any standard text book of particle physics; see for example refs. [4]-[8].

Chapter 2

Construction of the standard model

2.1 A gauge theory of weak interactions

Our starting point is the effective lagrangian that describes weak interaction processes at low energies. This lagrangian (often called the Fermi lagrangian) has the form of a sum of products of vector and axial vector currents. For example, the terms responsible for nucleon β decay and for muon decay are[1]

$$\mathcal{L} = -\frac{G^{(\beta)}}{\sqrt{2}}\overline{p}\gamma^\alpha(1-a\gamma_5)n\,\overline{e}\gamma_\alpha(1-\gamma_5)\nu_e - \frac{G^{(\mu)}}{\sqrt{2}}\overline{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\,\overline{e}\gamma_\alpha(1-\gamma_5)\nu_e. \tag{2.1.1}$$

From the experimental values of the muon and neutron lifetimes, one obtains

$$G^{(\mu)} \simeq 1.16639 \times 10^{-5}\ \mathrm{GeV}^{-2}; \quad G^{(\beta)} \simeq G^{(\mu)} \equiv G_F, \tag{2.1.2}$$

while the value

$$a = 1.239 \pm 0.09 \tag{2.1.3}$$

can be extracted from hyperon decays.

The field theory defined by the interaction in eq. (2.1.1) is manifestly not renormalizable, since it contains operators with mass dimension 6 (a necessary condition for perturbative renormalizability is that the lagrangian density contains operators with mass dimension less than or equal to 4, see Appendix 4.1), and it gives rise to a non-unitary S matrix (see Appendix 4.2). However, it contains all the physical information needed to build a renormalizable and unitary theory of weak interactions.

The idea is that of building a theory with local invariance under the action of some group of field transformations, a *gauge* theory, in analogy with quantum electrodynamics (see Appendix 4.3). We will then require that the new theory reduce to eq. (2.1.1) in the low-energy limit, in the sense that the local four-fermion interaction of the Fermi lagrangian will be interpreted as the interaction vertex that arises from the exchange of a massive vector boson with

[1] Throughout these lectures, particle fields will be denoted by the symbol usually adopted for the corresponding particle: e for the electron, ν_e for the electron neutrino, and so on.

momentum much smaller than its mass. In this way, both problems of renormalizability and unitarity will be solved, since gauge theories are known to be renormalizable, and the mass of the intermediate vector boson will act as a cut-off that stops the growth of cross sections with energy, thus ensuring unitarity of the scattering matrix.

In order to complete this program, we must choose the group of local invariance, and then assign particle fields to representations of this group. Both these steps can be performed with the help of the information contained in the Fermi lagrangian. Let us first consider the electron and the electron neutrino. They participate in the weak interaction via the current

$$J_\mu = \overline{\nu}_e \gamma_\mu \frac{1}{2}(1-\gamma_5) e. \tag{2.1.4}$$

We would like to rewrite J_μ in the form of a Noether current,

$$\overline{\psi}_i \, \gamma_\mu \, T^A_{ij} \, \psi_j, \tag{2.1.5}$$

where ψ_i are the components of some multiplet of the (as yet unknown) gauge group, and T^A_{ij} are the corresponding generators. In the case of J_μ, this can be done in the following way. We observe that the current J_μ can be written as

$$J_\mu = \overline{L} \gamma_\mu \tau^+ L, \tag{2.1.6}$$

where

$$L = \frac{1}{2}(1-\gamma_5) \begin{pmatrix} \nu_e \\ e \end{pmatrix} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \tag{2.1.7}$$

$$\tau^+ = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{2.1.8}$$

and τ_i are the usual Pauli matrices. The hermitian conjugate current

$$J^\dagger_\mu = \overline{L} \gamma_\mu \tau^- L; \qquad \tau^- = \frac{1}{2}(\tau_1 - i\tau_2) \tag{2.1.9}$$

will also participate in the interaction. The currents are in one-to-one correspondence with the generators of the symmetry group, which, in turn, form a closed set with respect to the commutation operation: the commutator of two generators is also a generator. Therefore, the current

$$J^\mu_3 = \overline{L} \gamma^\mu [\tau^+, \tau^-] L = \overline{L} \gamma^\mu \tau_3 L \tag{2.1.10}$$

will also be present. No other current must be introduced, since

$$[\tau_3, \tau^\pm] = 2\tau^\pm. \tag{2.1.11}$$

We have thus interpreted the current J_μ as one of the three conserved currents of a theory with $SU(2)$ gauge invariance, the Pauli matrices being the generators of $SU(2)$ in the fundamental representation, and we have assigned the left-handed neutrino and electron fields to an $SU(2)$ doublet. The right-handed neutrino and electron components, ν_{eR} and e_R, do not take part in the weak-interaction phenomena described by the Fermi lagrangian, so they must be assigned to the singlet (or scalar) representation. Of course, this is not the only possible choice, but it is the simplest possibility (and also the correct one, as we will see) since it does not require the introduction of fermion fields other than the observed ones.

The current J_3^μ is a *neutral* current: it contains creation and annihilation operators of particles with the same charge (actually, of the same particle). Neutral currents do not appear in the Fermi lagrangian; no neutral current phenomenon is observed in low-energy weak interactions. As we will see, the experimental observation of phenomena induced by weak neutral currents is a crucial test of the validity of the standard model. Notice also that the neutral current J_3^μ cannot be identified with the only other neutral current we know of, the electromagnetic current. This is for two reasons: first, the electromagnetic current involves both left-handed and right-handed fermion fields with the same weight; and second, the electromagnetic current does not contain a neutrino term, the neutrino being chargeless. We will come back later to the problem of neutral currents, that will end up with the inclusion of the electromagnetic current in the theory. For the moment, we go on with the construction of our $SU(2)$ gauge theory. We must introduce vector meson fields W_i^μ, one for each of the three $SU(2)$ generators, and build a covariant derivative

$$D^\mu = \partial^\mu - igW_i^\mu T_i, \tag{2.1.12}$$

where we have introduced, as is customary in gauge theories, a coupling constant g. The matrices T_i are generators of $SU(2)$ in the representation of the multiplet the covariant derivative is acting on. For example, when D^μ acts on the doublet L, we have $T_i \equiv \tau_i/2$, and when it acts on the gauge singlet e_R we have $T_i \equiv 0$. We are now ready to write the gauge-invariant lagrangian for the fermion fields (which we assume massless for the time being):

$$\begin{aligned}\mathcal{L} &= i\overline{L}\,\not{D}L + i\overline{\nu}_{eR}\,\not{D}\nu_{eR} + i\overline{e}_R\,\not{D}e_R \\ &= \mathcal{L}_{kin} + \mathcal{L}_c + \mathcal{L}_n\end{aligned} \tag{2.1.13}$$

where $\not{D} = \gamma_\mu D^\mu$. The lagrangian $\mathcal{L}$ contains the usual kinetic term for massless fermions,

$$\mathcal{L}_{kin} = i\overline{L}\,\not{\partial}\,L + i\overline{\nu}_{eR}\,\not{\partial}\,\nu_{eR} + i\overline{e}_R\,\not{\partial}\,e_R, \tag{2.1.14}$$

plus an interaction term $\mathcal{L}_c + \mathcal{L}_n$, where

$$\mathcal{L}_c = gW_1^\mu \overline{L}\gamma_\mu \frac{\tau_1}{2} L + gW_2^\mu \overline{L}\gamma_\mu \frac{\tau_2}{2} L \tag{2.1.15}$$

corresponds to charged-current interactions, and

$$\mathcal{L}_n = gW_3^\mu \overline{L}\gamma_\mu \frac{\tau_3}{2} L = \frac{g}{2} W_3^\mu \left(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} - \overline{e}_L\gamma_\mu e_L\right) \tag{2.1.16}$$

to neutral current interactions. The charged-current term $\mathcal{L}_c$ is usually expressed in terms of the fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2). \tag{2.1.17}$$

We find

$$\mathcal{L}_c = \frac{g}{\sqrt{2}}\overline{L}\gamma^\mu\tau^+LW^+_\mu + \frac{g}{\sqrt{2}}\overline{L}\gamma^\mu\tau^-LW^-_\mu . \tag{2.1.18}$$

We have already observed that the neutral current $J_3^\mu = \overline{L}\gamma^\mu\tau_3 L$ cannot be identified with the electromagnetic current, and correspondingly that the gauge vector boson W_3^μ cannot be interpreted as the photon. The construction of the model can therefore proceed in two different directions: either we modify the multiplet structure of the theory, in order to make J_3^μ equal to the electromagnetic current; or we admit the possibility of the existence of weak neutral currents, and we extend the gauge group in order to accommodate also the electromagnetic current in addition to J_3^μ. We proceed to describe the second possibility, which is the one that turns out to be correct, after the discovery of weak processes induced by neutral currents. Nevertheless, it should be kept in mind that this was not at all obvious to physicists before the observation of weak neutral-current effects.

The simplest way of extending the gauge group $SU(2)$ to include a second neutral generator is to include an abelian factor $U(1)$:

$$SU(2) \to SU(2)\otimes U(1). \tag{2.1.19}$$

We will require our lagrangian to be invariant also under the $U(1)$ gauge transformations

$$\psi \to \psi' = \exp\left[ig'\alpha\frac{Y(\psi)}{2}\right]\psi, \tag{2.1.20}$$

where ψ is a generic field of the theory, g' is the coupling constant associated with the $U(1)$ factor of the gauge group, and $Y(\psi)$ is a quantum number, usually called the *weak hypercharge*, to be specified for each field ψ. Since the $SU(2)$ factor of the gauge group acts in a different way on left-handed and right-handed fermions (it is a *chiral* group), it is natural to allow for the possibility of assigning different hypercharge quantum numbers to the left and right components of the same fermion field. A new gauge vector field B^μ must be introduced, and the covariant derivative becomes

$$D^\mu = \partial^\mu - igW_i^\mu T_i - ig'\frac{Y}{2}B^\mu, \tag{2.1.21}$$

where Y is a diagonal matrix with the hypercharge values in its diagonal entries. Y being diagonal, only the term $\mathcal{L}_n$ is modified. We have now

$$\begin{aligned}\mathcal{L}_n &= \frac{g}{2}W_3^\mu\left(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} - \overline{e}_L\gamma_\mu e_L\right)\\ &+ \frac{g'}{2}B^\mu\left[Y(L)\left(\overline{\nu}_{eL}\gamma_\mu\nu_{eL} + \overline{e}_L\gamma_\mu e_L\right) + Y(\nu_{eR})\overline{\nu}_{eR}\gamma_\mu\nu_{eR} + Y(e_R)\overline{e}_R\gamma_\mu e_R\right]. \end{aligned}\tag{2.1.22}$$

This can be written as

$$\mathcal{L}_n = g\,\overline{\Psi}\gamma_\mu T_3\Psi\, W_3^\mu + g'\,\overline{\Psi}\gamma_\mu\frac{Y}{2}\Psi\, B^\mu, \tag{2.1.23}$$

where Ψ is a column vector formed with all left-handed and right-handed fermionic fields in the theory, and $T_3 = \pm 1/2$ for ν_{eL} and e_L respectively, and $T_3 = 0$ for ν_{eR} and e_R. We can now assign the quantum numbers Y in such a way that the electromagnetic interaction term appear in eq. (2.1.22). To do this, we first perform a rotation by an angle θ_W in the space of the two neutral gauge fields W_3^μ, B^μ:

$$B^\mu = A^\mu \cos\theta_W - Z^\mu \sin\theta_W \tag{2.1.24}$$

$$W_3^\mu = A^\mu \sin\theta_W + Z^\mu \cos\theta_W \,. \tag{2.1.25}$$

In terms of the new vector fields A^μ, Z^ν, eq. (2.1.23) takes the form

$$\mathcal{L}_n = \overline{\Psi}\gamma_\mu \left(g \sin\theta_W T_3 + \frac{Y}{2} g' \cos\theta_W \right) \Psi \, A^\mu + \overline{\Psi}\gamma_\mu \left(g \cos\theta_W T_3 - \frac{Y}{2} g' \sin\theta_W \right) \Psi \, Z^\mu . \tag{2.1.26}$$

In order to identify one of the two neutral vector fields, say A^μ, with the photon field, we must choose $Y(L)$, $Y(\nu_{eR})$ and $Y(e_R)$ so that A^μ couples to the electromagnetic current

$$J^\mu_{em} = -e \left(\overline{e}_R \gamma^\mu e_R + \overline{e}_L \gamma^\mu e_L \right) \equiv e \overline{\Psi} \gamma^\mu Q \Psi , \tag{2.1.27}$$

where Q is the electromagnetic charge matrix in units of the positron charge e. In other words, it must be

$$T_3 \, g \sin\theta_W + \frac{Y}{2} \, g' \cos\theta_W = e \, Q \,. \tag{2.1.28}$$

The weak hypercharges Y appear in eq. (2.1.28) only through the combination Yg': thus, we have the freedom of rescaling the hypercharges by a common factor K, provided we rescale g' by $1/K$. This freedom can be used to fix arbitrarily the value of one of the three hypercharges $Y(L), Y(\nu_{eR}), Y(e_R)$. The conventionally adopted choice is

$$Y(L) = -1. \tag{2.1.29}$$

With this choice, eq. (2.1.28) restricted to the doublet of left-handed leptons is

$$+\frac{1}{2} g \sin\theta_W - \frac{1}{2} g' \cos\theta_W = 0 \tag{2.1.30}$$

$$-\frac{1}{2} g \sin\theta_W - \frac{1}{2} g' \cos\theta_W = -e, \tag{2.1.31}$$

which gives

$$g \sin\theta_W = g' \cos\theta_W = e. \tag{2.1.32}$$

(For a generic doublet of fermions with charges Q_1 and Q_2 the r.h.s. of eq. (2.1.32) becomes $e(Q_1 - Q_2)$, but gauge invariance of the charged coupling requires $Q_1 - Q_2 = 1$.) Equation (2.1.28) then reduces to

$$T_3 + \frac{Y}{2} = Q, \tag{2.1.33}$$

which is valid for any fermion. For example, we find

$$Y(\nu_{eR}) = 0; \qquad Y(e_R) = -2. \tag{2.1.34}$$

This completes the assignments of weak hypercharge values to all fermion fields. Notice that the right-handed neutrino has zero charge and zero hypercharge, and it is an $SU(2)$ singlet: it does not take part in electroweak interactions.

The second term in eq. (2.1.26) defines the weak neutral current coupled to the other neutral vector boson Z_μ. It can be written as

$$e\overline{\Psi}\gamma_\mu Q_Z \Psi Z^\mu \,, \tag{2.1.35}$$

where

$$Q_Z = \frac{1}{\cos\theta_W \sin\theta_W}\left(T_3 - Q\sin^2\theta_W\right). \tag{2.1.36}$$

The extension of the theory to more lepton doublets is straightforward.

We must now include hadrons in the theory. We will do this in terms of quark fields, taking as a starting point the hadronic current responsible for β decay and strange particle decays:

$$J^\mu_{had} = \cos\theta_c\,\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)d + \sin\theta_c\,\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)s, \tag{2.1.37}$$

where θ_c is the Cabibbo angle ($\theta_c \sim 13°$) and u, d, s are the up, down and strange quark fields respectively. We are tempted to proceed as in the case of leptons: define

$$Q = \frac{1}{2}(1-\gamma_5)\begin{bmatrix} u \\ d \\ s \end{bmatrix} \equiv \begin{bmatrix} u_L \\ d_L \\ s_L \end{bmatrix} \tag{2.1.38}$$

and

$$T^+ = \begin{bmatrix} 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1.39}$$

so that

$$J^\mu_{had} = \overline{Q}\gamma^\mu T^+ Q. \tag{2.1.40}$$

This leads to a system of currents which is in contrast with experimental observations. Indeed, we find that

$$T_3 = [T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta_c & -\cos\theta_c\sin\theta_c \\ 0 & -\cos\theta_c\sin\theta_c & -\sin^2\theta_c \end{bmatrix}. \tag{2.1.41}$$

The corresponding neutral current contains flavour-changing terms, such as e.g. $\overline{d}_L\gamma^\mu s_L$, with a weight of the same order of magnitude of flavour-conserving ones. These terms induce processes at a rate which is not compatible with experimental observation. For example, the ratio of the decay rates for the processes

$$K^+ \to \pi^0 e^+ \nu_e \tag{2.1.42}$$

$$K^+ \to \pi^+ e^+ e^- \tag{2.1.43}$$

is approximately

$$r = \left[\frac{\sin\theta_c}{\sin\theta_c\cos\theta_c}\right]^2 = \frac{1}{\cos^2\theta_c} \simeq 1.1, \tag{2.1.44}$$

while observations give

$$r_{\exp} \simeq 1.3 \times 10^5, \tag{2.1.45}$$

that is, the charged-current process ($s \to u$) is enhanced by five orders of magnitude with respect to the neutral-current ($s \to d$) one. Our theory should therefore be modified in order to avoid the introduction of flavour-changing neutral currents. The solution to this puzzle was found by S. Glashow, J. Iliopoulos and L. Maiani. They suggested to introduce a fourth quark c (for *charm*) with charge 2/3 like the up quark, and to assume that its couplings to down and strange quarks are given by

$$\begin{aligned} J^\mu_{had} &= \cos\theta_c\,\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)d + \sin\theta_c\,\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma_5)s \\ &- \sin\theta_c\,\overline{c}\gamma^\mu\frac{1}{2}(1-\gamma_5)d + \cos\theta_c\,\overline{c}\gamma^\mu\frac{1}{2}(1-\gamma_5)s. \end{aligned} \tag{2.1.46}$$

The c quark being not observed at the time, they had to assume that its mass was much larger than those of u, d and s quarks, and therefore outside the energy range of available experimental devices. The current J^μ_{had} can still be put in the form (2.1.40), where now

$$Q = \begin{bmatrix} u_L \\ c_L \\ d_L \\ s_L \end{bmatrix} \tag{2.1.47}$$

and

$$T^+ = \begin{bmatrix} 0 & 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & -\sin\theta_c & \cos\theta_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.1.48}$$

No flavour-changing neutral current is now present. In fact,

$$[T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{2.1.49}$$

thanks to the fact that the upper right 2×2 block of T^+ has been cleverly chosen to be an orthogonal matrix. The existence of the quark c was later confirmed by the discovery of the J/ψ particle. The current J^μ_{had} is usually written in the following form, analogous to the corresponding leptonic current:

$$J^\mu_{had} = (\overline{u}_L\overline{d}'_L)\gamma^\mu\tau^+\begin{pmatrix} u_L \\ d'_L \end{pmatrix} + (\overline{c}_L\overline{s}'_L)\gamma^\mu\tau^+\begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \tag{2.1.50}$$

where

$$\begin{pmatrix} d'_L \\ s'_L \end{pmatrix} = V \begin{pmatrix} d_L \\ s_L \end{pmatrix}, \qquad V = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix}. \tag{2.1.51}$$

The pairs (u, d), (c, s) are called *quark families.* Actually, there is a correspondence between quark and lepton families, whose origin will be investigated in section 3.3. The structure outlined above can be extended to an arbitrary number of quark families. With n families, V becomes an $n \times n$ matrix, and it must be unitary in order to ensure the absence of flavour-changing neutral currents.

The final form of the charged-current interaction term, including n families of leptons and quarks, is then

$$\mathcal{L}_c = \frac{g}{\sqrt{2}} \sum_{f=1}^{n} \left(\overline{L}_f \gamma^\mu \tau^+ L_f + \overline{Q}_f \gamma^\mu \tau^+ Q_f \right) W_\mu^+ + h.c., \tag{2.1.52}$$

where

$$L_f = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \ldots \tag{2.1.53}$$

$$Q_f = \begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \ldots. \tag{2.1.54}$$

An equivalent (and often more useful) form of eq. (2.1.52) is

$$\mathcal{L}_c = \frac{g}{\sqrt{2}} \left(\sum_{f=1}^{n} \bar{\nu}_L^f \, \gamma^\mu \, e_L^f + \sum_{f,g=1}^{n} \bar{u}_L^f \, \gamma^\mu \, V_{fg} \, d_L^g \right) W_\mu^+ + h.c. \tag{2.1.55}$$

The neutral-current lagrangian in eq. (2.1.23) is directly generalizable to include quark fields.

To conclude the construction of the standard model lagrangian, we must consider the pure Yang-Mills term

$$\mathcal{L}_{YM} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^i W_i^{\mu\nu}, \tag{2.1.56}$$

where

$$\begin{aligned} B^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu \\ W_i^{\mu\nu} &= \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g \epsilon_{ijk} W_j^\mu W_k^\nu . \end{aligned} \tag{2.1.57}$$

The corresponding expression in terms of the physical fields $W_\mu^\pm$, Z_μ and A_μ can be easily worked out with the help of eqs. (2.1.17), (2.1.24) and (2.1.25), which we rewrite here:

$$W_\mu^1 = \frac{1}{\sqrt{2}} (W_\mu^+ + W_\mu^-) \tag{2.1.58}$$

$$W_\mu^2 = \frac{i}{\sqrt{2}} (W_\mu^+ - W_\mu^-) \tag{2.1.59}$$

$$W_\mu^3 = A_\mu \sin\theta_W + Z_\mu \cos\theta_W \tag{2.1.60}$$

$$B_\mu = A_\mu \cos\theta_W - Z_\mu \sin\theta_W . \tag{2.1.61}$$

We get

$$
\begin{aligned}
W^1_{\mu\nu} &= \frac{1}{\sqrt{2}}\left[W^+_{\mu\nu} + ig\,\sin\theta_W(W^+_\mu A_\nu - W^+_\nu A_\mu) + ig\,\cos\theta_W(W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right] + \text{h.c.} \\
W^2_{\mu\nu} &= \frac{i}{\sqrt{2}}\left[W^+_{\mu\nu} + ig\,\sin\theta_W(W^+_\mu A_\nu - W^+_\nu A_\mu) + ig\,\cos\theta_W(W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right] + \text{h.c.} \\
W^3_{\mu\nu} &= F_{\mu\nu}\sin\theta_W + Z_{\mu\nu}\cos\theta_W - ig(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu) \\
B_{\mu\nu} &= F_{\mu\nu}\cos\theta_W - Z_{\mu\nu}\sin\theta_W, \qquad (2.1.62)
\end{aligned}
$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad (2.1.63)$$

$$Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu \qquad (2.1.64)$$

$$W^{\mu\nu}_\pm = \partial^\mu W^\nu_\pm - \partial^\nu W^\mu_\pm. \qquad (2.1.65)$$

It follows that

$$
\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W^+_{\mu\nu}W^{\mu\nu}_- \qquad (2.1.66) \\
&+ig\sin\theta_W\,(W^+_{\mu\nu}W^\mu_- A^\nu - W^-_{\mu\nu}W^\mu_+ A^\nu + F_{\mu\nu}W^\mu_+ W^\nu_-) \\
&+ig\cos\theta_W\,(W^+_{\mu\nu}W^\mu_- Z^\nu - W^-_{\mu\nu}W^\mu_+ Z^\nu + Z_{\mu\nu}W^\mu_+ W^\nu_-) \\
&-\frac{g^2}{2}\,(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\left[W^+_\mu W^-_\nu\,(A_\rho A_\sigma\sin^2\theta_W + Z_\rho Z_\sigma\cos^2\theta_W + 2A_\rho Z_\sigma\sin\theta_W\cos\theta_W) - \frac{1}{2}W^+_\mu W^+_\nu W^-_\rho W^-_\sigma\right]
\end{aligned}
$$

2.2 Masses

Masses for the gauge bosons

We will now show that, in order to make contact with the Fermi theory, which is known to correctly describe low-energy weak interactions, the gauge vector bosons of weak interactions must have a non-zero mass. We will also be able to set a lower bound to the mass of the W boson. Let us consider the amplitude for down-quark β decay. In the Fermi theory, it is simply given by

$$-\frac{G_F}{\sqrt{2}}\overline{u}\gamma^\mu(1-\gamma_5)d\,\overline{e}\gamma_\mu(1-\gamma_5)\nu_e. \qquad (2.2.1)$$

In the context of the standard model, the same process is induced by the exchange of a W boson, with amplitude

$$\left(\frac{g}{\sqrt{2}}\overline{u}_L\gamma^\mu d_L\right)\frac{1}{q^2-m_W^2}\left(\frac{g}{\sqrt{2}}\overline{e}_L\gamma_\mu\nu_{eL}\right), \qquad (2.2.2)$$

(we are neglecting Cabibbo mixing for simplicity). The virtuality q^2 of the exchanged vector boson is bounded from above by the square of the neutron-proton mass difference, $q^2 \leq (m_N -$

$m_P)^2 \sim (1.3\ \text{MeV})^2$. For eq. (2.2.2) to be equal to the Fermi amplitude in the $q^2 \to 0$ limit, m_W must be non zero, and

$$\frac{G_F}{\sqrt{2}} = \left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2}. \tag{2.2.3}$$

Recalling that $g = e/\sin\theta_W$, eq. (2.2.3) gives us the lower bound

$$m_W \geq 37.3\ \ \text{GeV}, \tag{2.2.4}$$

quite a large value, if compared with the present upper bound on the photon mass,

$$m_\gamma \leq 2 \cdot 10^{-16}\ \ \text{eV}. \tag{2.2.5}$$

So, we know since the beginning that if weak interactions are to be mediated by vector bosons, these must be very heavy. On the other hand, we also know that gauge theories are incompatible with mass terms for the vector bosons. One possibility is to break gauge invariance explicitly and insert a mass term for the W boson by hand, but this leads to a non-renormalizable theory. Let us investigate this point in more detail. Consider for simplicity the lagrangian of a pure abelian gauge theory, with a mass term for the gauge vector field:

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} m_\gamma^2 A^\mu A_\mu, \tag{2.2.6}$$

and work out the propagator $\Delta^{\mu\nu}$ for A^μ in momentum space. We get

$$\Delta^{\mu\nu} = \frac{i}{k^2 - m_\gamma^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_\gamma^2}\right). \tag{2.2.7}$$

The propagator $\Delta^{\mu\nu}$ has not the correct behaviour for large values of the momentum k: for $k \to \infty$ it becomes a constant, rather than vanishing as k^{-2}, thus violating power-counting and making the theory unrenormalizable.

A related problem of a massive vector boson theory, such as the one defined by eq. (2.2.6), is again unitarity of the scattering matrix. The amplitude for a generic physical process which involves the emission or the absorption of a vector boson with four-momentum k and polarization vector $\epsilon(k)$ has the form

$$\mathcal{M} = \mathcal{M}^\mu \epsilon_\mu(k). \tag{2.2.8}$$

A massive vector (contrary to a massless one) may be polarized longitudinally. In this case, choosing the z axis along the direction of the 3-momentum of the vector boson, the polarization is given by

$$\epsilon_L = (k/m_\gamma, 0, 0, E/m_\gamma) = k/m_\gamma + \mathcal{O}(m_\gamma^2/E^2), \tag{2.2.9}$$

where we have imposed the transversity condition $p \cdot \epsilon = 0$ and the normalization condition $\epsilon^2 = -1$. Clearly, the amplitude $\mathcal{M}$ will grow indefinitely with the energy E, unless some mechanism takes place to cut off this growth, and unitarity of the scattering matrix will eventually be violated.

To see how one can introduce a mass term for gauge vector bosons without spoiling renormalizability and unitarity, we first consider a simple example where this happens, and then we generalize our considerations to the standard model. The simple theory we consider is scalar electrodynamics, that is, a gauge theory based on $U(1)$ invariance, coupled to one complex scalar field ϕ with charge e. The lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^{\mu}\phi)^{\dagger}D_{\mu}\phi - V(\phi), \tag{2.2.10}$$

where $D^{\mu} = \partial^{\mu} - ieA^{\mu}$, and $V(\phi)$ is the so-called scalar potential, which is constrained by gauge invariance and renormalizability to be of the form

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4 . \tag{2.2.11}$$

We look for field configurations that minimize the energy of the system. Because of the requirement of translational invariance, they must be constant configurations, so we can neglect the derivative terms and look for the minimum of the potential V. Now, if $m^2 \geq 0$, then V has a minimum for $\phi = 0$. If, on the other hand, $m^2 < 0$, then m^2 can no longer be interpreted as a mass squared for the field ϕ; furthermore, the potential has now an infinite number of degenerate minima, given by all those field configurations for which

$$\mid \phi \mid^2 = -\frac{m^2}{2\lambda} \equiv \frac{1}{2}v^2. \tag{2.2.12}$$

All these minimum configurations (in the language of quantum theory, all these ground states) are connected by gauge transformations, that change the phase of the complex field ϕ without affecting its modulus. The system will choose one of the infinite possible minimum configurations. This phenomenon is usually called *spontaneous breaking of the gauge symmetry*, but the symmetry is not actually broken. In fact, the Lagrangian is still gauge invariant, and all the properties connected with this invariance (such as, for example, current conservation) are still there. It is important to stress this point, because at the quantum level this is essentially what guarantees the renormalizability of the theory, which would instead be lost in the case of an explicit breaking of the gauge symmetry.

Let us now expand the field ϕ around one of the infinite minimum configurations; we choose the one for which ϕ is real at the minimum, but of course any other choice would be equivalent. We introduce real scalar fields $H(x)$ and $G(x)$ by

$$\phi(x) = \frac{1}{\sqrt{2}}\left[v + H(x) + iG(x)\right], \tag{2.2.13}$$

where v is defined in eq. (2.2.12). In principle, the field G could have been removed from the lagrangian by an appropriate gauge transformation. In fact, we could have first applied a local gauge transformation to ϕ in order to make it real, and then shift it according to $\phi = (v+H)/\sqrt{2}$. For the moment, we keep both H and G in the lagrangian; we will come back to this point later. Up to an irrelevant constant, the scalar potential takes the form

$$\begin{aligned} V(\phi) &= (m^2 v + \lambda v^3)H + \frac{1}{2}(m^2 + 3\lambda v^2)H^2 + \frac{1}{2}(m^2 + \lambda v^2)G^2 \\ &\quad +\lambda v H(H^2 + G^2) + \frac{\lambda}{4}(H^2 + G^2)^2. \end{aligned} \tag{2.2.14}$$

Using eq. (2.2.12), $\lambda v^2 = -m^2$, we see that the terms proportional to H and G^2 vanish, which means that the field G is massless. The coefficient of the H^2 term is now $(-2m^2)/2$, and has therefore the correct sign to be interpreted as a mass term (remember that m^2 is negative).

After the reparametrization eq. (2.2.13), the $|D\phi|^2$ term takes the following form:

$$\begin{aligned}(D^\mu\phi)^\dagger D_\mu\phi &= \frac{1}{2}\partial^\mu H\partial_\mu H + \frac{1}{2}\partial^\mu G\partial_\mu G + \frac{1}{2}e^2(H^2+G^2+2vH)A^\mu A_\mu \\ &- eA_\mu(H\partial^\mu G - G\partial^\mu H) - evA^\mu\partial_\mu G + \frac{1}{2}e^2v^2A^\mu A_\mu. \end{aligned} \quad (2.2.15)$$

We see that the gauge field A_μ has acquired a mass $m_\gamma = ev$, precisely the result we were looking for. The term $-evA^\mu\partial_\mu G$ is unpleasant, because it mixes the gauge vector field A^μ with the unphysical field G; we will see in a moment how to get rid of it.

We must now check that the appearance of a mass term for A^μ via the spontaneous symmetry breaking mechanism has not spoiled the renormalizability of our theory, contrary to what happened when we tried to break the symmetry explicitly. It is well known that, in order to quantize a gauge theory, a gauge-fixing term must be added to the lagrangian (obviously, this was not necessary in the case of explicit gauge symmetry breaking). A convenient choice for the gauge-fixing term is

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu + ev\xi G)^2, \quad (2.2.16)$$

where ξ is an arbitrary constant (the gauge parameter). Equation (2.2.16) corresponds to the gauge-fixing condition

$$\partial^\mu A_\mu = -ev\xi G. \quad (2.2.17)$$

The gauge-fixing lagrangian (2.2.16) has been carefully chosen in order to cancel the term proportional to $A^\mu\partial_\mu G$ in eq. (2.2.15). Indeed, eq. (2.2.16) contains a term $-ev\partial^\mu AG$, which after partial integration cancels the unwanted term in eq. (2.2.15). Observe also that the gauge-fixing lagrangian introduces a term

$$-\frac{1}{2}\xi e^2v^2G^2 = -\frac{1}{2}\xi m_\gamma^2 G^2, \quad (2.2.18)$$

which gives a squared mass ξm_γ^2 to the unphysical field G.

Collecting the various terms, the lagrangian is given by:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\partial^\mu A^\nu\partial_\mu A_\nu - \partial^\mu A^\nu\partial_\nu A_\mu) + \frac{1}{2}m_\gamma^2A^\mu A_\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \\ &+\frac{1}{2}\partial^\mu H\partial_\mu H - \frac{1}{2}m_H^2H^2 + \frac{1}{2}\partial^\mu G\partial_\mu G - \frac{1}{2}\xi m_\gamma^2G^2 \\ &+\frac{1}{2}e^2(H^2+G^2+2vH)A^\mu A_\mu - eA_\mu(H\partial^\mu G - G\partial^\mu H) \\ &-\lambda vH(H^2+G^2) - \frac{\lambda}{4}(H^2+G^2)^2, \end{aligned} \quad (2.2.19)$$

where $m_\gamma = ev$ and $m_H^2 = 2\lambda v^2$. The propagators can be worked out from the quadratic terms, collected in the first two rows of eq. (2.2.19). We get

$$\Delta_\xi^{\mu\nu}(k) = \frac{i}{k^2 - m_\gamma^2}\left[-g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m_\gamma^2}\right] \quad (2.2.20)$$

for the photon propagator, and

$$\Delta_H(k) = \frac{i}{k^2 - m_H^2}; \qquad \Delta_G(k) = \frac{i}{k^2 - \xi m_\gamma^2} \tag{2.2.21}$$

for the two scalar propagators.

Observe that the photon propagator has now the correct behaviour $1/k^2$ at large momenta. However, in addition to the pole at $k^2 = m_\gamma^2$, an unphysical singularity at $k^2 = \xi m_\gamma^2$ has now appeared. This singularity is located at the mass squared of the unphysical scalar field G. One can prove that the contributions of this term of the photon propagator to physical quantities are exactly cancelled by the contributions of G exchange. It is easy to check this cancellation in specific cases, such as e.g. $H\gamma \to H\gamma$ scattering at tree level. In order to perform this kind of checks, it is useful to rewrite the propagator in eq. (2.2.20) in the form

$$\Delta_\xi^{\mu\nu}(k) = \frac{i}{k^2 - m_\gamma^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_\gamma^2}\right) - \frac{k^\mu k^\nu}{m_\gamma^2}\frac{i}{k^2 - \xi m_\gamma^2}, \tag{2.2.22}$$

where the G propagator appears explicitly.

When we let ξ tend to infinity, the photon propagator eq. (2.2.20) takes the form of eq. (2.2.7):

$$\lim_{\xi\to\infty} \Delta_\xi^{\mu\nu}(k) = \frac{i}{k^2 - m_\gamma^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_\gamma^2}\right). \tag{2.2.23}$$

The theory is still renormalizable, but in a hidden way: renormalizability must arise as a consequence of cancellations among different contributions to the same Green function, since the propagator does not obey the power-counting rule. The limit $\xi \to \infty$ is called the *unitary gauge*. The advantage of the unitary gauge is that the theory contains only physical degrees of freedom. In fact, when $\xi \to \infty$ the gauge-fixing condition reduces to $G(x) = 0$ (see eq. (2.2.16)); it corresponds to the gauge choice that eliminates G from the theory since the very beginning. The drawback is that in the unitary gauge renormalizability is not manifest at each intermediate step of a calculation.

Two common gauge choices are the Feynman gauge, $\xi = 1$, which gives

$$\Delta_F^{\mu\nu} = -\frac{ig^{\mu\nu}}{k^2 - m_\gamma^2} \tag{2.2.24}$$

and the Landau gauge, $\xi = 0$, for which

$$\Delta_L^{\mu\nu} = \frac{i}{k^2 - m_\gamma^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right). \tag{2.2.25}$$

One last observation about the field $G(x)$. It looks like we lost a degree of freedom, since we started with a complex scalar field and we end up with one real scalar. Actually, the number of degrees of freedom stays the same, since the photon is now massive, and has therefore three polarization states instead of two. The field $G(x)$ is called a *would-be Goldstone boson*. This terminology reflects the fact that, in the absence of gauge invariance and of the gauge-fixing term, G would have simply been a physical, zero-mass state, which is always present when spontaneous symmetry breaking occurs. This mechanism is known as the Higgs mechanism. It is possible to extend it to the standard model, with a few modifications that we now describe in detail.

We have learned that, in order to break spontaneously a gauge symmetry, we must introduce scalar fields in the game. How should we do this in the standard model? First, the scalar field must transform non-trivially under that part of the gauge group that we want to undergo spontaneous breaking. Secondly, we must be careful not to break the $U(1)$ invariance corresponding to electrodynamics, or, in other words, we want the photon to stay massless. This means that spontaneous symmetry breaking must take place in three of the four "directions" of the $SU(2) \times U(1)$ gauge group, the fourth one being that corresponding to electric charge. The simplest way to do this is to assign the scalar field ϕ to a doublet representation of $SU(2)$:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.2.26}$$

The Higgs mechanism takes place in analogy with scalar electrodynamics. The most general scalar potential consistent with gauge invariance and renormalizability is

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4, \tag{2.2.27}$$

which has a minimum at

$$\mid \phi \mid^2 = -\frac{m^2}{2\lambda} \equiv \frac{1}{2} v^2. \tag{2.2.28}$$

The value of the hypercharge of the scalar doublet ϕ is fixed by the requirement that the minimum configuration

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; \quad |v_1|^2 + |v_2|^2 = v^2 \tag{2.2.29}$$

is left unchanged by electromagnetic gauge transformations, that correspond to the subgroup $U(1)_{\mathrm{em}}$. This corresponds to the requirement

$$e^{ieQ\alpha} \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.2.30}$$

or equivalently

$$\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1/2 + Y/2 & 0 \\ 0 & -1/2 + Y/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.2.31}$$

where Q_1, Q_2 are the electric charges of ϕ_1, ϕ_2, and we have used eq. (2.1.33). There are two possibilities:

$$1) \quad v_1 = 0, \quad |v_2| = v, \quad Y = +1 \tag{2.2.32}$$

$$2) \quad v_2 = 0, \quad |v_1| = v, \quad Y = -1. \tag{2.2.33}$$

We will adpot the first choice, with $Y = +1$ and therefore $Q_1 = 1, Q_2 = 0$. We will further assume that v_2 is real and positive.

We can reparameterize ϕ in the following way:

$$\phi = \frac{1}{\sqrt{2}} e^{i\tau^i \theta^i(x)/v} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{2.2.34}$$

with $\theta^i(x)$ and $H(x)$ real. This parametrization is not suited for renormalizable gauges, because it is non-linear and contains all powers of the fields θ_i. It is convenient, however, if we work in the unitary gauge; in fact, it is apparent that the fields θ_i can be transformed away by an $SU(2)$ gauge transformation. In this section, we will use the unitary gauge $\theta_i = 0$. The standard model lagrangian in a generic renormalizable gauge is given in Appendix 4.4.

The scalar potential takes the form

$$V = \frac{1}{2}(2\lambda v^2)H^2 + \lambda v H^3 + \frac{1}{4}\lambda H^4; \tag{2.2.35}$$

the Higgs scalar H has a squared mass $m_H^2 = 2\lambda v^2$. The term $(D^\mu\phi)^\dagger D_\mu \phi$ can be worked out using eq. (2.2.34) with $\theta^i = 0$. We get

$$\begin{aligned} D^\mu \phi &= \left(\partial^\mu - i\frac{g}{2}\tau^i W_\mu^i - i\frac{g'}{2}B_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H(x) + v \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} - \frac{i}{2}(H+v)\frac{1}{\sqrt{2}}\begin{pmatrix} g(W_1^\mu - iW_2^\mu) \\ -gW_3^\mu + g'B^\mu \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} - \frac{i}{2}(H+v)\begin{pmatrix} gW^{\mu +} \\ -\sqrt{(g^2+g'^2)/2}Z^\mu \end{pmatrix}, \end{aligned} \tag{2.2.36}$$

where in the last step we have used eqs. (2.1.17), (2.1.24), (2.1.25) and (2.1.32). We have therefore

$$(D^\mu\phi)^\dagger D_\mu\phi = \frac{1}{2}\partial^\mu H \partial_\mu H + \left[\frac{1}{4}g^2 W^{\mu +}W_\mu^- + \frac{1}{8}(g^2+g'^2)Z^\mu Z_\mu\right](H+v)^2. \tag{2.2.37}$$

We see that the W and Z bosons have acquired masses

$$m_W^2 = \frac{1}{4}g^2 v^2 \tag{2.2.38}$$

$$m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2. \tag{2.2.39}$$

Note that the photon stays massless. With the scalar field ϕ transforming as a doublet of $SU(2)$, there is always a linear combination of B^μ and W_3^μ that does not receive a mass term, but only if $Y(\phi) = 1$ (or -1) does this linear combination coincide with the one in eq. (2.1.24). The lagrangian in a generic renormalizable gauge is much more complicated, since it also involves kinetic and interaction terms for non-physical Higgs scalars, the would-be Goldstone bosons. It is described in Appendix 4.4.

The value of v, the *vacuum expectation value* of the neutral component of the Higgs doublet, can be obtained combining eqs. (2.2.3) and (2.2.38), and using the measured valued of the Fermi constant. We get

$$v = \sqrt{\frac{1}{G_F\sqrt{2}}} \simeq 246.22 \text{ GeV}. \tag{2.2.40}$$

The value of the Higgs quartic coupling λ (or equivalently the Higgs mass) is not fixed by our present knowledge.

Masses for hadrons and flavour-mixing

Fermion masses are also forbidden by the gauge symmetry of the standard model. In fact, the mass term for a fermion field ψ has the form

$$-m\overline{\psi}\psi = -m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L), \tag{2.2.41}$$

and cannot be invariant under a chiral transformation, that is, a transformation that acts differently on left-handed and right-handed fields. The gauge transformations of the standard model are precisely of this kind. Again, this difficulty can be circumvented by means of the Higgs doublet ϕ.

We first consider the hadronic sector. We have seen in section 2.1 that the interaction lagrangian is not diagonal in terms of quark fields with definite flavours. Let us call u'^f and d'^f the fields that bring the interaction terms diagonal (the index f runs over the n fermion generations); in principle, there is no reason why only down-type quarks should be rotated. We also define

$$Q'^f = \begin{pmatrix} u'^f_L \\ d'^f_L \end{pmatrix} \qquad U'^f = u'^f_R \quad D'_i = d'^f_R. \tag{2.2.42}$$

A Yukawa interaction term can be added to the lagrangian:

$$\mathcal{L}_Y^{hadr} = -(\bar{Q}'\,\phi\,h'_D\,D' + \bar{D}'\,\phi^\dagger\,h'^\dagger_D\,Q') - (\bar{Q}'\,\phi_c\,h'_U\,U' + \bar{U}'\,\phi_c^\dagger\,h'^\dagger_U\,Q'), \tag{2.2.43}$$

where h'_U and h'_D are generic $n \times n$ constant complex matrices in the generation space, and

$$\phi_c = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}. \tag{2.2.44}$$

It easy to check that $\mathcal{L}_Y^{hadr}$ is Lorentz-invariant, gauge-invariant[2] and renormalizable, and therefore it can (actually, it must) be included in the lagrangian. The matrices h'_U and h'_D can be diagonalized by means of bi-unitary transformations:

$$h_U \equiv V_L^{U\dagger} h'_U V_R^U \tag{2.2.45}$$

$$h_D \equiv V_L^{D\dagger} h'_D V_R^D, \tag{2.2.46}$$

[2] If ϕ transforms as an $SU(2)$ doublet, so does $\phi_c = \epsilon\phi^*$, where ϵ is the antisymmetric tensor; check it as an exercise.

where $V^{\rm U,D}_{\rm L,R}$ are unitary matrices, chosen so that are diagonal with real, non-negative entries. Now, we define new quark fields u and d by

$$u'_L = V_L^U u_L, \quad u'_R = V_R^U u_R \tag{2.2.47}$$

$$d'_L = V_L^D d_L, \quad d'_R = V_R^D d_R, \tag{2.2.48}$$

In the unitary gauge, eq. (2.2.43) becomes

$$\mathcal{L}_Y^{hadr} = -\frac{1}{\sqrt{2}}(v+H)\sum_{f=1}^{n}(h_D^f\,\bar{d}^f d^f + h_U^f\,\bar{u}^f u^f), \tag{2.2.49}$$

where $h^f_{U,D}$ are the diagonal entries of the matrices $h_{U,D}$. We can now identify the quark masses with

$$m_U^f = \frac{v h_U^f}{\sqrt{2}}, \qquad m_D^f = \frac{v h_D^f}{\sqrt{2}}. \tag{2.2.50}$$

Since the matrices $V^{\rm U,D}_{\rm L,R}$ are constant in space-time, eqs. (2.2.47,2.2.48) are obviously global symmetry transformations of the free quark lagrangian. They also leave unchanged the neutral-current interaction term, because of the universality of the couplings of fermions of different families to the photon and the Z. The only term in the lagrangian which is affected by eqs. (2.2.47,2.2.48) is the charged-current interaction, because the up and down components of the same left-handed doublet are transformed in a different way. Indeed, we find

$$J^\mu_{hadr} = \sum_{f=1}^{n}\bar{Q}'^f\,\gamma^\mu\,\tau^+\,Q'^f = \sum_{f,g=1}^{n}\bar{u}_L^f\,\gamma^\mu\,V_{fg}\,d_L^g, \tag{2.2.51}$$

where

$$V = V_L^{U\dagger} V_L^D\,. \tag{2.2.52}$$

The matrix V is usually called the *Cabibbo-Kobayashi-Maskawa* (CKM) matrix. It is a unitary matrix, and its unitarity guarantees the suppression of flavour changing neutral currents, as we already discussed in section 2 in the case of two fermion families. The matrix V enters the standard model lagrangian as a fundamental parameter, on the same step as masses and gauge couplings. The values of its entries must be determined from experiments.

To conclude this subsection, we now determine the number of independent parameters in the CKM matrix. A generic $n\times n$ unitary matrix is formed with n^2 independent real parameters. Some (n_A) of them can be thought of as rotation angles in the n-dimensional space of generations, and there are as many as the coordinate planes in N dimensions:

$$n_A = \begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{1}{2}n(n-1). \tag{2.2.53}$$

The remaining parameters are just complex phases; their number is

$$\hat{n}_P = n^2 - n_A = \frac{1}{2}n(n+1). \tag{2.2.54}$$

Some of the $\hat{n}_P$ complex phases, however, can be eliminated by redefining the left-handed quark fields. This means that $2n-1$ phases are eliminable: in fact, there are n up-type quarks and n down-type quarks, that can be rotated to eliminate the phase of one row and one column of V, and the -1 accounts for the fact that the entry corresponding to the intersection of the row and the column cannot be rotated twice. The number of really independent complex phases in V is therefore

$$n_P = \hat{n}_P - (2n-1) = \frac{1}{2}(n-1)(n-2). \tag{2.2.55}$$

Observe that, with one or two fermion families, the CKM matrix can be made real. The first case with non-trivial phases is $n=3$, which corresponds to $n_P = 1$. In the standard model with three fermion families, the CKM matrix has four independent parameters: three rotation angles and one complex phase. In the general case, the total number of independent parametersi in the CKM matrix is

$$n_A + n_P = (n-1)^2. \tag{2.2.56}$$

Masses for leptons

The same procedure can be applied to the leptonic sector. Everything is formally unchanged: up-quarks are replaced by neutrinos and down-quarks are replaced by charged leptons (e^-, μ^- and τ^-). There is however an important difference, which leads to considerable simplifications: as we have seen, right-handed neutrinos have no interactions. Therefore, there is no Yukawa coupling involving the conjugate scalar field ϕ_c, and there is only one matrix of Yukawa couplings, h'_E:

$$\mathcal{L}_Y^{lept} = -(\overline{L'}\phi h'_E E' + \overline{E'}\phi^\dagger h'^\dagger_E L')\,, \tag{2.2.57}$$

which can be diagonalized by means of a biunitary transformation

$$h_E = V_L^{E\dagger} h'_E V_R^E\,. \tag{2.2.58}$$

The difference with respect to the case of quarks is that now we have the freedom of redefining the left-handed neutrino fields using *the same* matrix V_L^E that rotates charged leptons:

$$\nu'_L = V_L^E \nu_L \tag{2.2.59}$$

$$e'_L = V_L^E e_L, \quad e'_R = V_R^E e_R\,. \tag{2.2.60}$$

This puts the Yukawa interaction in diagonal form,

$$\mathcal{L}_Y^{lept} = -\sum_{f=1}^{n} h_E^f\,(\bar{L}^f\,\phi\,e_R^f + \bar{e}_R^f\,\phi^\dagger\,L^i), \tag{2.2.61}$$

but, contrary to what happens in the quark sector, leaves the charged interaction term unchanged, since

$$J^\mu_{lept} = \bar{L}'\,\gamma^\mu\,\tau^+\,L' = \bar{L}\,\gamma^\mu\,\tau^+\,L = \sum_{f=1}^{n} \bar{\nu}_L^f\,\gamma^\mu\,e_L^f. \tag{2.2.62}$$

In other words, in the leptonic sector there is no mixing among different generations, because the Yukawa coupling matrix can be diagonalized by a global transformation under which the full lagrangian is invariant. As a consequence, not only the overall leptonic number, but also individual leptonic flavors are conserved. This is due to the absence of right-handed neutrinos.

The values of the Yukawa couplings h_E^f are determined by the values of the observed lepton masses. In fact, using eq. (2.2.34), we find

$$\mathcal{L}_Y^{lept} = -\sum_{f=1}^{n} \frac{h_E^f}{\sqrt{2}}(v+H)\bar{e}_f e_f, \tag{2.2.63}$$

thus allowing the identifications

$$m_E^f = \frac{v h_E^f}{\sqrt{2}}. \tag{2.2.64}$$

As in the case of vector bosons, in renormalizable gauges there are also interaction terms between quarks and non-physical scalars; the details are given in Appendix 4.4.

2.3 Summary

To summarize, the standard model lagrangian in the unitary gauge is given by

$$\mathcal{L}_{SM} = \mathcal{L}_{kin} + \mathcal{L}_{em} + \mathcal{L}_c + \mathcal{L}_n + \mathcal{L}_{YM} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}, \tag{2.3.1}$$

where

- $\mathcal{L}_{kin}$ is the free fermion lagrangian:

$$\mathcal{L}_{kin} = \sum_{f=1}^{n} \left[\bar{\nu}^f \, i\partial\!\!\!/ \, \nu^f + \bar{e}^f \left(i\partial\!\!\!/ - m_E^f \right) e^f + \bar{u}^f \left(i\partial\!\!\!/ - m_U^f \right) u^f + \bar{d}^f \left(i\partial\!\!\!/ - m_D^f \right) d^f \right]. \tag{2.3.2}$$

  The index f labels the n fermion families. Neutrinos are assumed massless.

- $\mathcal{L}_{em}$ is the electromagnetic coupling:

$$\mathcal{L}_{em} = e \sum_{f=1}^{n} \left(-\bar{e}^f \, \gamma_\mu \, e^f + \frac{2}{3} \bar{u}^f \, \gamma_\mu \, u^f - \frac{1}{3} \bar{d}^f \, \gamma_\mu \, d^f \right) \, A^\mu, \tag{2.3.3}$$

- $\mathcal{L}_c$ is the charged-current interaction term:

$$\begin{aligned} \mathcal{L}_c &= \frac{g}{2\sqrt{2}} \left[\sum_{f=1}^{n} \bar{\nu}^f \, \gamma^\mu (1-\gamma_5) \, e^f + \sum_{f,g=1}^{n} \bar{u}^f \, \gamma^\mu (1-\gamma_5) \, V_{fg} \, d^g \right] W_\mu^+ \\ &+ \frac{g}{2\sqrt{2}} \left[\sum_{f=1}^{n} \bar{e}^f \, \gamma^\mu (1-\gamma_5) \, \nu^f + \sum_{f,g=1}^{n} \bar{d}^f \, \gamma^\mu (1-\gamma_5) \, V_{fg}^* \, u^g \right] W_\mu^- . \end{aligned} \tag{2.3.4}$$

- $\mathcal{L}_n$ is the neutral-current interaction term:

$$\begin{aligned} \mathcal{L}_n &= \frac{e}{4\cos\theta_W \sin\theta_W} \sum_{f=1}^{n} \left[\bar{\nu}^f \, \gamma_\mu (1-\gamma_5) \, \nu^f + \bar{e}^f \, \gamma_\mu \left(-1 + 4\sin^2\theta_W + \gamma_5 \right) \, e^f \right. \\ &+ \left. \bar{u}^f \, \gamma_\mu \left(1 - \frac{8}{3}\sin^2\theta_W - \gamma_5 \right) \, u^f + \bar{d}^f \, \gamma_\mu \left(-1 + \frac{4}{3}\sin^2\theta_W + \gamma_5 \right) \, d^f \right] Z^\mu . \end{aligned} \tag{2.3.5}$$

- $\mathcal{L}_{YM}$ is the pure Yang-Mills lagrangian:

$$\begin{aligned} \mathcal{L}_{YM} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{2} W_{\mu\nu}^+ W_-^{\mu\nu} \\ & + ig \sin\theta_W \left(W_{\mu\nu}^+ W_-^\mu A^\nu - W_{\mu\nu}^- W_+^\mu A^\nu + F_{\mu\nu} W_+^\mu W_-^\nu \right) \\ & + ig \cos\theta_W \left(W_{\mu\nu}^+ W_-^\mu Z^\nu - W_{\mu\nu}^- W_+^\mu Z^\nu + Z_{\mu\nu} W_+^\mu W_-^\nu \right) \\ & - \frac{g^2}{2} \left(2 g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) \\ & \left[W_\mu^+ W_\nu^- \left(A_\rho A_\sigma \sin^2\theta_W + Z_\rho Z_\sigma \cos^2\theta_W + 2 A_\rho Z_\sigma \sin\theta_W \cos\theta_W \right) - \frac{1}{2} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- \right] \end{aligned} \tag{2.3.6}$$

  where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu; \quad Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu; \quad W_\pm^{\mu\nu} = \partial^\mu W_\pm^\nu - \partial^\nu W_\pm^\mu . \tag{2.3.7}$$

- The Higgs sector provides a term

$$\mathcal{L}_{Higgs} = \frac{1}{2}\,\partial^\mu H\,\partial_\mu H + \left(m_W^2\,W^{\mu+}W_\mu^- + \frac{1}{2}\,m_Z^2\,Z^\mu Z_\mu\right)\left(1+\frac{H}{v}\right)^2 - \frac{1}{2}\,m_H^2\,H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4. \tag{2.3.8}$$

- The Yukawa coupling $\mathcal{L}_{Yukawa}$ is given by

$$\mathcal{L}_{Yukawa} = -\frac{1}{\sqrt{2}}\frac{H}{v}\sum_{f=1}^{n}(m_D^f\,\bar{d}^f d^f + m_U^f\,\bar{u}^f u^f + m_E^f\,\bar{e}^f e^f). \tag{2.3.9}$$

The parameters appearing in $\mathcal{L}_{SM}$ are not all independent. The gauge-Higgs sector is entirely specified by the four parameters

$$g, \quad g', \quad v, \quad m_H, \tag{2.3.10}$$

since

$$m_W^2 = \frac{1}{4}g^2v^2, \quad m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2, \quad \lambda = \frac{m_H^2}{2v^2}, \quad \tan\theta_W = \frac{g'}{g} \tag{2.3.11}$$

and $g\sin\theta_W = g'\cos\theta_W = e$. However, g, g', v are often eliminated in favour of the electromagnetic coupling α_{em}, the Fermi constant G_F and the Z^0 mass m_Z, which are measured with high accuracy. We have

$$\alpha_{em} = \frac{g^2g'^2}{4\pi(g^2+g'^2)}, \quad G_F = \frac{1}{\sqrt{2}v^2}, \quad m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2. \tag{2.3.12}$$

The free parameters in the fermionic sector are the $3n$ masses m_U^f, m_D^f, m_E^f, and the $(n-1)^2$ independent parameters in the Cabibbo-Kobayashi-Maskawa matrix V. This gives a total of 17 free parameters for the standard model with three fermion generations.

Chapter 3

Special topics

3.1 The scalar sector beyond the tree level

Effective action and effective potential

In this section we will study the scalar sector of the standard model, and in particular the phenomenon of spontaneous breaking of the gauge symmetry, beyond the classical level. This is most conveniently done in the context of the generating functional formalism, which we briefly recall. One introduces the functional

$$Z[J] = \langle 0|Te^{i\int d^4x\, J(x)\phi(x)}|0\rangle = \langle 0|0\rangle_J, \tag{3.1.1}$$

where $J(x)$ is a classical source with the appropriate gauge transformation properties (we are only interested in the scalar sector, so we do not introduce here sources for the other fields in the theory). Functional derivatives of $Z[J]$ with respect to J at $J = 0$ give the Green's functions of the theory; for this reason, $Z[J]$ is called the generating functional. It can be shown that the functional

$$W[J] = -i \log Z[J] \tag{3.1.2}$$

is the generating functional for connected Green's functions. One then defines the classical field ϕ_c as

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} \tag{3.1.3}$$

and the effective action $\Gamma[\phi_c]$ as

$$\Gamma[\phi_c] = W[J] - \int d^4x\, J(x)\phi_c(x). \tag{3.1.4}$$

The effective action has an expansion in powers of the classical field,

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n\, \phi_c(x_1)\dots\phi_c(x_n)\Gamma_n(x_1, \dots x_n), \tag{3.1.5}$$

whose coefficients $\Gamma_n(x_1,\ldots x_n)$ can be shown to be the connected, one-particle irreducible Green's functions of the theory. The functional $\Gamma[\phi_c]$ is the appropriate tool to study spontaneous symmetry breaking. In fact, the condition for spontaneous symmetry breaking is that ϕ_c is different from zero even when the source J is set equal to zero, as can be read off eq. (3.1.3). On the other hand, for $J=0$, one has

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c}=0. \tag{3.1.6}$$

We conclude that spontaneous symmetry breaking takes place when the classical field that minimizes the effective action is different from zero.

Consider now the Fourier transforms of the functions $\Gamma_n(x_1,\ldots x_n)$:

$$\Gamma_n(x_1,\ldots x_n)=\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\,e^{i(p_1x_1+\ldots p_nx_n)}\,(2\pi)^4\delta(p_1+\ldots+p_n)\tilde{\Gamma}_n(p_1,\ldots p_n), \tag{3.1.7}$$

and expand $\tilde{\Gamma}_n$ in powers of momenta around $p_i=0$,

$$\tilde{\Gamma}_n(p_1,\ldots,p_n)=\tilde{\Gamma}_n(0)+\ldots\,. \tag{3.1.8}$$

The effective action becomes

$$\begin{aligned}\Gamma[\phi_c] &= \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1\ldots d^4x_n\,\phi_c(x_1)\ldots\phi_c(x_n)\\ &\quad\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\,e^{i(p_1x_1+\ldots p_nx_n)}\int d^4x\,e^{-ix(p_1+\ldots+p_n)}\left[\tilde{\Gamma}_n(0)+\ldots\right]\\ &= \int d^4x\,\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\Gamma}_n(0)\phi_c^n(x)+\ldots\,.\end{aligned} \tag{3.1.9}$$

The first term in this expansion is usually written as

$$-\int d^4x\,V(\phi_c), \tag{3.1.10}$$

where

$$V(\phi_c)=-\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\Gamma}_n(0)\phi_c^n \tag{3.1.11}$$

is called the *effective potential* of the theory, since it does not contain derivatives of the classical field. The following terms, originating from higher powers of momenta in the expansion of $\tilde{\Gamma}_n$, contain instead two or more derivatives of ϕ_c. The minimum condition eq. (3.1.6) reduces to

$$\frac{\delta}{\delta\phi_c}\int d^4x\,V(\phi_c)=\frac{dV(\phi_c)}{d\phi_c}=0 \tag{3.1.12}$$

if we require translational invariance of the vacuum state.

Effective potential for a real scalar field

The effective potential can be computed directly, by taking the sum of all diagrams with an arbitrary number of external scalar lines and zero external momenta. Consider for example a theory with a single real scalar field ϕ, and a tree-level potential given by

$$V_0(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4. \tag{3.1.13}$$

The one-loop Green's functions at zero external momenta are given by

$$\tilde{\Gamma}_{2n}(0) = -i\,S_n\left(-4!\,\frac{i\lambda}{4}\right)^n \int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{k^2-m^2+i\eta}\right)^n, \tag{3.1.14}$$

while Green's functions with an odd number of external lines are obviously zero. The combinatorial factor S_n is

$$S_n = \frac{(2n)!}{2^n 2n}, \tag{3.1.15}$$

and can be determined in the following way: there are $(2n)!$ ways of assigning the external momenta to the vertices; this number must be divided by 2^n because there are two external lines for each vertex, and by $2n$ because there are $2n$ identical vertices in the diagram. The one-loop correction to the scalar potential is therefore given by

$$V_1(\phi_c) = \frac{i}{2}\sum_{n=1}^{\infty}\left(3\lambda\phi_c^2\right)^n \frac{1}{n}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\eta)^n}. \tag{3.1.16}$$

One sees immediately that the terms corresponding to $n=1$ and $n=2$ are divergent. This is no surprise: these terms are proportional to ϕ_c^2 and ϕ_c^4 respectively, and the divergences must undergo the usual procedure of mass and coupling constant renormalization. Let us first take care of the finite part. The loop integrals can be performed using eq. (4.5.2); we find

$$V_1^{\rm finite} = \frac{i}{2}\frac{i}{(4\pi)^2}\sum_{n=3}^{\infty}\left(3\lambda\phi_c^2\right)^n \frac{(-1)^n}{n}\frac{\Gamma(n-2)}{\Gamma(n)}\,m^{4-2n}, \tag{3.1.17}$$

or, using the properties of the Γ function and defining $z = 3\lambda\phi_c^2/m^2$,

$$\begin{aligned} V_1^{\rm finite} &= -\frac{m^4}{32\pi^2}\sum_{n=3}^{\infty}\frac{(-1)^n z^n}{n(n-1)(n-2)} \\ &= -\frac{m^4}{64\pi^2}\sum_{n=3}^{\infty}(-1)^n z^n\left[\frac{1}{n}-\frac{2}{n-1}+\frac{1}{n-2}\right]. \end{aligned} \tag{3.1.18}$$

It is now easy to sum the series by shifting the summation index to $n+1$ and $n+2$ in the second and third term, and by adding and subtracting the missing $n=1,2$ terms. We get

$$\begin{aligned} V_1^{\rm finite} &= \frac{m^4}{64\pi^2}\left[(1+z)^2\log(1+z) - z - \frac{3}{2}z^2\right] \\ &= \frac{1}{64\pi^2}\left[\left(m^2+3\lambda\phi_c^2\right)^2\log\frac{m^2+3\lambda\phi_c^2}{m^2} - 3\lambda\phi_c^2 m^2 - \frac{3}{2}(3\lambda\phi_c^2)^2\right]. \end{aligned} \tag{3.1.19}$$

Let us now consider the divergent part:

$$V_1^{\rm div} \frac{i}{2}\left[\left(3\lambda\phi_c^2\right)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\eta}+\frac{1}{2}\left(3\lambda\phi_c^2\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\eta)^2}\right]. \tag{3.1.20}$$

The renormalization procedure requires that a regularization prescription is given in order to give mathematical meaning to the divergent integrals. Then, one must add suitable counterterms in order to cancel the divergences; the renormalizability of the theory manifests itself in the fact that the only divergent diagrams correspond to terms which are already present in the bare lagrangian. The finite parts of the counterterms are arbitrary; different choices correspond to different renormalization schemes, and consequently to different definitions of the renormalized parameters.

We notice that the first term in eq. (3.1.20) is quadratically divergent: if we were to regularize the integrals by simply imposing an ultraviolet cut-off Λ on the modulus of the loop momentum k, we would find a term proportional to $\lambda\Lambda^2\phi_c^2$, which corresponds to a quadratically divergent radiative correction to the mass of the scalar field ϕ_c. This fact is characteristic of scalar mass parameters.

In general, after regularization, the divergent part of the one-loop potential takes the form

$$V_1^{\rm div} = A\phi_c^2 + B\phi_c^4, \tag{3.1.21}$$

where A and B are functions of λ, m and of some parameter which defines the regularization prescription; both are divergent in the physical limit, e.g. $\Lambda\to\infty$ for the cut-off regularization, or $d\to 4$ in dimensional regularization. We must give some renormalization prescription to fix the finite counterterms. For example, we could require that

$$\tilde\Gamma_2(0) = -m^2; \quad \tilde\Gamma_4(0) = -6\lambda. \tag{3.1.22}$$

Since eqs. (3.1.22) hold for the tree-level potential, and since the finite part of the one-loop corrections starts with ϕ_c^6, this prescription simply means that the counterterms must be exactly equal and opposite to the divergent part, namely

$$V_1^{\rm ct} = -A\phi_c^2 - B\phi_c^4, \tag{3.1.23}$$

so that in this case

$$V_1 = V_1^{\rm finite}. \tag{3.1.24}$$

Another possibility is to perform the so-called minimal subtraction (MS). This prescription amounts to computing the divergent part in dimensional regularization, and then fixing the counterterms in such a way that only the pole in $d-4$ is subtracted. A modified version of this renormalization prescription ($\overline{\rm MS}$) consists in subtracting the term proportional to

$$\frac{1}{\epsilon} - \gamma + \log(4\pi), \tag{3.1.25}$$

where the space-time dimension is $d = 4-2\epsilon$. In this case, we have to compute explicitly the loop integrals in eq. (3.1.20). Using again eq. (4.5.2), we find

$$V_1^{\rm div} = -\frac{1}{64\pi^2}\left[6\lambda\phi_c^2 m^2 + 6\lambda\phi_c^2\left(m^2+\frac{3}{2}\lambda\phi_c^2\right)\left(\frac{1}{\epsilon}-\gamma+\log(4\pi)+\log\frac{\mu^2}{m^2}\right)\right], \tag{3.1.26}$$

where μ is an arbitrary mass parameter which must be introduced in dimensional regularization in order to keep the coupling constant λ dimensionless. Now, we simply subtract the term proportional to $1/\epsilon - \gamma + \log(4\pi)$. Adding all together, we find

$$V_1^{\overline{\rm MS}} = \frac{1}{64\pi^2}\left(m^2 + 3\lambda\phi_c^2\right)^2\left[\log\frac{m^2 + 3\lambda\phi_c^2}{\mu^2} - \frac{3}{2}\right], \tag{3.1.27}$$

where we have used the identity

$$6\lambda\phi_c^2\left(m^2 + \frac{3}{2}\lambda\phi_c^2\right) = \left(m^2 + 3\lambda\phi_c^2\right)^2 - m^4 \tag{3.1.28}$$

and we have dropped constant terms.

In more complicated theories, like the standard model, the effective potential receives contributions also from fermion and vector loops. These contributions can be computed in the same way as the scalar one, but the calculations are quite tedious and complicated. Fortunately, there is a much cleverer technique, which allows one to obtain all contributions to the one-loop scalar potential in a very simple way. Consider a new theory, obtained from the original one by shifting the scalar field by an arbitrary quantity ω:

$$\phi \to \phi + \omega. \tag{3.1.29}$$

The corresponding effective potential is

$$V'(\phi_c) = -\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\Gamma}_n(0)\,(\phi_c + \omega)^n = -\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\Gamma}'_n(\omega, 0)\,\phi_c^n, \tag{3.1.30}$$

where the Green's functions $\tilde{\Gamma}'_n$ can be computed in terms of $\tilde{\Gamma}_n$. From eq. (3.1.30) we find

$$\tilde{\Gamma}'_1(\omega, 0) = \sum_{n=1}^{\infty}\frac{1}{n!}\tilde{\Gamma}_n(0)\,n\omega^{n-1} \tag{3.1.31}$$

and therefore

$$\int_0^{\phi_c} d\omega\,\tilde{\Gamma}'_1(\omega, 0) = \sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\Gamma}_n(0)\,\phi_c^n = -V(\phi_c). \tag{3.1.32}$$

Equation (3.1.32) tells us that the effective potential of the original theory can be obtained by computing the one-particle (or *tadpole*) amplitude of the shifted theory and integrating it with respect to the shift. Let us see explicitly how this works. The tree-level potential of the shifted theory is

$$V'_0(\phi) = \frac{1}{2}m^2(\phi + \omega)^2 + \frac{1}{4}\lambda(\phi + \omega)^4. \tag{3.1.33}$$

The tree-level tadpole is therefore

$$-m^2\omega - \lambda\omega^3, \tag{3.1.34}$$

which, integrated in ω between 0 and ϕ_c gives minus the tree-level potential (3.1.13) as expected. We now turn to the one-loop term. There is only one diagram to be computed, with one external line and one internal propagator. In the shifted theory, the mass of the ϕ_c field is $m^2 + 3\lambda\omega^2$, and the ϕ_c^3 vertex is $-3\lambda\omega$ (the factor 3 is due to the fact that the three lines are identical), and therefore

$$\tilde{\Gamma}_1'(\omega, 0) = -3\lambda\omega \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2 - 3\lambda\omega^2}. \tag{3.1.35}$$

Using the results of appendix 4.5 we readily find

$$\begin{aligned} \tilde{\Gamma}_1'(\omega, 0) &= -3\lambda\omega \frac{(4\pi)^\epsilon}{(4\pi)^2} \Gamma(-1+\epsilon)(m^2 + 3\lambda\omega^2)^{1-\epsilon} \\ &= \frac{3\lambda\omega}{(4\pi)^2}(m^2 + 3\lambda\omega^2)\left[\frac{1}{\epsilon} - \gamma + \log(4\pi) - \log\frac{m^2 + 3\lambda\omega^2}{\mu^2} + 1\right] + \mathcal{O}(\epsilon), \end{aligned} \tag{3.1.36}$$

where μ is the renormalization scale introduced by dimensional regularization. After performing the $\overline{\text{MS}}$ subtraction, we find

$$\begin{aligned} V_1(\phi_c) &= \frac{1}{(4\pi)^2} \int_0^{\phi_c} d\omega\, 3\lambda\omega(m^2 + 3\lambda\omega^2)\left(\log\frac{m^2 + 3\lambda\omega^2}{\mu^2} - 1\right) \\ &= \frac{1}{64\pi^2}(m^2 + 3\lambda\phi_c^2)^2 \left(\log\frac{m^2 + 3\lambda\phi_c^2}{\mu^2} - \frac{3}{2}\right). \end{aligned} \tag{3.1.37}$$

which is the same result obtained with the direct calculation, eq. (3.1.27).

The effective potential in the standard model

The procedure outlined at the end of the previous subsection can be applied to the standard model. The scalar field is now a complex doublet, which we write in terms of four real scalar fields ϕ_i:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}. \tag{3.1.38}$$

In the standard model, the effective potential receives contributions from the scalar sector, the vector boson sector, the Faddeev-Popov ghost sector and the fermion sector:

$$V_1(\phi) = V_S(\phi) + V_V(\phi) + V_g(\phi) + V_F(\phi) \tag{3.1.39}$$

(we drop the suffix c on from now on).

The effective potential is a gauge-dependent quantity. It can be shown that the gauge dependence of the effective potential is governed by the equation

$$\left[\xi\frac{\partial}{\partial\xi} + C(\phi,\xi)\frac{\partial}{\partial\phi}\right] V(\phi,\xi) = 0\ , \tag{3.1.40}$$

where ξ is the gauge parameter and $C(\phi,\xi)$ is a function which can be computed order by order in perturbation theory. Equation (3.1.40), in particular, tells us that V is gauge-independent at its minimum, where $\partial V/\partial\phi = 0$. We will compute $V(\phi)$ in the Landau gauge $\xi = 0$; in this case, the ghost contribution $V_g(\phi)$ vanishes.

We begin by computing the scalar contribution. After the shift $\phi_i \to \phi_i + \omega_i$, the tree-level potential

$$V_0(\phi) = m^2\,|\phi|^2 + \lambda\,|\phi|^4 \tag{3.1.41}$$

becomes

$$\begin{aligned} V_0'(\phi) &= \phi_i\omega_i(m^2+\lambda\omega^2) + \frac{1}{2}\left[(m^2+\lambda\omega^2)\delta_{ij} + 2\lambda\omega_i\omega_j\right]\phi_i\phi_j \\ &\quad +\lambda\omega_i\phi_i\phi_j\phi_j + \frac{1}{4}\lambda(\phi_i\phi_i)^2, \end{aligned} \tag{3.1.42}$$

where $\omega^2 = \omega_i\omega_i$. It is immediate to check that integrating the tree-level tadpole with respect to ω_i and summing over the index i gives back the tree-level potential. The one-loop contribution is obtained in the same way as in the case of the real scalar field, that is by computation of the one-loop tadpole diagram. A complication arises here, due to the fact that the mass term in eq. (3.1.42) is not diagonal. A simple way to circumvent this difficulty is to choose $\omega_i = 0$ for all i except one of them, say $\omega_3 = \omega$ (the reason of this choice will become clear later; of course, it does not affect the final result). This choice simplifies considerably the calculation, since now eq. (3.1.42) describes three real scalars, ϕ_1, ϕ_2 and ϕ_4, with mass $m^2+\lambda\omega^2$, and one real scalar, ϕ_3, with mass $m^2+3\lambda\omega^2$. The trilinear couplings $\phi_3\phi_j\phi_j$ are simply $-\lambda\omega$ for $i \neq 3$ and $-3\lambda\omega$ for $j = 3$. The calculation is now exactly analogous to that of a single scalar field, except that all four contributions must be taken into account. The result is therefore

$$\begin{aligned} V_S(\phi) &= \frac{1}{64\pi^2}(m^2+3\lambda\phi^2)^2\left[\log\frac{m^2+3\lambda\phi^2}{\mu^2} - \frac{3}{2}\right] \\ &\quad +\frac{3}{64\pi^2}(m^2+\lambda\phi^2)^2\left[\log\frac{m^2+\lambda\phi^2}{\mu^2} - \frac{3}{2}\right], \end{aligned} \tag{3.1.43}$$

where $\phi^2 = \phi_i\phi_i$. Some comments are in order. First of all, we observe that the same result could have been obtained without any specific assumption about the shift variables ω_i. Secondly, we stress the fact that the result in eq. (3.1.43) (as well as all the other contributions, to be computed below) is independent of the values of m^2 and λ. More specifically, this result holds in both the $m^2 > 0$ and $m^2 < 0$ cases. In the first case, there is no spontaneous breaking of the gauge symmetry, the vacuum expectation values of the fields ϕ_i are all zero, and the scalar masses are all equal to m^2. In the $m^2 < 0$ case, the minimum of the tree-level potential lies at $\phi^2 = v^2$, and eq. (3.1.43) is easily intrrerpreted: there is a contribution coming from the physical Higgs boson, with mass $m^2+3\lambda v^2$, and a contribution from the three would-be Goldstone bosons, whose masses vanish at the minimum of the tree-level potential. In both cases, the one-loop effective potential has the same form. Note that the masses of the unphysical scalars vanish because we are working in the Landau gauge.

We now turn to the contribution of vector bosons, $V_V(\phi)$. The only term of the lagrangian we need is the scalar-scalar-vector-vector term that appears in the squared covariant derivative of the Higgs doublet. In fact, after shifting the fields ϕ_i, this term contains both the mass terms for the vector bosons and the scalar-vector-vector vertices needed to compute the one-loop tadpole. With the help of the results in 4.4 we find that the relevant term in the shifted lagrangian is

$$\mathcal{L} = (\omega_i\omega_i + 2\phi_i\omega_i)\left[\frac{1}{4}g^2 W^{+\mu}W^-_\mu + \frac{1}{8}(g^2+g'^2)Z^\mu Z_\mu\right], \tag{3.1.44}$$

where again we have chosen $\omega_i = 0$ for $i \neq 3$ and $\omega_3 = \omega$. Therefore, the one-loop tadpole receives one contribution from a loop of a W vector boson with mass $g^2\omega^2/4$ and couplings $g^2\omega_i g_{\mu\nu}/2$ to the scalar fields ϕ_i, and a contribution from the Z boson with mass $(g^2+g'^2)\omega^2/4$ and couplings $(g^2+g'^2)\omega_i g_{\mu\nu}/4$. The corresponding contributions to the effective potential are easily computed with the help of eq. (3.1.36), recalling that a factor $g_{\mu\nu}(-g^{\mu\nu} + k^\mu k^\nu/k^2) = -3+2\epsilon$ must now be included because of the form of the vector boson propagators in the Landau gauge. The final result is

$$\begin{aligned} V_V(\phi) &= \frac{3}{64\pi^2}\left[\frac{1}{4}(g^2+g'^2)\phi^2\right]^2\left[\log\frac{(g^2+g'^2)\phi^2/4}{\mu^2} - \frac{5}{6}\right]. \\ &+\frac{6}{64\pi^2}\left(\frac{1}{4}g^2\phi^2\right)^2\left[\log\frac{g^2\phi^2/4}{\mu^2} - \frac{5}{6}\right]. \end{aligned} \tag{3.1.45}$$

Finally, we must consider the contribution of fermions. For simplicity, we consider only the contribution of the top quark, since all other Yukawa couplings in the standard model are negligibly small. With the choice of ω adopted above, the relevant piece of the shifted lagrangian is

$$\mathcal{L} = -\frac{h_t}{\sqrt{2}}(\phi_3+\omega)\bar{t}t, \tag{3.1.46}$$

and proceeding as above we find

$$V_F(\phi) = -\frac{12}{64\pi^2}\left(\frac{1}{2}h_t^2\phi^2\right)^2\left[\log\frac{h_t^2\phi^2/2}{\mu^2} - \frac{3}{2}\right], \tag{3.1.47}$$

where we have included a factor of three for the colour quantum number, and a minus sign because of the fermion loop.

To summarize our results, we have computed the one-loop effective potential of the standard model in the $\overline{\mathrm{MS}}$ subtraction scheme. The result is

$$\begin{aligned} V(\phi) &= \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda(\phi^2)^2 \\ &+\frac{1}{64\pi^2}\left[H^2\left(\log\frac{H}{\mu^2} - \frac{3}{2}\right) + 3G^2\left(\log\frac{G}{\mu^2} - \frac{3}{2}\right)\right. \\ &\left.+6W^2\left(\log\frac{W}{\mu^2} - \frac{5}{6}\right) + 3Z^2\left(\log\frac{Z}{\mu^2} - \frac{5}{6}\right) - 12T^2\left(\log\frac{T}{\mu^2} - \frac{3}{2}\right)\right], \end{aligned} \tag{3.1.48}$$

where

$$H = m^2 + 3\lambda\phi^2; \quad G = m^2 + \lambda\phi^2; \quad W = \frac{1}{4}g^2\phi^2; \quad Z = \frac{1}{4}(g^2+g'^2)\phi^2; \quad T = \frac{1}{2}h_t^2\phi^2. \tag{3.1.49}$$

The quantities defined in eq. (3.1.49) are usually called the field dependent squared masses of the theory; there is one such function for each particle in the spectrum, and its value at $\phi^2 = v^2$ equals the squared mass of the corresponding particle. We may denote these functions collectively with the symbol

$$\mathcal{M}_i^2(\phi^2) \tag{3.1.50}$$

with the index i running over all particles in the theory, and rewrite the one-loop correction to the scalar potential as

$$V_1(\phi) = \frac{1}{64\pi^2} \sum_i (-1)^{2s_i} (2s_i + 1) \mathcal{M}_i^4(\phi^2) \left[\log \frac{\mathcal{M}_i^2(\phi^2)}{\mu^2} - c_i \right], \tag{3.1.51}$$

where s_i is the spin of particle i, $c_i = 3/2$ for scalars and fermions, and $c_i = 5/6$ for vectors.

A number of interesting things can be done with the one-loop effective potential (the original work of S. Coleman and E. Weinberg is particularly instructive). We will concentrate on some of them. Let us consider for example the dependence on the renormalization scale μ. From eq. (3.1.11), we have

$$\frac{dV(\phi)}{dt} = 0, \tag{3.1.52}$$

where $t = \log \mu^2$. In fact, the one-particle irreducible Green's functions obey the Callan-Symanzik equations

$$\left(\frac{\partial}{\partial t} + \beta_\lambda \frac{\partial}{\partial \lambda} + m^2 \gamma_m \frac{\partial}{\partial m^2} + n\gamma \right) \tilde{\Gamma}_n = 0, \tag{3.1.53}$$

where

$$\frac{d\lambda}{dt} = \beta_\lambda, \tag{3.1.54}$$

$$\frac{dm^2}{dt} = \gamma_m m^2, \tag{3.1.55}$$

$$\frac{d\phi^2}{dt} = 2\gamma\phi^2, \tag{3.1.56}$$

and β_λ, γ_m and γ are functions of the coupling constants, and are computable in perturbation theory. Using eqs. (3.1.53) in eq. (3.1.11), eq. (3.1.52) is immediately obtained.

On the other hand, dV/dt can be computed explicitly by differentiating eq. (3.1.48) with respect to $\log \mu^2$ and neglecting two-loop effects. We find

$$\begin{aligned} \frac{dV(\phi)}{dt} &= \frac{1}{4}\phi^4 \left\{ \beta_\lambda + 4\lambda\gamma - \frac{1}{16\pi^2} \left[12\lambda^2 + \frac{3}{8} g^4 + \frac{3}{16} (g^2 + g'^2)^2 - 3h_t^4 \right] \right\} \\ &\quad + \frac{1}{2} m^2 \phi^2 \left[\gamma_m + 2\gamma - \frac{12\lambda}{32\pi^2} \right], \end{aligned} \tag{3.1.57}$$

and therefore

$$\beta_\lambda + 4\lambda\gamma = \frac{1}{16\pi^2} \left[12\lambda^2 + \frac{3}{8} g^4 + \frac{3}{16} (g^2 + g'^2)^2 - 3h_t^4 \right] \tag{3.1.58}$$

$$\gamma_m + 2\gamma = \frac{12\lambda}{32\pi^2}. \tag{3.1.59}$$

Observe that eqs. (3.1.58,3.1.59) are not quite enough to compute all the anomalous dimensions of the scalar sector, but almost so: infact, it is sufficient to compute explicitly one of them, for example γ, to obtain the others.

We will now study the behaviour of the effective potential for large values of the classical fields ϕ_i. We will be interested in discovering under which conditions $V(\phi) \to +\infty$ for large ϕ^2, a necessary condition for the existence of a minimum of $V(\phi)$ for finite ϕ^2. We therefore assume that $\phi^2 \sim \Lambda^2$, where Λ is some energy scale much larger than the electroweak scale. Under this assumptions, the effective potential is approximately given by

$$\begin{aligned} V(\phi) \;&\simeq\; \frac{1}{4}\phi^4 \left\{ \lambda + \frac{1}{16\pi^2}\left[12\lambda^2 + \frac{3}{8}g^4 + \frac{3}{16}(g^2+g'^2)^2 - 3h_t^4\right]\log\frac{\Lambda^2}{\mu^2}\right\} \\ &+\frac{1}{2}m^2\phi^2\left[1+\frac{12\lambda}{32\pi^2}\log\frac{\Lambda^2}{\mu^2}\right], \end{aligned} \tag{3.1.60}$$

or, using eqs. (3.1.58,3.1.59),

$$V(\phi) \simeq \frac{1}{4}\phi^4\left[\lambda + (\beta_\lambda + 4\lambda\gamma)\log\frac{\Lambda^2}{\mu^2}\right] + \frac{1}{2}m^2\phi^2\left[1+(\gamma_m+2\gamma)\log\frac{\Lambda^2}{\mu^2}\right]. \tag{3.1.61}$$

We now observe that the renormalization group equations (3.1.54-3.1.56) have the approximate solutions

$$\lambda(\Lambda) \simeq \lambda + \beta_\lambda \log\frac{\Lambda^2}{\mu^2} \tag{3.1.62}$$

$$m^2(\Lambda) \simeq m^2\left(1+\gamma_m\log\frac{\Lambda^2}{\mu^2}\right) \tag{3.1.63}$$

$$\phi^2(\Lambda) \simeq \phi^2\left(1+2\gamma\log\frac{\Lambda^2}{\mu^2}\right), \tag{3.1.64}$$

with $\lambda = \lambda(\mu)$, $m^2 = m^2(\mu)$, $\phi^2 = \phi^2(\mu)$. It is now immediate to show that eq. (3.1.61) is just the expansion of the renormalization group improved effective potential

$$V_{RG}(\phi) = \frac{1}{2}m^2(\Lambda)\phi^2(\Lambda) + \frac{1}{4}\lambda(\Lambda)\phi^4(\Lambda). \tag{3.1.65}$$

We see that the stability condition for the potential is simply the positivity of the running coupling constant $\lambda(\Lambda)$ at large scales.

The stability condition can be translated into a lower limit for the Higgs boson mass. To see this, we need the explicit form of the one-loop renormalization group equation for $\lambda(\mu)$:

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\left[12\lambda^2 + \frac{3}{8}g^4 + \frac{3}{16}(g^2+g'^2)^2 - 3h_t^4 - 3\lambda g^2 - \frac{3}{2}\lambda(g^2+g'^2) + 6\lambda h_t^2\right]. \tag{3.1.66}$$

This equation must be solved together with the one-loop renormalization group equations for gauge and Yukawa coupling constants, which in the standard model are given by

$$\frac{dg}{dt} = \frac{1}{32\pi^2}\left(-\frac{19}{6}g^3\right) \tag{3.1.67}$$

$$\frac{dg'}{dt} = \frac{1}{32\pi^2}\frac{41}{6}g'^3 \tag{3.1.68}$$
$$\frac{dg_S}{dt} = \frac{1}{32\pi^2}(-7g_S^3) \tag{3.1.69}$$
$$\frac{dh_t}{dt} = \frac{1}{32\pi^2}\left[\frac{9}{2}h_t^3 - \left(8g_S^2 + \frac{9}{4}g^2 + \frac{17}{12}g'^2\right)h_t\right], \tag{3.1.70}$$

where g_S is the strong interaction coupling constant, and the $\overline{\mathrm{MS}}$ scheme is adopted. This system of coupled first-order differential equations can be easily solved numerically. The result for $\lambda(\mu)$ is shown in fig. 3.1 for different values of the initial condition $\lambda(\mu = m_z)$. Namely, we have chosen $\lambda(m_z)$ corresponding to $m_H = 60, 100, 130, 150, 190$ and 210 GeV, where

$$m_H^2 \simeq 2\lambda(m_z)v^2 . \tag{3.1.71}$$

The interpretation of fig. 3.1 in connection with the problem of the stability of the effective

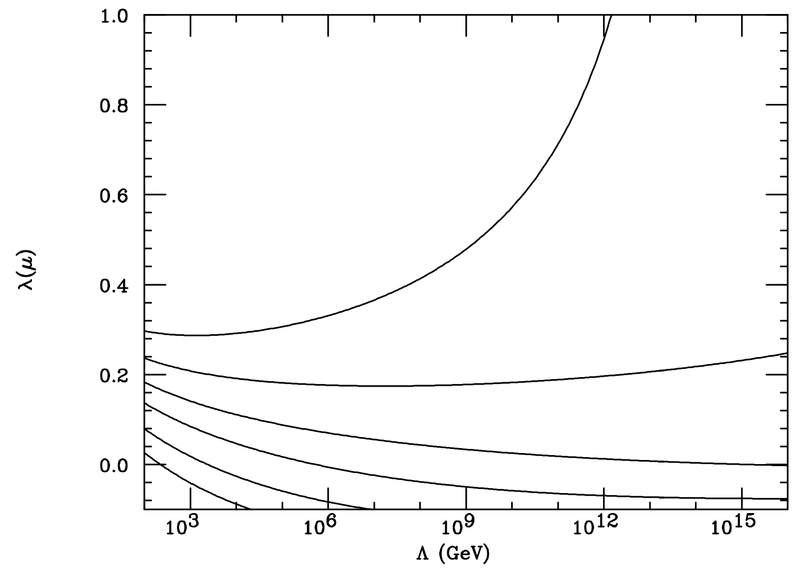

Figure 3.1: *The running coupling constant $\lambda(\mu)$ for different values of $\lambda(m_z)$, as explained in the text.*

potential is as follows. We see that if the initial condition at $\mu = v$ is small, then $\lambda(\mu)$ becomes negative for some value of the renormalization scale. Conversely, the requirement that $\lambda(\mu)$ stay positive at least up to a given value of μ, $\mu \sim \Lambda$, translates into a lower limit on $\lambda(v)$, or equivalently on m_H. This lower bound depends on Λ; we see for example that if we ask $\lambda(\mu) > 0$ up to the grand unification scale, $\sim 10^{16}$ GeV, the Higgs boson mass cannot go below ~ 150 GeV (fig. 3.1 is obtained for $m_t = 175$ GeV). This lower limit becomes less stringent if we require $\lambda(\mu) > 0$ in a smaller range of μ.

There is another lesson to be learned from fig. 3.1. We observe that, for large values of the Higgs boson mass, the coupling constant λ grows with increasing μ, and eventually leaves the perturbative domain, $\lambda < 1$. This is because the solution of the renormalization group equation for λ has a singularity in μ, known as the Landau singularity. Also in this case, for the theory to make sense up to a given scale Λ, we must ask $\lambda(\mu) < 1$ (or something like that) for $\mu \leq \Lambda$. This in turns implies an upper bound on the Higgs boson mass, which is approximately 180 GeV for $\Lambda \sim 10^{16}$ GeV and $m_t = 175$ GeV.

The upper limit on the standard model Higgs boson mass is often referred to as the *triviality limit*. The reason for this is that the existence of a Landau singularity in the running coupling constant λ would imply $\lambda(v) = 0$ if we require that the theory be valid for all values of the scale μ, that is, the theory would be non-interacting, or *trivial*, in the scalar sector. Therefore, we are forced to require the consistency of the theory only up to some finite value of μ, and to assume that some new phenomena become relevant at higher energy scales. Notice however that no rigorous proof of the triviality of the standard model has been given so far; there are only some indications of this, coming from studies and lattice simulations of simplified theories.

Both the triviality upper bound and the stability lower bound on the Higgs mass are shown in fig. 3.2, as functions of Λ. As Λ increases, the allowed range for m_H becomes narrower. Recent

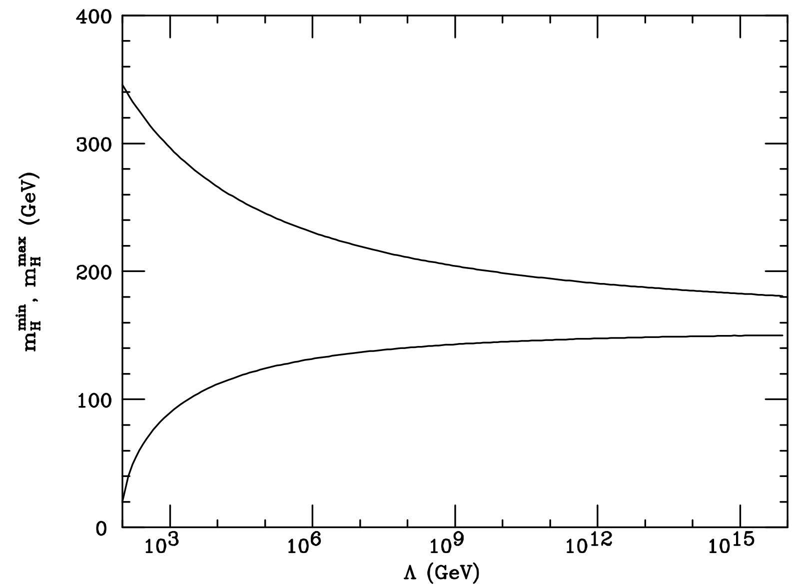

Figure 3.2: *Theoretical upper and lower bounds on the Higgs mass.*

LEP and SLD precision data allow to estimate, although with a large uncertainty, the value of the standard model Higgs mass, that affects various observables (like the W boson mass, or forward-backward asymmetries) through radiative corrections. The central values of these fits are between 100 and 200 GeV. It is interesting to notice that a value of m_H in this range is compatible with Λ close to the unification scale, $\sim 10^{16}$ GeV.

3.2 The $SU(2)$ custodial symmetry

We have seen in section 3 that in the standard model at tree level the weak vector boson masses m_W and m_Z are related by

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1. \tag{3.2.1}$$

Equation (3.2.1) could in principle be modified at higher orders in perturbation theory. Actually, the measured value of ρ is very close to 1:

$$\rho_{\rm exp} = 1.0048 \pm 0.0022, \tag{3.2.2}$$

thus suggesting that some symmetry property prevents the quantity ρ from receiving large radiative corrections. We will now show that this is indeed the case.

Preliminarly, we observe that, even after the inclusion of radiative corrections, the most general vector boson mass term is given by

$$\mathcal{L}_{\rm mass} = \frac{1}{2} m_W^2 (W^{1\mu} W^1_\mu + W^{2\mu} W^2_\mu) + \frac{1}{2} (W_3^\mu \;\; B^\mu) \begin{bmatrix} M^2 & M'^2 \\ M'^2 & M''^2 \end{bmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix}. \tag{3.2.3}$$

Furthermore, the condition that the photon stays massless gives us $M'^2 = MM''$, and $M^2 + M''^2 = m_Z^2$. Therefore, the mass matrix in the neutral sector is completely fixed by the value of one parameter, say M^2, and it is diagonalized by a rotation of an angle θ_W given by

$$\tan\theta_W = \frac{\sqrt{m_Z^2 - M^2}}{M}. \tag{3.2.4}$$

This in turn implies that

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = \frac{m_W^2}{M^2}, \tag{3.2.5}$$

that is, $\rho = 1$ only if $M^2 = m_W^2$.

Next we notice that the scalar potential

$$V(\phi) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4 \tag{3.2.6}$$

is invariant under a group of transformations which is larger than the standard model $SU(2)_L \times U(1)_Y$. In fact, defining

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{3.2.7}$$

we see that

$$\mid \phi \mid^2 = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) \tag{3.2.8}$$

can be interpreted as the squared length of a real four vector. Therefore, the scalar potential has an $O(4) \sim SU(2) \times SU(2)$ invariance. This symmetry property can be implemented in the following way. We define a 2×2 matrix

$$H = \begin{bmatrix} \phi^+ & \phi^{0*} \\ \phi^0 & -\phi^- \end{bmatrix}. \tag{3.2.9}$$

Recalling that the field $\phi_c = (\phi^{0*}, -\phi^-)^T$ transforms as an $SU(2)$ doublet, it follows that, under the action of a generic $SU(2)_L$ transformation U, we have

$$H \to UH. \tag{3.2.10}$$

On the other hand, it is easy to check that the scalar potential can be written in terms of H as

$$V(\phi) = \frac{1}{2} m^2 \, \mathrm{Tr}\left(H^\dagger H\right) + \frac{1}{2} \lambda \, \mathrm{Tr}\left(H^\dagger H\right)^2, \tag{3.2.11}$$

which is invariant under the $SU(2)_L \times SU(2)$ transformation

$$H \to UHV^\dagger, \tag{3.2.12}$$

where V is a second $SU(2)$ constant matrix, independent of U. This is possible because the structure of H in eq. (3.2.9) is preserved also by right multiplication with an $SU(2)$ matrix. Equation (3.2.12) is a representation of the $O(4)$ symmetry we mentioned above. Is it possible to write also the kinetic term for the field ϕ in an $O(4)$-invariant way? The natural candidate is of course

$$\frac{1}{2} \, \mathrm{Tr}\left(D_\mu H\right)^\dagger D^\mu H, \tag{3.2.13}$$

which is invariant under the transformations (3.2.12) since $D^\mu \to U D^\mu U^\dagger$. However, one readily realizes that (3.2.13) is not equal to $(D_\mu \phi)^\dagger D^\mu \phi$ (prove this statement as an exercise); this is because ϕ and ϕ_c have opposite values of the hypercharge quantum number. We conclude that the $O(4)$ symmetry is violated by the hypercharge interaction term contained in the covariant derivative. Let us therefore neglect for the moment the hypercharge factor of the gauge group, which amounts to setting $g' = 0$, in order to work with an $O(4)$-invariant theory.

Due to spontaneous breaking of $SU(2)_L$, the ground state is not invariant under $O(4)$; however, there is a residual $O(3) \sim SU(2)$ symmetry under transformations of the kind

$$H \to UH(\tau_1 U^\dagger \tau_1), \tag{3.2.14}$$

that leave the vacuum expectation value $< H >= \sqrt{2} v \tau_1$ unchanged (U is now x-independent). We are almost at the end of the road: in fact, it is easy to check that the only mass term for the W^i_μ fields allowed by the symmetry in eq. (3.2.14) is of the form $W^i_\mu W^\mu_i$, that is, a scalar product in $O(3)$. In other words, $M^2 = m^2_W$ in the notation of eq. (3.2.3).

We have proven that $\rho = 1$ is a consequence of the so-called *custodial* $SU(2)$ symmetry defined in eq. (3.2.14), and therefore it is not spoiled by radiative corrections. The inclusion of the hypercharge interaction, that breaks $O(4)$ explicitly, does not change this conclusion, since radiative corrections to ρ due to the hypercharge coupling are very small.

Of course, fermion mass terms do not preserve the custodial symmetry; we expect corrections to eq. (3.2.1) of the order of $G_\mu m^2_f$. More precisely, one finds

$$\rho \simeq 1 + \frac{3 G_\mu m_t^2}{8 \pi^2 \sqrt{2}}, \tag{3.2.15}$$

where we have included only the contribution from the top quark, for obvious reasons.

3.3 Axial anomaly cancellation

We have seen in the previous sections that the renormalizability of the standard model is strictly connected with gauge invariance. In particular, we have seen that the massive vector boson propagators show unphysical singularities, that are cancelled by the presence of would-be Goldstone bosons. In turn, gauge invariance manifests itself in the form of identities between Green functions, called Slavnov-Taylor identities, that are consequences of current conservation, and that must hold at all perturbative orders for the theory to be renormalizable. In this section, we will show that this might not be the case for theories with axial currents, as the standard model itself. It might happen that current conservation is spoiled at the quantum level, unless the spectrum of the theory satisfies particular conditions. In the language of quantum field theory, terms that spoil the validity of Slavnov-Taylor identities are called *anomalies*. We will illustrate the problem of anomalies in the context of a simple example, and we will then state under which conditions the standard model is anomaly-free and renormalizable.

We consider quantum electrodynamics with one massive fermion, ψ with electric charge e and mass m. We consider the operators

$$J_V^\mu = \overline{\psi}\gamma^\mu\psi \tag{3.3.1}$$
$$J_A^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \tag{3.3.2}$$
$$J_P = \overline{\psi}\gamma_5\psi. \tag{3.3.3}$$

It is easy to show, using the equations of motion, that

$$\partial_\mu J_V^\mu = 0 \tag{3.3.4}$$
$$\partial_\mu J_A^\mu = 2imJ_P\,. \tag{3.3.5}$$

The interpretation of eqs. (3.3.4) and (3.3.5) is well known. Equation (3.3.4) is simply the conservation of the electromagnetic current, which reflects the gauge-invariance of the theory. The current J_A^μ, on the other hand, is associated with axial transformations,

$$\psi \to e^{i\alpha\gamma_5}\psi. \tag{3.3.6}$$

The lagrangian of massive QED is not invariant under axial transformations because of the presence of the mass term, and as a consequence the associated current J_A^μ is not conserved. Equation (3.3.5) precisely states this fact. Exact axial-current conservation is obviously recovered in the $m \to 0$ limit.

Now consider the Green function

$$T^{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1 d^4x_2 e^{ik_1x_1+ik_2x_2}\langle 0|T[J_V^\mu(x_1)J_V^\nu(x_2)J_A^\rho(0)]|0\rangle\,, \tag{3.3.7}$$

which can be easily shown to be related to the matrix element of the axial current between the vacuum state and a two-photon state by the relation

$$\langle\gamma(k_1,\epsilon_1)\gamma(k_1,\epsilon_1)|J_A^\rho(0)|0\rangle = 2ie^2\,(\epsilon_1^*)^\mu\,(\epsilon_2^*)^\nu\,T_{\mu\nu\rho}(k_1,k_2)\,. \tag{3.3.8}$$

Formally, it obeys the Slavnov-Taylor identities

$$k_1^\mu T_{\mu\nu\rho} = k_2^\nu T_{\mu\nu\rho} = 0 \tag{3.3.9}$$

$$q^\rho T_{\mu\nu\rho} = 2m T_{\mu\nu}, \tag{3.3.10}$$

where $q = k_1 + k_2$ and

$$T^{\mu\nu}(k_1,k_2) = i\int d^4x_1 d^4x_2 e^{ik_1x_1+ik_2x_2}\langle 0|T[J_V^\mu(x_1)J_V^\nu(x_2)J_P(0)]|0\rangle\,. \tag{3.3.11}$$

The identities in eqs. (3.3.9,3.3.10) can be worked out by exploiting eqs. (3.3.4) and (3.3.5), and the canonical commutation relations. We will now check explicitly whether eqs. (3.3.9,3.3.10) are satisfied in perturbation theory or not. At the one-loop order, the diagrams to be computed are those of fig. 3.3. We have

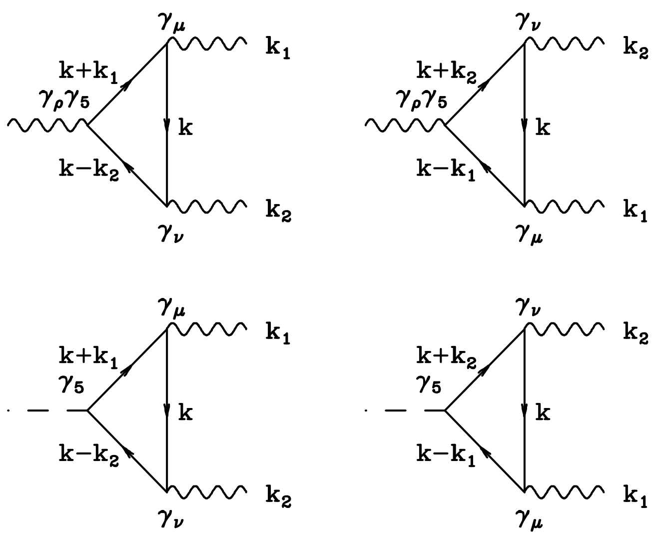

Figure 3.3: *Diagrams contributing to* $T^{\mu\nu\rho}(k_1,k_2)$ *and* $T^{\mu\nu}(k_1,k_2)$.

$$T^{\mu\nu\rho}(k_1,k_2) = T_1^{\mu\nu\rho}(k_1,k_2) + T_2^{\mu\nu\rho}(k_1,k_2) \tag{3.3.12}$$

$$T^{\mu\nu}(k_1,k_2) = T_1^{\mu\nu}(k_1,k_2) + T_2^{\mu\nu}(k_1,k_2), \tag{3.3.13}$$

where

$$T_1^{\mu\nu\rho} = -i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not k+\not k_1-m}\gamma^\rho\gamma_5\frac{i}{\not k-\not k_2-m}\gamma^\nu\frac{i}{\not k-m}\gamma^\mu\right] \tag{3.3.14}$$

$$T_1^{\mu\nu} = -i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not k+\not k_1-m}\gamma_5\frac{i}{\not k-\not k_2-m}\gamma^\nu\frac{i}{\not k-m}\gamma^\mu\right] \tag{3.3.15}$$

and

$$T_2^{\mu\nu\rho}(k_1,k_2) = T_1^{\nu\mu\rho}(k_2,k_1) \tag{3.3.16}$$

$$T_2^{\mu\nu}(k_1,k_2) = T_1^{\nu\mu}(k_2,k_1). \tag{3.3.17}$$

The overall minus sign is due to the presence of a fermion loop.

The loop integrals in eqs. (3.3.14) and (3.3.15) are superficially divergent. We must therefore choose a regularization scheme before proceeding. Dimensional regularization is not suited here, because of the presence of γ_5, which has an intrinsically four-dimensional meaning and cannot be generalized to other space-time dimensions in a simple way. We will make a different choice, keeping in mind, however, that it is possible, although quite complicated, to treat this problem within dimensional regularization. The regularization scheme we choose is the following. We subtract from each integrand in eqs. (3.3.14) and (3.3.15) the same expression, but with m replaced by an arbitrary regularization parameter M. In the limit $M \to \infty$ the original expression is recovered, while, for finite M, the integrals are now convergent. We will indicate with a subscript M the regularized quantities.

Equations (3.3.9), that state the conservation of the vector current, are satisfied by $T^{\mu\nu\rho}$ as given in eqs. (3.3.12) and (3.3.14). In fact, writing

$$\not{k}_1 = (\not{k} + \not{k}_1 - m) - (\not{k} - m) \tag{3.3.18}$$

in $T_1^{\mu\nu\rho}$, and

$$\not{k}_1 = (\not{k} - m) - (\not{k} - \not{k}_1 - m) \tag{3.3.19}$$

in $T_2^{\mu\nu\rho}$ (and similarly in the regularizing part of the integrands), we find

$$\begin{aligned}
[k_1^\mu T_{\mu\nu\rho}]_M &= -i \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \left[\frac{i}{\not{k} + \not{k}_1 - m} \gamma_\rho \gamma_5 \frac{i}{\not{k} - \not{k}_2 - m} \gamma_\nu \frac{i}{\not{k} - m} \not{k}_1 \right. \\
&\quad \left. + \frac{i}{\not{k} + \not{k}_2 - m} \gamma_\rho \gamma_5 \frac{i}{\not{k} - \not{k}_1 - m} \not{k}_1 \frac{i}{\not{k} - m} \gamma_\nu - (m \to M) \right] \\
&= \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \left[\gamma_\rho \gamma_5 \frac{i}{\not{k} - \not{k}_2 - m} \gamma_\nu \frac{i}{\not{k} - m} - \frac{i}{\not{k} + \not{k}_1 - m} \gamma_\rho \gamma_5 \frac{i}{\not{k} - \not{k}_2 - m} \gamma_\nu \right. \\
&\quad \left. + \frac{i}{\not{k} + \not{k}_2 - m} \gamma_\rho \gamma_5 \frac{i}{\not{k} - \not{k}_1 - m} \gamma_\nu - \frac{i}{\not{k} + \not{k}_2 - m} \gamma_\rho \gamma_5 \frac{i}{\not{k} - m} \gamma_\nu - (m \to M) \right].
\end{aligned} \tag{3.3.20}$$

Now, shifting $k \to k + k_2$ in the first term and shifting $k \to k - k_1 + k_2$ in the second one, they cancel against the fourth and second terms, respectively. We have therefore

$$[k_1^\mu T_{\mu\nu\rho}]_M = 0, \tag{3.3.21}$$

and also

$$[k_2^\nu T_{\mu\nu\rho}]_M = 0 \tag{3.3.22}$$

by an analogous argument. The limit $M \to \infty$ can then be taken safely, thus obtaining the announced results.

We may use a similar procedure to check the identity in eq. (3.3.10). Using

$$\not{q} \gamma_5 = 2m\gamma_5 + (\not{k} + \not{k}_1 - m)\gamma_5 + \gamma_5(\not{k} - \not{k}_2 - m) \tag{3.3.23}$$

and

$$\not{q}\gamma_5 = 2m\gamma_5 + (\not{k} + \not{k}_2 - m)\gamma_5 + \gamma_5(\not{k} - \not{k}_1 - m) \tag{3.3.24}$$

in $q_\rho T_1^{\mu\nu\rho}$ and $q_\rho T_2^{\mu\nu\rho}$ respectively (and making similar replacements in the terms with $m \to M$), we get

$$\left[q_\rho T^{\mu\nu\rho}\right]_M = \left[2mT^{\mu\nu}\right]_M + \left[R^{\mu\nu}\right]_M , \tag{3.3.25}$$

where

$$R^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{k} + \not{k}_1 - m}\gamma_5\gamma^\nu \frac{i}{\not{k} - m}\gamma^\mu - \frac{i}{\not{k} - \not{k}_2 - m}\gamma_5\gamma^\nu \frac{i}{\not{k} - m}\gamma^\mu \right.$$
$$\left. + \frac{i}{\not{k} + \not{k}_2 - m}\gamma_5\gamma^\mu \frac{i}{\not{k} - m}\gamma^\nu - \frac{i}{\not{k} - \not{k}_1 - m}\gamma_5\gamma^\mu \frac{i}{\not{k} - m}\gamma^\nu\right]. \tag{3.3.26}$$

It is now easy to see that $\left[R^{\mu\nu}\right]_M$ vanishes. In fact, by shifting the loop momentum k to $k + k_2$ in the second term, and to $k + k_1$ in the fourth, they cancel against the third and the first respectively. The important point here is that these shifts in the integration variable can be performed only after regularizing the integrals. Therefore,

$$\left[q_\rho T^{\mu\nu\rho}\right]_M = \left[2mT^{\mu\nu}\right]_M . \tag{3.3.27}$$

Let us now compute $\left[2mT^{\mu\nu}\right]_M$ explicitly. Using the Feynman parametrization

$$\begin{aligned} \frac{1}{d_1^{\alpha_1} \dots d_n^{\alpha_n}} &= \frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)} \\ &\times \int_0^1 dx_1 \dots \int_0^1 dx_n \frac{x_1^{\alpha_1 - 1}\dots x_n^{\alpha_n - 1}\delta(1 - x_1 - \dots - x_n)}{(x_1 d_1 + \dots + x_n d_n)^{\alpha_1 + \dots + \alpha_n}}, \end{aligned} \tag{3.3.28}$$

we find

$$\left[2mT_1^{\mu\nu}\right]_M = 2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^4k}{(2\pi)^4}\left[\frac{-8im^2\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma}{\left[k^2 + 2k(k_1 x - k_2 y) - m^2\right]^3} - (m \to M)\right], \tag{3.3.29}$$

where we have set $k_1^2 = k_2^2 = 0$. The simple expression in the numerator is obtained by dropping all products of γ_5 with two, three and five γ matrices, and exploiting the antisymmetry of $\epsilon_{\rho\nu\sigma\mu}$. The integration over the loop momentum k can be easily performed by shifting the integration variable

$$k \to k - k_1 x + k_2 y \tag{3.3.30}$$

with the result

$$\left[2mT_{\mu\nu}\right]_M = \frac{1}{\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma \int_0^1 dx \int_0^{1-x} dy \left[\frac{m^2}{m^2 - q^2 xy} - \frac{M^2}{M^2 - q^2 xy}\right]. \tag{3.3.31}$$

Notice that the RHS of eq. (3.3.31) is finite when $M \to \infty$. The limit can now be taken safely, giving

$$q^\rho T_{\mu\nu\rho} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}k_1^\rho k_2^\sigma. \qquad (3.3.32)$$

The effect of the regularization is that the Slavnov-Taylor identity in eq. (3.3.10) is spoiled by an *anomalous* term, which is usually called the *axial anomaly*, or the Adler-Bardeen-Jackiw anomaly. This term arises because of the impossibility of regularizing the theory in a way that preserves both the vector and axial vector classical current divergence relations; one of the two must be given up. The anomalous term is finite; however, a regularization procedure is needed in order to prove the cancellation of integrals with two propagators, which are divergent.

The anomalous term can be taken into account by modifying eq. (3.3.5) at the one-loop level in the following way:

$$\partial_\mu J_A^\mu = 2imJ_P + \frac{1}{(4\pi)^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}, \qquad (3.3.33)$$

where $F^{\mu\nu}$ is the field-strength tensor of QED. In other words, the axial current is not conserved, at the quantum level, even if $m = 0$. Notice in fact that the anomaly is independent of the fermion mass. Furthermore, it can be proved that higher-order corrections do not modify the one-loop expression of the anomaly.

The result in eq. (3.3.33) can be immediately generalized to a theory with n fermion fields ψ_i, $i = 1, \ldots, n$ with masses m_i, vector charges Q_i and axial charges Q_i^5:

$$\partial_\mu J_A^\mu = \sum_{i=1}^n Q_i^5 Q_i^2 \left[2im_i J_P^i + \frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right], \qquad (3.3.34)$$

where now

$$J_A^\mu = \sum_{i=1}^n Q_i^5 \overline{\psi}_i\gamma^\mu\gamma_5\psi_i; \qquad J_P^i = \overline{\psi}_i\gamma_5\psi_i\,. \qquad (3.3.35)$$

The above considerations can be extended to the case of a theory with non-abelian gauge invariance. In this case, also fermion loops with four and five internal lines contribute to the anomaly. It can be shown that the anomalous term of the axial vector current in a non-abelian theory is proportional to

$$\mathrm{Tr}\,(\{T^a, T^b\}T^c), \qquad (3.3.36)$$

where T^a are the gauge group generators. In the standard model, fermions are either in the doublet or in the singlet representation of $SU(2)$; this means that the four quantities

$$\mathrm{Tr}\,(\{\tau^a, \tau^b\}\tau^c) \qquad (3.3.37)$$
$$\mathrm{Tr}\,(\{\tau^a, \tau^b\}Y) \qquad (3.3.38)$$
$$\mathrm{Tr}\,(Y^2\tau^c) \qquad (3.3.39)$$
$$\mathrm{Tr}\,(Y^3) \qquad (3.3.40)$$

must all vanish, for the axial anomaly to be cancelled. The first quantity is obviously zero:

$$\mathrm{Tr}\,(\{\tau^a, \tau^b\}\tau^c) = 2\delta^{ab}\mathrm{Tr}\,(\tau^c) = 0. \qquad (3.3.41)$$

The second quantity requires more care. Since $\tau^a = 0$ for right-handed fermions, we have

$$\mathrm{Tr}\,(\{\tau^a, \tau^b\}Y) = 2\delta^{ab}\mathrm{Tr}\,(Y_L), \tag{3.3.42}$$

where Y_L is the hypercharge matrix restricted to left-handed fermions. Since $Y = 1/3$ for the doublets of left-handed quarks, and $Y = -1$ for the doublets of left-handed leptons, we find

$$\mathrm{Tr}\,(Y_L) = n_q \times 3 \times 2 \times \frac{1}{3} + n_l \times 2 \times (-1) = 2(n_q - n_l), \tag{3.3.43}$$

where n_q (n_l) is the number of quark (lepton) families. The factor of 3 in front of the quark term is due to the colour degree of freedom, and the overall factor of 2 is due to the fact that left-handed fermions are $SU(2)$ doublets. We see that the cancellation of the axial anomaly requires that the numbers of quark and lepton families are equal! This is an important prediction of the standard model, which has been recently confirmed by the discovery of the *top* quark.

The third condition, $\mathrm{Tr}\,(Y^2\tau^c) = 0$, is again trivially satisfied, since Y has the same value for both components of each doublet, and $\mathrm{Tr}\,(\tau^c) = 0$ (for singlets, we have simply $\tau^c = 0$).

The last condition, $\mathrm{Tr}\,(Y^3) = 0$, is also satisfied provided $n_q = n_l$. To show this, it is convenient to write the axial current as

$$\bar{\psi}\gamma^\mu\gamma_5\psi = \bar{\psi}\gamma^\mu\frac{1}{2}(1+\gamma_5)\psi - \bar{\psi}\gamma^\mu\frac{1}{2}(1-\gamma_5)\psi. \tag{3.3.44}$$

In this way, it is clear that left-handed fermions and right-handed fermions contribute to the axial anomaly with opposite signs. We have therefore

$$\mathrm{Tr}\,(Y^3) = \mathrm{Tr}\,(Y_L^3) - \mathrm{Tr}\,(Y_R^3). \tag{3.3.45}$$

Using $Y = 2(Q - T_3)$ we find

$$\mathrm{Tr}\;(Y_L^3) = 6n_q\left(\frac{1}{3}\right)^3 + 2n_l(-1)^3 \tag{3.3.46}$$

$$\mathrm{Tr}\;(Y_R^3) = 3n_q\left[\left(\frac{4}{3}\right)^3 + \left(-\frac{2}{3}\right)^3\right] + n_l(-2)^3, \tag{3.3.47}$$

and therefore

$$\mathrm{Tr}\;(Y^3) = -6(n_q - n_l). \tag{3.3.48}$$

It is easy to prove that, because of the axial anomaly, the currents associated with the leptonic and baryonic numbers,

$$L^\mu = \sum_{i=1}^{n_l}\left[\bar{e}_i\gamma^\mu e_i + \bar{\nu}_i\gamma^\mu\nu_i\right] \tag{3.3.49}$$

$$B^\mu = \frac{1}{3}\sum_{i=1}^{n_q}\left[\bar{u}_i\gamma^\mu u_i + \bar{d}_i\gamma^\mu d_i\right] \tag{3.3.50}$$

are anomalous. In order to prove this statement, let us consider the case of only one generation (the extension to more than one generation is trivial), and let us rewrite the leptonic current as

$$L^\mu = L^\mu_{L} + L^\mu_{R}, \tag{3.3.51}$$

where

$$L^\mu_{L} = (\bar\nu_L, \bar e_L)\gamma^\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{3.3.52}$$

$$L^\mu_{R} = \bar e_R \gamma^\mu e_R. \tag{3.3.53}$$

We now consider triangle diagrams with L^μ_{L} or L^μ_{R} on one vertex, and weak vector bosons on the the two remaining vertices. Clearly, only left-handed (right-handed) fermions circulate in the loop with L^μ_{L} (L^μ_{R}). This is easily seen by working out the Dirac structure of the loop integrand:

$$\gamma_\mu P_{L} \hat k (a\gamma^\nu P_{L} + b\gamma^\nu P_{R})\hat k'(a'\gamma^\nu P_{L} + b'\gamma^\nu P_{R}) = aa'\gamma_\mu P_{L}\hat k\gamma^\nu \hat k'\gamma^\nu. \tag{3.3.54}$$

Thus,

$$\partial_\mu L^\mu_{L} = -\frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}\left[g'^2 B_{\mu\nu}B_{\rho\sigma}\mathrm{Tr}\left\{\frac{Y_L}{2},\frac{Y_L}{2}\right\} + g^2 W^i_{\mu\nu}W^j_{\rho\sigma}\mathrm{Tr}\left\{\frac{\tau^i}{2},\frac{\tau^j}{2}\right\}\right]. \tag{3.3.55}$$

The minus sign arises because γ_5 appears in L^μ_{L} with a minus sign. Using $Y_L = -1$ and the anticommutation relations among the Pauli matrices, we find

$$\partial_\mu L^\mu_{L} = -\frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}\left[g'^2 B_{\mu\nu}B_{\rho\sigma} + g^2 W^i_{\mu\nu}W^i_{\rho\sigma}\right]. \tag{3.3.56}$$

By a similar argument, we get

$$\partial_\mu L^\mu_{R} = \frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}g'^2 B_{\mu\nu}B_{\rho\sigma}\mathrm{Tr}\left\{\frac{Y_R}{2},\frac{Y_R}{2}\right\} = \frac{2}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}g'^2 B_{\mu\nu}B_{\rho\sigma}, \tag{3.3.57}$$

since $Y_R = -2$, and therefore

$$\partial_\mu L^\mu = \frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}\left[g'^2 B_{\mu\nu}B_{\rho\sigma} - g^2 W^i_{\mu\nu}W^i_{\rho\sigma}\right]. \tag{3.3.58}$$

This results in a (numerically negligible) non-conservation of leptonic and baryonic numbers L and B, due to instanton effects. The difference $B - L$ is however conserved. Indeed, we may write for the baryonic current

$$B^\mu = B^\mu_{L} + B^\mu_{R}, \tag{3.3.59}$$

where

$$B^\mu_{L} = \frac{1}{3}(\bar u_L, \bar d_L)\gamma^\mu \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{3.3.60}$$

$$B^\mu_{R} = \frac{1}{3}\bar u_R\gamma^\mu u_R + \frac{1}{3}\bar d_R\gamma^\mu d_R, \tag{3.3.61}$$

and compute $\partial_\mu B^\mu$ as in the case of the leptonic current. We find

$$\begin{aligned}\partial_\mu B^\mu &= -\frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}\left[g'^2 B_{\mu\nu}B_{\rho\sigma}\mathrm{Tr}\left\{\frac{Y_Q}{2},\frac{Y_Q}{2}\right\}+g^2 W^i_{\mu\nu}W^j_{\rho\sigma}\mathrm{Tr}\left\{\frac{\tau^i}{2},\frac{\tau^j}{2}\right\}\right]\\ &\quad+\frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}g'^2 B_{\mu\nu}B_{\rho\sigma}\left[\mathrm{Tr}\left\{\frac{Y_{u_R}}{2},\frac{Y_{u_R}}{2}\right\}+\mathrm{Tr}\left\{\frac{Y_{d_R}}{2},\frac{Y_{d_R}}{2}\right\}\right].\end{aligned} \tag{3.3.62}$$

The global factor of $1/3$ is cancelled by a factor of 3 from color. Using the known values of quark hypercharges

$$Y_Q=\frac{1}{3}\quad Y_{u_R}=\frac{4}{3}\quad Y_{d_R}=-\frac{2}{3} \tag{3.3.63}$$

we get

$$\begin{aligned}\partial_\mu B^\mu &= \frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}\left[-g'^2 B_{\mu\nu}B_{\rho\sigma}\left(Y_Q^2-\frac{Y_{u_R}^2}{2}-\frac{Y_{d_R}^2}{2}\right)-g^2 W^i_{\mu\nu}W^i_{\rho\sigma}\right]\\ &= \frac{1}{(4\pi)^2}\epsilon^{\mu\nu\rho\sigma}\left[g'^2 B_{\mu\nu}B_{\rho\sigma}-g^2 W^i_{\mu\nu}W^i_{\rho\sigma}\right]\end{aligned} \tag{3.3.64}$$

which is exactly equal to $\partial_\mu L^\mu$. This shows that the current $B^\mu - L^\mu$ is conserved.

3.4 Accidental symmetries

The need for a Yukawa interaction term of fermion fields with scalar fields can be motivated in a different way. Consider the standard model with only one generation of quarks and leptons, and no scalar fields. The lagrangian for fermion fields can be written in the following compact form:

$$\mathcal{L}=\sum_{k=1}^{5}\bar{\psi}_k\, \not{D}\psi_k, \tag{3.4.1}$$

where the sum runs over the five different irreducible representations of $SU(2)_L\otimes U(1)_Y$ of the fermions in a generation:

$$\begin{aligned}\psi_1 &= e_R \sim (\mathbf{1},-2)\\ \psi_2 &= L \sim (\mathbf{2},-1)\\ \psi_3 &= u_R \sim (\mathbf{1},4/3)\\ \psi_4 &= d_R \sim (\mathbf{1},-2/3)\\ \psi_5 &= Q \sim (\mathbf{2},1/3).\end{aligned}$$

Here, the symbol $\sim$ means "transforms as", and the two numbers in brackets stand for the $SU(2)$ representation (**2** for the doublet, **1** for the scalar) and for the hypercharge quantum number, respectively. Mass terms are forbidden by the gauge symmetry.

In addition to the assumed gauge symmetry, the lagrangian in eq. (3.4.1) is manifestly invariant under a large class of global transformations: namely, the fermion fields within each representation can be multiplied by an arbitrary constant phase

$$\psi_k \to e^{i\phi_k}\psi_k \tag{3.4.2}$$

without affecting $\mathcal{L}$. This $[U(1)]^5$ global symmetry was not imposed at the beginning: it is just a consequence of the assumed gauge symmetry and of the renormalizability condition. It is therefore called an *accidental* symmetry.

Let us take a closer look to the accidental symmetry. The five conserved currents corresponding to the global transformations (3.4.2) are

$$\begin{aligned}
J_1^\mu &= \bar{e}_R \gamma^\mu e_R \\
J_2^\mu &= \bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L \\
J_3^\mu &= \bar{u}_R \gamma^\mu u_R \\
J_4^\mu &= \bar{d}_R \gamma^\mu d_R \\
J_5^\mu &= \bar{u}_L \gamma^\mu u_L + \bar{d}_L \gamma^\mu d_L
\end{aligned}$$

Equivalently, one could define the accidental symmetry transformations in such a way that the corresponding currents are five independent linear combinations of $J_1^\mu, \dots, J_5^\mu$. Consider for example the choice

$$\begin{aligned}
J_Y^\mu &= \sum_{k=1}^{5} \frac{Y_k}{2} J_k^\mu \\
J_\ell^\mu &= J_1^\mu + J_2^\mu \equiv \bar{\nu}\gamma^\mu \nu + \bar{e}\gamma^\mu e \\
J_{\ell 5}^\mu &= J_1^\mu - J_2^\mu \equiv \bar{\nu}\gamma^\mu \gamma_5 \nu + \bar{e}\gamma^\mu \gamma_5 e \\
J_b^\mu &= \frac{1}{3}(J_3^\mu + J_4^\mu + J_5^\mu) \equiv \frac{1}{3}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d) \\
J_{b5}^\mu &= J_3^\mu + J_4^\mu - J_5^\mu \equiv \bar{u}\gamma^\mu \gamma_5 u + \bar{d}\gamma^\mu \gamma_5 d.
\end{aligned}$$

The current J_Y is the hypercharge current, which corresponds to a local invariance of the theory. The true accidental symmetry is therefore $[U(1)]^4$, rather than $[U(1)]^5$.

The currents J_ℓ and J_b are immediately recognized to be the leptonic and baryonic number currents, respectively. The invariance of the lagrangian under the corresponding global symmetries is certainly good news, since baryonic and leptonic number are known to be conserved to an extremely high accuracy.

On the other hand, experiments show no sign of the conservation of $J_{\ell 5}$ and J_{b5}; in a realistic theory, the corresponding symmetries should be broken. In fact, they are incompatible with mass terms, and they are broken by the Yukawa interaction terms that generate fermion masses via the Higgs mechanism.

When the theory is extended to include more fermion generations, the accidental symmetry gets much larger, since also mixing among different generation is allowed. The Yukawa interaction terms of the previous subsection break this larger accidental symmetry too, leaving however baryonic and leptonic numbers conserved. Individual leptonic numbers are separately conserved, while only the total baryonic number is conserved, because of flavour mixing.

To conclude this subsection, let us briefly review the most important experimental evidences of baryon and lepton number conservation. The most obvious test of baryon number conservation is proton stability. The experimental lower bound on the proton lifetime is at present

$$\tau_p > 1.6 \cdot 10^{25}\, y\,. \tag{3.4.3}$$

The most accurate tests of lepton number conservation are provided by the following observables:

$$B(\mu \to e\gamma) \leq 1.2 \cdot 10^{-11} \tag{3.4.4}$$

$$B(\mu \to 3e) \leq 1 \cdot 10^{-12} \tag{3.4.5}$$

$$\frac{\Gamma(\mu Ti \to e\ Ti)}{\Gamma(\mu Ti \to all)} \leq 4 \cdot 10^{-12} \tag{3.4.6}$$

$$B(\tau \to \mu\gamma) \leq 2.7 \cdot 10^{-6}\,. \tag{3.4.7}$$

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Chapter 4

Appendices

4.1 Renormalizability and power counting

In this appendix, we describe the power-counting criterion for renormalizability of local field theories. Consider a Feynman diagram containing

– L loops;
– V vertices;
– I_f internal fermionic lines;
– E_f external fermionic lines;
– I_b internal bosonic lines;
– E_b external bosonic lines.

Let us assume that there are different types of vertices, each type being labelled by the index i, and that

$$V = \sum_i V^i, \tag{4.1.1}$$

where V^i is the number of vertices of type i. Finally, let n_f^i, n_b^i, d^i be the number of fermionic lines, bosonic lines and field derivatives in type-i vertices, respectively. The following relations hold:

$$2I_f + E_f = \sum_i n_f^i V^i \tag{4.1.2}$$

$$2I_b + E_b = \sum_i n_b^i V^i. \tag{4.1.3}$$

The number L of loops is equal to the number of independent internal momenta, which in turn is equal to the total number of internal lines $I = I_f + I_b$ minus the number of independent momentum conservation equations. Therefore, we have

$$L = I_f + I_b - (V - 1). \tag{4.1.4}$$

We now define the degree of superficial divergence D of the diagram as the power of momenta in the numerator minus the power of momenta in the denominator of the Feynman diagram.

Clearly,

$$D = dL - I_f - 2I_b + \sum_i d^i V^i, \tag{4.1.5}$$

since fermion propagators behave as k^{-1}, boson propagator behave as k^{-2}, each field derivative corresponds to one power of momentum, and d powers of momentum are carried by each loop integration in d-dimensional space-time. Now, replacing eqs. (4.1.1) and (4.1.4) in eq. (4.1.5) and eliminating I_f and I_b via eqs. (4.1.2) and (4.1.3), we find

$$D = d - \frac{d-1}{2} E_f - \frac{d-2}{2} E_b + \sum_i V^i \left(d^i + \frac{d-1}{2} n_f^i + \frac{d-2}{2} n_b^i - d \right). \tag{4.1.6}$$

If $D \geq 0$, the Feynman amplitude will be ultraviolet divergent. On the other hand, $D < 0$ is not a sufficient condition for convergence, since there can still be subdiagrams with $D \geq 0$. However, we notice that D decreases with increasing number of external lines. Therefore, if the last term in the r.h.s. of eq. (4.1.6) is zero or negative, then only a finite number of diagrams have $D \geq 0$, and the whole theory can be made finite by renormalizing only these *primitively divergent* amplitudes, at any order in perturbation theory. The condition for renormalizability then becomes

$$d^i + \frac{d-1}{2} n_f^i + \frac{d-2}{2} n_b^i \leq d \tag{4.1.7}$$

and it must hold for each i separately (a diagram can contain only vertices of one type). Notice that the l.h.s. of eq. (4.1.7) is just the mass dimension of the operator that corresponds to type i vertices: in fact, fermion fields have dimension 3/2, boson fields have dimension 1 and derivatives have dimension 1. For this reason, the condition in eq. (4.1.7) can be rephrased in terms of coupling constant dimensionality: a renormalizable theory can contain only constants with mass dimension ≥ 0.

4.2 Non-unitarity of the Fermi theory

In this Appendix we will work out the restrictions imposed on scattering amplitudes by the unitarity condition of the scattering matrix, and we will show that the Fermi theory violates this unitarity bound at sufficiently high energy. Writing the scattering matrix as

$$S = I + iT, \tag{4.2.1}$$

the unitarity condition $S^\dagger S = I$ gives

$$T^\dagger T = -i(T - T^\dagger). \tag{4.2.2}$$

For generic states a, b we have

$$\langle a|T^\dagger T|b\rangle = -i\left(\langle a|T|b\rangle - \langle a|T^\dagger|b\rangle\right). \tag{4.2.3}$$

Now define the invariant amplitude $\mathcal{M}_{af}$ for the process $a \to f$ by

$$\langle f|T|a\rangle = \mathcal{M}_{af}\,(2\pi)^4\,\delta^{(4)}(P_a - P_f), \tag{4.2.4}$$

and insert the identity operator between $T^\dagger$ and T in the l.h.s. of eq. (4.2.3):

$$I = \sum_f \prod_i \int \frac{d^3 P_i^f}{(2\pi)^3 2E_i^f}\,|f\rangle\langle f| \tag{4.2.5}$$

where P_i^f is the momentum of particle i in the state f. We get

$$\begin{aligned}\sum_f \prod_i \int \frac{d^3 P_i^f}{(2\pi)^3 2E_i^f}\,(2\pi)^4\,\delta^{(4)}(P_a - \sum_i P_i^f)\,(2\pi)^4\,\delta^{(4)}(P_b - \sum_i P_i^f)\,\mathcal{M}_{bf}\mathcal{M}^*_{af} \\ = -i\,(\mathcal{M}_{ba} - \mathcal{M}^*_{ab})\,(2\pi)^4\delta^{(4)}(P_a - P_b),\end{aligned} \tag{4.2.6}$$

or

$$\sum_f \prod_i \int \frac{d^3 P_i^f}{(2\pi)^3 2E_i^f}\,(2\pi)^4\,\delta^{(4)}(P_a - \sum_i P_i^f)\,\mathcal{M}_{bf}\mathcal{M}^*_{af} = -i\,(\mathcal{M}_{ba} - \mathcal{M}^*_{ab}). \tag{4.2.7}$$

For $a = b$, eq. (4.2.7) gives

$$\sum_f \prod_i \int \frac{d^3 P_i^f}{(2\pi)^3 2E_i^f}\,(2\pi)^4\,\delta^{(4)}(P_a - \sum_i P_i^f)\;|\mathcal{M}_{af}|^2 = 2\,\mathrm{Im}\,\mathcal{M}_{aa}, \tag{4.2.8}$$

which is the so-called optical theorem: the total cross section for the process $a \to f$ is proportional to the imaginary part of the forward invariant amplitude $\mathcal{M}_{aa}$.

Let us now assume that $|a\rangle$ is a state of two particles of the same species, with momenta p_1, p_2; furthermore, let us assume that only elastic scattering is allowed. Under these conditions, the states $|f\rangle$ are also two-particle states of the same species as those in $|a\rangle$, and the amplitudes

$\mathcal{M}_{af}$ depend on the initial and final states through the two independent Mandelstam variables s, t:

$$\mathcal{M}_{af} \equiv \mathcal{M}(s,t), \tag{4.2.9}$$

where

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - P_1)^2. \tag{4.2.10}$$

In the center-of-mass frame,

$$t = -\frac{s}{2}(1 - \cos\theta) \ \rightarrow \ \cos\theta = 1 + \frac{2t}{s}, \tag{4.2.11}$$

where θ is the scattering angle. Thus, for a given value of the center-of mass squared energy s, the amplitude $\mathcal{M}(s,t)$ is a function of $\cos\theta$ only, and can be expanded on the basis of the Legendre polynomials

$$P_J(z) = \frac{1}{J!2^J}\frac{d^J}{dz^J}(z^2-1)^J. \tag{4.2.12}$$

The Legendre polynomials obey the orthogonality conditions

$$\int_{-1}^{1} dz\, P_J(z)\, P_K(z) = \frac{2}{2J+1}\,\delta_{JK} \tag{4.2.13}$$

and the normalization conditions

$$P_J(1) = 1. \tag{4.2.14}$$

We find

$$\mathcal{M}(s,t) = 16\pi \sum_J (2J+1)\, a_J(s)\, P_J(\cos\theta), \tag{4.2.15}$$

where the partial-wave amplitudes a_J are given by

$$a_J(s) = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, P_J(\cos\theta)\, \mathcal{M}(s,t). \tag{4.2.16}$$

Replacing eq. (4.2.15) in the l.h.s. of eq. (4.2.8) we get

$$\begin{aligned}
&\int \frac{d^3P_1}{(2\pi)^3\, 2E_1}\frac{d^3P_2}{(2\pi)^3\, 2E_2}\,(2\pi)^4\delta^{(4)}(p_1+p_2-P_1-P_2)\ |\mathcal{M}(s,t)|^2 \\
= \ & \frac{1}{16\pi}\int_{-1}^{1} d\cos\theta \left[16\pi\sum_J(2J+1)\, a_J(s)\, P_J(\cos\theta)\right]\left[16\pi\sum_K(2K+1)\, a_K^*(s)\, P_K(\cos\theta)\right] \\
= \ & 32\pi\sum_J(2J+1)\ |a_J(s)|^2\,,
\end{aligned} \tag{4.2.17}$$

while the r.h.s. is given by

$$2\,\mathrm{Im}\,\mathcal{M}(s,0) = 32\pi\sum_J(2J+1)\,\mathrm{Im}\, a_J(s), \tag{4.2.18}$$

where we have set $t = 0$, or equivalently $\cos\theta = 1$, as appropriate for a forward amplitude, and we have used the normalization condition (4.2.14). Therefore, unitarity of the scattering matrix requires

$$|a_J(s)|^2 = \mathrm{Im}\, a_J(s) \qquad (4.2.19)$$

for all partial amplitudes. Equation (4.2.19) provides the unitarity bound

$$|a_J(s)| \leq 1. \qquad (4.2.20)$$

Let us now consider a specific process, namely the scattering

$$e^-(p_1) + \nu_\mu(p_2) \to \mu^-(P_1) + \nu_e(P_2) \qquad (4.2.21)$$

within the Fermi theory. The relevant amplitude is

$$\mathcal{M}(s,t) = -\frac{G_F}{\sqrt{2}} \bar{u}(P_2)\,\gamma^\alpha(1-\gamma_5)\,u(p_1)\,\bar{u}(P_1)\,\gamma_\alpha(1-\gamma_5)\,u(p_2), \qquad (4.2.22)$$

which gives

$$\begin{aligned} |\mathcal{M}(s,t)|^2 &= \frac{G_F^2}{2}\,\mathrm{Tr}\left[\gamma^\alpha(1-\gamma_5)\,\not{p}_1\,\gamma^\beta(1-\gamma_5)\,\not{k}_2\right]\ \mathrm{Tr}\left[\gamma_\alpha(1-\gamma_5)\,\not{p}_2\,\gamma_\beta(1-\gamma_5)\,\not{k}_1\right] \\ &= 32 G_F^2 s^2, \end{aligned} \qquad (4.2.23)$$

where a sum over polarizations is understood. We see that only the partial amplitude $a_0(s)$ is nonzero, since there is no t dependence at all. Using the definition eq. (4.2.16) we obtain

$$|a_0(s)| = \frac{G_F\, s}{2\sqrt{2}\pi}. \qquad (4.2.24)$$

The unitarity bound eq. (4.2.20) is therefore violated at

$$\sqrt{s} = \sqrt{\frac{2\sqrt{2}\pi}{G_F}} \simeq 875\,\mathrm{GeV}. \qquad (4.2.25)$$

From eq. (4.2.23) we obtain the total cross section

$$\sigma = \frac{G_F^2 s}{2\pi}. \qquad (4.2.26)$$

Let us now repeat the same calculation in the context of a theory with an interacting vector boson W with mass m_W and coupling $g/(2\sqrt{2})$ to left-handed fermions (the coupling g is dimensionless; the numerical factor is conventional). The squared amplitude in this theory is obtained from the result in eq. (4.2.23) by performing the replacement

$$-\frac{G_F}{\sqrt{2}} \to \frac{g^2}{8}\frac{1}{t - m_W^2}. \qquad (4.2.27)$$

We get

$$|\mathcal{M}(s,t)|^2 = 32 \left(\frac{g^2\sqrt{2}}{8} \frac{1}{t - m_W^2} \right)^2 s^2 = \frac{g^4 s^2}{(t - m_W^2)^2}. \tag{4.2.28}$$

The total cross section is now given by

$$\sigma = \frac{g^4}{64\pi m_W^2} \frac{s}{s + m_W^2}. \tag{4.2.29}$$

For $s \ll m_W^2$, this expression reduces to the result obtained in the Fermi theory, eq. (4.2.26), with the identification

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}. \tag{4.2.30}$$

In this case, however, the linear growth of the cross section with s is cut off at $s \sim m_W^2$. At very large energy we have

$$\sigma \to \frac{g^4}{64\pi m_W^2} = \frac{G_F^2 m_W^2}{2\pi}. \tag{4.2.31}$$

The value of m_W is related to the size of the coupling g through eq. (4.2.30). If m_W were close to the energy at which the Fermi theory breaks down, about 900 GeV, then g would take a value close to 10, far from the perturbative domain. The fact that the measured value m_W is instead much smaller, $m_W \simeq 80$ GeV, is a signal of the fact that a theory of weak interactions with an intermediate vector boson can be treated perturbatively: indeed, in this case we get $g \sim 0.7$.

4.3 Gauge theories

The abelian case

The Dirac free lagrangian for a massive fermion

$$\mathcal{L} = \overline{\psi}(i\hat{\partial} - m)\psi \tag{4.3.1}$$

is invariant under the global (or first kind) $U(1)$ gauge transformation

$$\begin{aligned} \psi &\to \psi' = e^{ie\alpha}\psi \\ \overline{\psi} &\to \overline{\psi}' = e^{-ie\alpha}\overline{\psi}, \end{aligned} \tag{4.3.2}$$

where α is a real constant. The constant e plays the role of the conserved charge associated with this invariance property. We want to promote this global symmetry to a local one, that is, we want to modify $\mathcal{L}$ in order to make it invariant under the field transformation (4.3.2) with $\alpha = \alpha(x)$. The derivative term is not invariant:

$$\overline{\psi}\partial^\mu\psi \to e^{-ie\alpha}\overline{\psi}\partial^\mu(e^{ie\alpha}\psi) = \overline{\psi}\partial^\mu\psi + ie\overline{\psi}(\partial^\mu\alpha)\psi. \tag{4.3.3}$$

The ordinary derivative must be replaced by a *covariant* derivative,

$$D^\mu = \partial^\mu - ieA^\mu, \tag{4.3.4}$$

where A^μ is a real vector field. The transformation property of A^μ must be fixed in such a way that

$$D^\mu\psi \to e^{ie\alpha}D^\mu\psi. \tag{4.3.5}$$

This gives

$$\begin{aligned} &(\partial^\mu - ieA'^\mu)\psi' = e^{ie\alpha}(\partial^\mu - ieA^\mu)\psi \\ &(\partial^\mu - ieA'^\mu)e^{ie\alpha}\psi = e^{ie\alpha}(\partial^\mu - ieA^\mu)\psi \\ &\partial^\mu\psi + ie(\partial^\mu\alpha)\psi - ieA'^\mu\psi = \partial^\mu\psi - ieA^\mu\psi \\ &(\partial^\mu\alpha)\psi - A'^\mu\psi = -A^\mu\psi \end{aligned} \tag{4.3.6}$$

which implies

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu\alpha. \tag{4.3.7}$$

The lagrangian

$$\mathcal{L} = \overline{\psi}(i\hat{D} - m)\psi \tag{4.3.8}$$

is invariant under the local (or second kind) gauge transformation

$$\begin{aligned} \psi &\to \psi' = e^{ie\alpha(x)}\psi \\ \overline{\psi} &\to \overline{\psi}' = e^{-ie\alpha(x)}\overline{\psi}, \\ A^\mu &\to A'^\mu = A^\mu + \partial^\mu\alpha(x). \end{aligned} \tag{4.3.9}$$

Notice that the requirement of local gauge invariance generates the interaction term $e\overline{\psi}\gamma_\mu\psi A^\mu$.

A kinetic term, involving derivatives of the vector field A^μ, must now be introduced. It is uniquely fixed by the requirements of Lorentz and gauge invariance, and by assuming the standard normalization of the propagator for A^μ. It is given by

$$\mathcal{L}^{YM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{4.3.10}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{4.3.11}$$

Notice that

$$(D^\mu D^\nu - D^\nu D^\mu)\psi = -ieF^{\mu\nu}\psi, \tag{4.3.12}$$

and that $F^{\mu\nu}$ is invariant under a gauge transformation. Notice also that gauge invariance forbids the presence of a mass term for the gauge field A^μ. Finally, we observe that no self-interaction term for the vector field A^μ is present in the lagrangian. This is connected with the abelian nature of the invariance group.

The non-abelian case

Let us consider now the case when the invariance group of the theory is non-abelian. For definiteness, we consider the group $SU(N)$ of $N \times N$ unitary matrices with unit determinant. This group has $N^2 - 1$ hermitian traceless generators t^A, that obey the commutation relations

$$[t^A, t^B] = if^{ABC}t^C, \qquad A, B, C = 1, ..., N^2 - 1, \tag{4.3.13}$$

where f^{ABC} is completely antisymmetric. A generic element U of $SU(N)$ can be expressed in terms of the generators t^A and of a set of real functions $\alpha^A(x)$ by

$$U \equiv U(\alpha) = \exp(ig\alpha^A t^A); \qquad U^{-1} = U^\dagger, \tag{4.3.14}$$

where we have inserted a coupling constant g in analogy with the abelian case. The covariant derivative is now given by

$$D^\mu = \partial^\mu I - igA^\mu, \tag{4.3.15}$$

where I is the unity matrix in the representation space, and the vector field A^μ is now a hermitian matrix

$$A^\mu = A^\mu_A t^A. \tag{4.3.16}$$

It is easy to show, in analogy with the abelian case, that the transformation law

$$A^\mu \to A'^\mu = UA^\mu U^{-1} + \frac{i}{g}U(\partial^\mu U^{-1}) \tag{4.3.17}$$

ensures that

$$D^\mu \to UD^\mu U^{-1}. \tag{4.3.18}$$

Consider now an infinitesimal gauge transformation

$$U(\alpha) = I + ig\alpha^A t^A + \mathcal{O}(\alpha^2). \tag{4.3.19}$$

To first order in α, eq. (4.3.17) becomes

$$\begin{aligned} A'^\mu &= A^\mu + ig[\alpha^A t^A, A^\mu] - \frac{i}{g} ig\partial^\mu \alpha^A t^A \\ &= A^\mu_C t^C - g\alpha^A A^\mu_B f^{ABC} t^C + \partial^\mu \alpha^C t^C, \end{aligned} \tag{4.3.20}$$

or

$$A'^\mu_C = A^\mu_C - g\alpha^A A^\mu_B f^{ABC} + \partial^\mu \alpha^C. \tag{4.3.21}$$

A kinetic term for the gauge fields can be built in analogy with the abelian case. We have Recalling eq. (4.3.12), we define a field tensor $F^{\mu\nu}$ through

$$(D^\mu D^\nu - D^\nu D^\mu)\psi = -igF^{\mu\nu}\psi, \tag{4.3.22}$$

where ψ is a multiplet of some $SU(N)$ representation, and $F^{\mu\nu} = F^{\mu\nu}_A t^A$. We find

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu], \\ F^{\mu\nu}_A &= \partial^\mu A^\nu_A - \partial^\nu A^\mu_A + gf^{ABC} A^\mu_B A^\nu_C. \end{aligned} \tag{4.3.23}$$

The kinetic term is then given by

$$-\frac{1}{4} F^{\mu\nu}_A F^A_{\mu\nu}. \tag{4.3.24}$$

In the non-abelian case, self-interaction terms among the gauge fields are present. This is related to the fact that, contrary to the abelian case, the field strength $F^{\mu\nu}$ transforms non-trivially under a gauge transformation:

$$F^{\mu\nu} \to F'^{\mu\nu} = UF^{\mu\nu}U^{-1}. \tag{4.3.25}$$

For an infinitesimal gauge transformation, we find

$$F'^{\mu\nu}_A = F^{\mu\nu}_A - gf^{ABC}\alpha^B F^{\mu\nu}_C, \tag{4.3.26}$$

which means that the components $F^{\mu\nu}_A$ form a multiplet in the adjoint representation of the gauge group.

4.4 The standard model lagrangian in renormalizable gauges

Let us consider the following part of the standard model lagrangian:

$$\mathcal{L}_D - V(\phi) + \mathcal{L}_{GF}, \tag{4.4.1}$$

where

$$\mathcal{L}_D = (D^\mu \phi)^\dagger D_\mu \phi \tag{4.4.2}$$

$$V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4 \tag{4.4.3}$$

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left[\partial^\mu W^i_\mu - \xi f^i(\phi)\right]^2 - \frac{1}{2\xi}\left[\partial^\mu B_\mu - \xi f(\phi)\right]^2. \tag{4.4.4}$$

For the moment, we do not specify the value of the hypercharge quantum number Y of the Higgs doublet ϕ. We define

$$\phi = \phi_1 + \phi_2, \tag{4.4.5}$$

where

$$\phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} G^+ \\ (H + iG)/\sqrt{2} \end{pmatrix} \tag{4.4.6}$$

and v_1, v_2 are arbitrary complex numbers, only restricted by the minimization condition

$$|v_1|^2 + |v_2|^2 \equiv v^2 = -\frac{m^2}{\lambda}. \tag{4.4.7}$$

We have

$$\begin{aligned} \mathcal{L}_D &= \left[\partial^\mu \phi^\dagger + \frac{i}{2}\phi^\dagger\left(g W^\mu_i \tau^i + g' Y B^\mu\right)\right]\left[\partial_\mu \phi - \frac{i}{2}\left(g W^j_\mu \tau^j + g' Y B_\mu\right)\phi\right] \\ &\equiv \mathcal{L}_{\phi\phi} + \mathcal{L}_{\phi\phi VV} + \mathcal{L}_{\phi\phi V}. \end{aligned} \tag{4.4.8}$$

The first term is simply the kinetic term for ϕ,

$$\mathcal{L}_{\phi\phi} = (\partial^\mu \phi)^\dagger \partial_\mu \phi = \partial^\mu G^+ \partial_\mu G^- + \frac{1}{2}\partial^\mu H \partial_\mu H + \frac{1}{2}\partial^\mu G \partial_\mu G. \tag{4.4.9}$$

Next, we consider the $\phi\phi VV$ term:

$$\begin{aligned} \mathcal{L}_{\phi\phi VV} &= \frac{1}{4}(g^2 W^\mu_i W^i_\mu + g'^2 Y^2 B^\mu B_\mu)\phi^\dagger \phi + \frac{1}{2} g g' Y B^\mu W^i_\mu \phi^\dagger \tau^i \phi \\ &= \frac{1}{4}(W^\mu_i \;\; B^\mu)\begin{bmatrix} g^2 \phi^\dagger \phi \delta^{ij} & g g' Y \phi^\dagger \tau^i \phi \\ g g' Y \phi^\dagger \tau^j \phi & g'^2 Y^2 \phi^\dagger \phi \end{bmatrix}\begin{pmatrix} W_{j\mu} \\ B_\mu \end{pmatrix}. \end{aligned} \tag{4.4.10}$$

Equation (4.4.10) contains a mass term for the vector fields, that can be isolated by replacing ϕ with ϕ_1:

$$\mathcal{L}_{\text{mass}} = (W^\mu_i \;\; B^\mu)\, \mathcal{M}^2 \begin{pmatrix} W_{j\mu} \\ B_\mu \end{pmatrix}, \tag{4.4.11}$$

where

$$\mathcal{M}^2 = \frac{1}{4}\left[\begin{array}{cc} g^2\phi_1^\dagger\phi_1\delta^{ij} & gg'Y\phi_1^\dagger\tau^i\phi_1 \\ gg'Y\phi_1^\dagger\tau^j\phi_1 & g'^2Y^2\phi_1^\dagger\phi_1 \end{array}\right]. \tag{4.4.12}$$

Observe that the square mass matrix in eq. (4.4.11) has zero determinant:

$$\det\mathcal{M}^2 = \frac{1}{16}\,g^2g'^2\,Y^2\,|\phi_1|^4\left(\phi_1^\dagger\tau^j\phi_1\,\phi_1^\dagger\tau^j\phi_1 - |\phi_1|^4\right) \tag{4.4.13}$$

which is seen to vanish by means of the identity

$$\tau^j_{ab}\tau^j_{cd} = 2\left(\delta_{ad}\delta_{bc} - \frac{1}{2}\delta_{ab}\delta_{cd}\right). \tag{4.4.14}$$

In other words, with only one scalar doublet of any hypercharge, one of the four physical vector boson has always zero mass. This is because it is always possible to find a $U(1)$ subgroup of the gauge group which leaves the vacuum expectation value ϕ_1 invariant.

Let us now diagonalize $\mathcal{M}^2$. This is easily done by choosing $v_1 = 0, v_2 = v$, which is allowed because all the degenerate vacuum configurarions are connected by gauge transformations. We find

$$\mathcal{L}_{mass} = \frac{1}{4}\,g^2v^2\,W^{+\mu}W^-_\mu + \frac{1}{8}v^2(W_3^\mu\;\;B^\mu)\left[\begin{array}{cc} g^2 & -gg'Y \\ -gg'Y & g'^2Y^2 \end{array}\right]\left(\begin{array}{c} W_{3\mu} \\ B_\mu \end{array}\right). \tag{4.4.15}$$

The first term is already in diagonal form, and tells us that the charged vector bosons

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu) \tag{4.4.16}$$

are mass eigenstates, with masses

$$m_W^2 = \frac{1}{4}g^2v^2. \tag{4.4.17}$$

The second term in eq. (4.4.15) is diagonalized by the rotation

$$\left(\begin{array}{c} W_3^\mu \\ B^\mu \end{array}\right) = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right]\left(\begin{array}{c} Z^\mu \\ A^\mu \end{array}\right); \qquad \tan\theta = \frac{g'Y}{g}, \tag{4.4.18}$$

where the combination A^μ corresponds to the zero-mass vector boson. We see immediately that, for $Y = 1$, A^μ is precisely equal to the photon field coupled to the electromagnetic current, and $\theta \equiv \theta_W$. The eigenvalue corresponding to Z^μ is

$$m_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2. \tag{4.4.19}$$

In terms of $W^\pm_\mu$, A_μ and Z_μ eq. (4.4.10) becomes

$$\mathcal{L}_{\phi\phi VV} \;=\; W^{+\mu}W^-_\mu\left(m_W + \frac{1}{2}gH\right)^2 + \frac{1}{2}Z^\mu Z_\mu\left(m_Z + \frac{1}{2}\frac{g}{\cos\theta_W}H\right)^2$$

$$+\frac{1}{2}g^2W^{+\mu}W^-_\mu(G^+G^- + \frac{1}{2}G^2) + \frac{1}{8}\frac{g^2}{\cos^2\theta_W}Z^\mu Z_\mu G^2$$
$$+\frac{1}{4}\frac{g^2}{\cos^2\theta_W}(A^\mu\sin 2\theta_W + Z^\mu\cos 2\theta_W)^2G^+G^-$$
$$+g\sin\theta_W(m_W A^\mu - m_Z Z^\mu\sin\theta_W)\left(G^-W^+_\mu + G^+W^-_\mu\right)$$
$$+\frac{1}{2}g^2\sin\theta_W(A^\mu - Z^\mu\tan\theta_W)\left[G^-W^+_\mu(H+iG) + G^+W^-_\mu(H-iG)\right]. \tag{4.4.20}$$

The third term in $\mathcal{L}_D$ must be considered in conjunction with the gauge-fixing term. We have

$$\begin{aligned}\mathcal{L}_{\phi\phi V} &= -\frac{i}{2}gW^i_\mu\left[(\partial^\mu\phi_2)^\dagger\tau^i\phi_1 - \phi_1^\dagger\tau^i\partial^\mu\phi_2\right] - \frac{i}{2}g'B_\mu\left[(\partial^\mu\phi_2)^\dagger\phi_1 - \phi_1^\dagger\partial^\mu\phi_2\right]\\ &\quad -\frac{i}{2}gW^i_\mu\left[(\partial^\mu\phi_2)^\dagger\tau^i\phi_2 - \phi_2^\dagger\tau^i\partial^\mu\phi_2\right] - \frac{i}{2}g'B_\mu\left[(\partial^\mu\phi_2)^\dagger\phi_2 - \phi_2^\dagger\partial^\mu\phi_2\right].\end{aligned} \tag{4.4.21}$$

Exploiting the fact that $\partial^\mu\phi_1 = 0$, we can integrate by parts the first row. Adding $\mathcal{L}_{GF}$, we find

$$\begin{aligned}\mathcal{L}_{\phi\phi V} + \mathcal{L}_{GF} &= -\frac{i}{2}gW^i_\mu\left[(\partial^\mu\phi_2)^\dagger\tau^i\phi_2 - \phi_2^\dagger\tau^i\partial^\mu\phi_2\right] - \frac{i}{2}g'B_\mu\left[(\partial^\mu\phi_2)^\dagger\phi_2 - \phi_2^\dagger\partial^\mu\phi_2\right]\\ &\quad +\partial^\mu W^i_\mu\left[\frac{i}{2}g(\phi_2^\dagger\tau^i\phi_1 - \phi_1^\dagger\tau^i\phi_2) + f^i(\phi)\right]\\ &\quad +\partial^\mu B_\mu\left[\frac{i}{2}g'(\phi_2^\dagger\phi_1 - \phi_1^\dagger\phi_2) + f(\phi)\right]\\ &\quad -\frac{1}{2\xi}(\partial^\mu W^i_\mu)^2 - \frac{1}{2\xi}(\partial^\mu B_\mu)^2 - \frac{1}{2}\xi f^i(\phi)f^i(\phi) - \frac{1}{2}\xi f(\phi)f(\phi).\end{aligned} \tag{4.4.22}$$

With the choices

$$f^i(\phi) = -\frac{i}{2}g(\phi_2^\dagger\tau^i\phi_1 - \phi_1^\dagger\tau^i\phi_2) \tag{4.4.23}$$
$$f(\phi) = -\frac{i}{2}g'(\phi_2^\dagger\phi_1 - \phi_1^\dagger\phi_2) \tag{4.4.24}$$

the mixing between vector bosons and scalars disappears, and we remain with

$$\begin{aligned}\mathcal{L}_{\phi\phi V} + \mathcal{L}_{GF} &= -\frac{i}{2}gW^+_\mu\left[(H+iG)\partial^\mu G^- - G^-\partial^\mu(H+iG)\right]\\ &\quad +\frac{i}{2}gW^-_\mu\left[(H-iG)\partial^\mu G^+ - G^+\partial^\mu(H-iG)\right]\\ &\quad -\frac{i}{2}\left[2g\sin\theta_W A^\mu + (g\cos\theta_W - g'\sin\theta_W)Z^\mu\right](G^+\partial_\mu G^- - G^-\partial_\mu G^+)\\ &\quad -\frac{1}{2}(g\cos\theta_W + g'\sin\theta_W)Z^\mu(G\partial_\mu H - H\partial_\mu G)\\ &\quad -\frac{1}{2\xi}(\partial^\mu W^i_\mu)^2 - \frac{1}{2\xi}(\partial^\mu B_\mu)^2 - \xi m_W^2 G^+G^- - \frac{1}{2}\xi m_Z^2 G^2.\end{aligned} \tag{4.4.25}$$

We see that the would-be Goldstone bosons $G^\pm$ and G have acquired squared masses equal to ξm_W^2 and ξm_Z^2, respectively, as is necessary in order to cancel the unphysical singularities in the vector boson propagators. These masses vanish in the Landau gauge, $\xi = 0$.

The last term to be considered is the scalar potential $V(\phi)$. After some algebra, we find

$$V(\phi) = \frac{1}{2} m_H^2 \left[H + \frac{H^2 + 2G^+G^- + G^2}{2v} \right]^2 , \tag{4.4.26}$$

where

$$m_H^2 = 2\lambda v^2. \tag{4.4.27}$$

We consider now the interaction between fermions and scalars. From eqs. (2.2.43-2.2.46) and the definition in eq. (2.2.52), we get

$$\begin{aligned} \mathcal{L}_Y^{hadr} &= -G^+ \left(\overline{u}_L V h_D d_R - \overline{u}_R h_U V d_L \right) - G^- \left(\overline{d}_R h_D V^\dagger u_L - \overline{d}_L V^\dagger h_U u_R \right) \\ & -\frac{1}{\sqrt{2}} (v + H) \left(\overline{d} h_D d + \overline{u} h_U u \right) - \frac{iG}{\sqrt{2}} \left(\overline{d} h_D \gamma_5 d - \overline{u} h_U \gamma_5 u \right) , \end{aligned} \tag{4.4.28}$$

and

$$\mathcal{L}_Y^{lept} = -\frac{1}{\sqrt{2}} (v + H) \overline{e} h_L e - G^+ \overline{\nu} h_L e_R - G^- \overline{e}_R h_L \nu , \tag{4.4.29}$$

where sums over generation indices are understood.

4.5 Dimensional regularization

A convenient way of regularizing divergent integrals, like those appearing when computing loop diagrams in perturbation theory, is that of modifying the dimension of the integration space (space-time in our case): the integral of $1/(k^2-m^2)^2$ is logarithmically divergent at large momenta in four-dimensional space-time, while it would be convergent if space-time dimensions are lowered to 3, for example. More generally, one computes the integral in a d-dimensional space-time, with d chosen in such a way that the integral converges, and then continues analitycally the result in the complex d plane. Divergences will therefore appear as poles in $d-4$. Dimensional regularization is particularly useful because it preserves Lorentz invariance and gauge invariance of the theory.

In the following, I will show how to compute ultraviolet-divergent loop integrals in dimensional regularization. After Feynman reduction of the denominators and appropriate shifts in the loop variable, loop integrals can be reduced to the form

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu_1} \dots q^{\mu_k}}{(q^2-m^2+i\eta)^n}, \tag{4.5.1}$$

where k is an even integer and m^2 is a function of external momenta, masses, and Feynman parameters. For $k=0,2,4$ we find

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2-m^2+i\eta)^n} = (-1)^n \frac{i(4\pi)^\epsilon}{(4\pi)^2} \frac{\Gamma(n-2+\epsilon)}{\Gamma(n)} (m^2)^{-(n-2+\epsilon)} \tag{4.5.2}$$

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2-m^2+i\eta)^n} = (-1)^{n-1} \frac{i(4\pi)^\epsilon}{2(4\pi)^2} \frac{\Gamma(n-3+\epsilon)}{\Gamma(n)} (m^2)^{-(n-3+\epsilon)} g^{\mu\nu} \tag{4.5.3}$$

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu q^\rho q^\sigma}{(q^2-m^2+i\eta)^n} = (-1)^n \frac{i(4\pi)^\epsilon}{4(4\pi)^2} \frac{\Gamma(n-4+\epsilon)}{\Gamma(n)} (m^2)^{-(n-4+\epsilon)} \times (g^{\mu\nu} g^{\rho\sigma} + g^{\nu\rho} g^{\mu\sigma} + g^{\mu\rho} g^{\nu\sigma}), \tag{4.5.4}$$

where we have set, as usual,

$$d = 4 - 2\epsilon. \tag{4.5.5}$$

The Euler Γ function is defined by

$$\Gamma(z) = \int_0^{+\infty} dt\, e^{-t} t^{z-1}. \tag{4.5.6}$$

The properties

$$\Gamma(z+1) = z\Gamma(z); \qquad \Gamma(1) = 1; \qquad \Gamma(1/2) = \sqrt{\pi} \tag{4.5.7}$$

follow from the definition. Furthermore, it can be shown that $\Gamma(z)$ is analytic in the whole compex plane z, except when z is 0 or a negative integer, where it has simple poles. One finds

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon) \right], \tag{4.5.8}$$

where

$$\psi(s) = \frac{d}{ds} \log \Gamma(s) \tag{4.5.9}$$

and

$$\begin{aligned} \psi(n+1) &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma, \\ \psi(1) &= -\gamma = -0.5772\ldots \end{aligned} \tag{4.5.10}$$

We now compute explicitly the integral in eq. (4.5.2). Equations (4.5.3,4.5.4) (and similar formulae with higher powers of q in the numerator) can be obtained by shifting $q \to q + k$ and taking derivatives with respect to k at $k = 0$. By virtue of the analiticity properties of the integrand in the complex q_0 plane, the q_0 integral along the closed path C shown in fig. 4.1 is

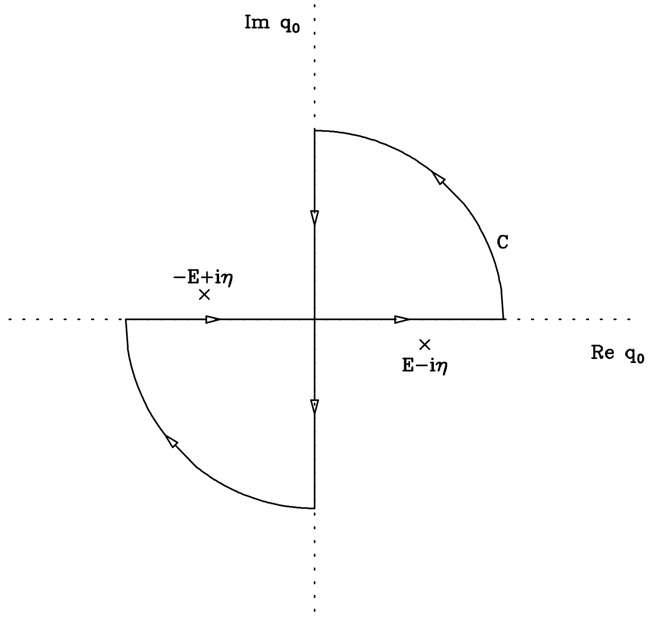

Figure 4.1: *Integration in the complex q_0 plane. Crosses indicate the singularities of the Feynman integrands at $q_0 = \pm(E - i\eta)$, with $E = \sqrt{\vec{q}^2 + m^2}$.*

equal to zero. We have therefore

$$\int_{-\infty}^{+\infty} dq_0 \, \frac{1}{(q_0^2 - \vec{q}^2 - m^2 + i\eta)^n} + \int_{+i\infty}^{-i\infty} dq_0 \, \frac{1}{(q_0^2 - \vec{q}^2 - m^2 + i\eta)^n} = 0. \tag{4.5.11}$$

With the variable change $q_0 = iq_4$ in the second term of eq. (4.5.11), we find

$$\int_{-\infty}^{+\infty} dq_0 \, \frac{1}{(q_0^2 - \vec{q}^2 - m^2 + i\eta)^n} = i(-1)^n \int_{-\infty}^{+\infty} dq_4 \, \frac{1}{(q_4^2 + \vec{q}^2 + m^2)^n} \tag{4.5.12}$$

Notice that the $+i\eta$ prescription is now immaterial, since the integration is performed along the imaginary axis. We have therefore

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2 + i\eta)^n} = i(-1)^n \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^n}, \tag{4.5.13}$$

where q in the r.h.s. is a vector in a 4-dimensional Euclidean space. We first observe that the integrand does not depend on angular variables, which can therefore be integrated directly. The integral over the d-dimensional solid angle can be obtained in the following way. We have

$$\int d^d q \, e^{-q^2} = \int d\Omega_d \int_0^{+\infty} dq \, q^{d-1} e^{-q^2} = \frac{1}{2} \int_0^{+\infty} dq^2 \, (q^2)^{d/2-1} e^{-q^2} = \frac{1}{2}\Gamma(d/2), \tag{4.5.14}$$

where we have used polar coordinates and the definition of $\Gamma(z)$. On the other hand, the usual gaussian integration formula gives

$$\int d^d q \, e^{-q^2} = \pi^{d/2}. \tag{4.5.15}$$

Thus,

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{4.5.16}$$

For $d = 2, 3$ the familiar results $\int d\Omega_2 = 2\pi, \int d\Omega_3 = 4\pi$ are recovered. Using this result, we have

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^n} = \frac{1}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{2} \int_0^{+\infty} dq^2 \frac{(q^2)^{\frac{d-2}{2}}}{(q^2 + m^2)^n}. \tag{4.5.17}$$

The integral can be performed with the change of integration variable

$$x = \frac{m^2}{q^2 + m^2}, \tag{4.5.18}$$

which gives

$$\begin{aligned} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{(m^2)^{-n+d/2}}{\Gamma(d/2)} \int_0^1 dx \, x^{n-d/2-1} (1-x)^{d/2-1} \\ &= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} (m^2)^{-n+d/2}, \end{aligned} \tag{4.5.19}$$

where we have used

$$\int_0^1 dx \, x^{a-1} (1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{4.5.20}$$

By replacing $d = 4 - 2\epsilon$, we finally obtain

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2 + i\eta)^n} = i(-1)^n \frac{(4\pi)^\epsilon}{(4\pi)^2} \frac{\Gamma(n - 2 + \epsilon)}{\Gamma(n)} (m^2)^{-(n-2+\epsilon)}, \tag{4.5.21}$$

which is the announced result. Notice in particular that the integral vanishes when $m^2 = 0$. This happens, for example, when one computes on-shell amplitudes in a massless theory.

The use of dimensional regularization poses some special problems in calculations where the γ_5 matrix is involved. In fact, γ_5 (or equivalently the antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$) is a quantity whose definition is strictly connected to the fact that space-time is four-dimensional, and a definition in d dimensions requires special care. It is tempting to define γ_5 simply by requiring that its four-dimensional properties

$$\gamma_5^2 = I; \quad \{\gamma_5, \gamma_\mu\} = 0 \tag{4.5.22}$$

hold in d dimensions as well. It is easy to prove that this assumption, together with the circular property of the trace operator, leads to inconsistent results. To see this important fact explicitly, consider the trace of γ_5 times an even number of γ matrices:

$$T = \text{Tr}\ \gamma_5 \gamma_{\mu_1} \ldots \gamma_{\mu_{2n}}. \tag{4.5.23}$$

We can use the anticommutation rules $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ to bring, for example, γ_{μ_1} at the right of the product; this requires $2n-1$ steps, and at each step a trace with $2n-2$ γ matrices appears. We denote by C_{2n-1} the sum of such terms. At the end of the procedure, using the circularity property of the trace and eq. (4.5.22), the trace can brought to its original form, and we get

$$T = T + C_{2n-1} \tag{4.5.24}$$

or

$$C_{2n-1} = 0. \tag{4.5.25}$$

For $n = 1$ eq. (4.5.25) gives

$$g_{\mu_1\mu_2} \text{Tr}\ \gamma_5 = 0 \tag{4.5.26}$$

and, for n=2,

$$g_{\mu_1\mu_2} \text{Tr}\ \gamma_5\gamma_{\mu_3}\gamma_{\mu_4} - g_{\mu_1\mu_3} \text{Tr}\ \gamma_5\gamma_{\mu_2}\gamma_{\mu_4} + g_{\mu_1\mu_4} \text{Tr}\ \gamma_5\gamma_{\mu_2}\gamma_{\mu_3} = 0. \tag{4.5.27}$$

Using eq. (4.5.26), eq. (4.5.27) implies

$$(d-2)\text{Tr}\ \gamma_5\gamma_{\mu_1}\gamma_{\mu_2} = 0. \tag{4.5.28}$$

Repeating the same procedure for $n = 3$ one gets

$$(d-2)(d-4)\text{Tr}\ \gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4} = 0. \tag{4.5.29}$$

For $d = 4$, eq. (4.5.29) is satisfied for any value of Tr $\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}$, which in fact is non-zero (and proportional to the axial current anomalous term, by the way); however, if we require eq. (4.5.29) to hold for any value of d, then we are forced to conclude that

$$\text{Tr}\ \gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4} = 0 \tag{4.5.30}$$

which is manifestly an inconsistent result, since it does not give the correct answer when d tends to 4. In particular, one would conclude that there is no axial current anomaly! We conclude that the definition of γ_5 cannot be based on eq. (4.5.22).

The correct way to define γ_5 in dimensional regularization is the following. We decompose all γ matrices into a four-dimensional and an extra-dimensional component:

$$\gamma_\mu = \bar{\gamma}_\mu + \hat{\gamma}_\mu, \tag{4.5.31}$$

where $\bar{\gamma}_\mu$ is non-zero only when μ takes the ordinary values 0,1,2,3 and $\hat{\gamma}_\mu$ vanishes in the ordinary dimensions. Correspondingly, the matrix tensor $g^{\mu\nu}$ has a four-dimensional and an extra-dimensional part,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \hat{g}^{\mu\nu}; \tag{4.5.32}$$

mixed components obviously vanish. The anticommutation relations become

$$\{\bar{\gamma}_\mu, \bar{\gamma}_\nu\} = 2\bar{g}_{\mu\nu}; \quad \{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\hat{g}_{\mu\nu}; \quad \{\bar{\gamma}_\mu, \hat{\gamma}_\nu\} = 0. \tag{4.5.33}$$

Then, we simply define γ_5 as in four dimensions, that is

$$\gamma_5 = i\bar{\gamma}_0\bar{\gamma}_2\bar{\gamma}_2\bar{\gamma}_3. \tag{4.5.34}$$

It is easy to check that the definition (4.5.34) implies

$$\{\gamma_5, \bar{\gamma}_\mu\} = 0; \quad [\gamma_5, \hat{\gamma}_\mu] = 0, \tag{4.5.35}$$

or, in a more compact form,

$$\{\gamma_5, \gamma_\mu\} = 2\gamma_5\hat{\gamma}_\mu. \tag{4.5.36}$$

The identities

$$\mathrm{Tr}\ \gamma_5 = 0; \quad \mathrm{Tr}\ \gamma_5\gamma_\mu\gamma_\nu = 0 \tag{4.5.37}$$

can be shown to hold, regardless of the value of d (this result is nontrivial; it can be obtained by the same way of reasoning that leads to eqs. (4.5.26) and (4.5.27). Prove it as an exercise). Furthermore, one sees immediately that the quantity

$$\mathrm{Tr}\ \gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4} \tag{4.5.38}$$

vanishes if at least one of the indices has a value in the extra dimensions. We have therefore

$$\mathrm{Tr}\ \gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4} = \mathrm{Tr}\ \gamma_5\bar{\gamma}_{\mu_1}\bar{\gamma}_{\mu_2}\bar{\gamma}_{\mu_3}\bar{\gamma}_{\mu_4} = 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4}, \tag{4.5.39}$$

which is the correct four-dimensional result.

The use of the definition (4.5.34) requires special attention, because it introduces an explicit violation of chiral invariance, which must therefore be restored by means of finite renormalization. I will not discuss this point in detail here.

In the following, I will show that the computation of the axial current anomaly, performed in sect. 3.3 in the Pauli-Villars regularization scheme, can also be performed in dimensional regularization. I will present the computation in the massless case; the extension to massive fermions is straightforward. From eq. (3.3.14) we have

$$(k_1 + k_2)_\rho T_1^{\mu\nu\rho} = -\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\ \gamma_5(\not{k} - \not{k}_2)\gamma^\nu\not{k}\gamma^\mu(\not{k} + \not{k}_1)(\not{k}_1 + \not{k}_2)}{k^2(k - k_2)^2(k + k_1)^2}, \tag{4.5.40}$$

where the integral is made convergent by dimensional regularization. The numerator of the integrand contains terms which are linear, quadratic or cubic in the loop momentum k. The linear term is convergent, and it gives a vanishing contribution:

$$\mathrm{Tr}\ \gamma_5 \not{k}_2 \gamma^\nu \not{k} \gamma^\mu \not{k}_1 (\not{k}_1 + \not{k}_2) = 0 \tag{4.5.41}$$

because $k_1^2 = k_2^2 = 0$.

The quadratic term requires more work. We have

$$\begin{aligned} &-\mathrm{Tr}\ \gamma_5 \not{k} \gamma^\nu \not{k} \gamma^\mu \not{k}_1 (\not{k}_1 + \not{k}_2) + \mathrm{Tr}\ \gamma_5 \not{k}_2 \gamma^\nu \not{k} \gamma^\mu \not{k} (\not{k}_1 + \not{k}_2) \\ &= 2k^2 \mathrm{Tr}\ \gamma_5 \gamma^\nu \gamma^\mu \not{k}_1 \not{k}_2 - 2k^\nu k^\alpha \mathrm{Tr}\ \gamma_5 \gamma^\alpha \gamma^\mu \not{k}_1 \not{k}_2 - 2k^\alpha k^\mu \mathrm{Tr}\ \gamma_5 \gamma^\nu \gamma^\alpha \not{k}_1 \not{k}_2. \end{aligned} \tag{4.5.42}$$

The first term contributes to the final result with

$$2I \mathrm{Tr}\ \gamma_5 \gamma^\nu \gamma^\mu \not{k}_1 \not{k}_2, \tag{4.5.43}$$

where

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k - k_2)^2 (k + k_1)^2}. \tag{4.5.44}$$

The second and third terms in eq. (4.5.42) involve the integral

$$I^{\nu\alpha} = \int \frac{d^d k}{(2\pi)^d} \frac{k^\nu k^\alpha}{k^2 (k - k_2)^2 (k + k_1)^2}, \tag{4.5.45}$$

which can be written in the form

$$I^{\nu\alpha} = A g^{\nu\alpha} + B(k_1^\nu k_2^\alpha + k_2^\nu k_1^\alpha) + C(k_1^\nu k_1^\alpha + k_2^\nu k_2^\alpha), \tag{4.5.46}$$

exploiting symmetry under $k_1 \leftrightarrow k_2$ and $\nu \leftrightarrow \alpha$. It is clear from eq. (4.5.42) that only the term $A g^{\nu\alpha}$ contributes to the result. In order to compute A, we observe that eq. (4.5.46) gives

$$\begin{aligned} I_\nu^\nu &\equiv I = dA + 2k_1 k_2 B \\ k_1^\nu k_2^\alpha I_{\nu\alpha} &= k_1 k_2 A + (k_1 k_2)^2 B. \end{aligned} \tag{4.5.47}$$

Now, using the identities $k_1 k = ((k + k_1)^2 - k^2)/2$, $k_2 k = (k^2 - (k - k_2)^2)/2$, one can show that

$$k_1^\nu k_2^\alpha I_{\nu\alpha} = -\frac{1}{4} J, \tag{4.5.48}$$

where

$$J = \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k - k_2)^2 (k + k_1)^2}. \tag{4.5.49}$$

Solving the system (4.5.47), one gets

$$A = \frac{1}{d - 2} \left(I + \frac{J}{2k_1 k_2} \right). \tag{4.5.50}$$

Finally, we come to the cubic term:

$$-\mathrm{Tr}\ \gamma_5 \not{k}\gamma^\nu \not{k}\gamma^\mu \not{k}(\not{k}_1 + \not{k}_2) = k^2 \mathrm{Tr}\ \gamma_5 \not{k}\gamma^\nu \gamma^\mu (\not{k}_1 + \not{k}_2), \tag{4.5.51}$$

since the anticommutator term gives zero contribution because of antisymmetry. We must therefore compute

$$I^\alpha = \int \frac{d^d k}{(2\pi)^d} \frac{k^\alpha}{(k-k_2)^2 (k+k_1)^2} = D(k_1^\alpha - k_2^\alpha), \tag{4.5.52}$$

and taking the product $I_\alpha k_1^\alpha$ one easily obtains

$$D = \frac{J}{2k_1 k_2}. \tag{4.5.53}$$

Collecting all our results, we finally obtain

$$(k_1 + k_2)_\rho T^{\mu\nu\rho} = -4\, \frac{d-4}{d-2} \left(I + \frac{J}{2k_1 k_2} \right)\ \mathrm{Tr}\ \gamma_5 \gamma^\nu \gamma^\mu \not{k}_1 \not{k}_2, \tag{4.5.54}$$

where a factor of 2 has been inserted to take into account the contribution of $T_2^{\mu\nu\rho}$. The final result is ultraviolet-finite: indeed, in dimensional regularization at one loop ultraviolet divergences manifest themselves as simple poles in $d-4$, and there is a $d-4$ factor in front of the divergent integrals. It is now easy to compute $(d-4)I$ and $(d-4)J$ for $d=4$ with the help of the formulae obtained earlier in this Appendix, and recover the result of eq. (3.3.33).