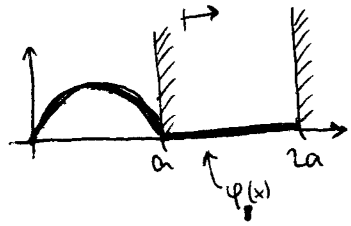
(es. 6B) "Expanding Box": se siamo nel ground state e istantaneamente la buca raddoppia, cioè $a \mapsto 2a$, calcolare $Prob(\psi \mapsto \tilde{\varphi}_2)$ dove $\tilde{\varphi}_n(x)$ sono gli autostati per la buca larga $2a$.



$$\psi = \varphi_1(x) = \sqrt{\frac{2}{a}}\ \mathrm{sen}\left(\frac{\pi x}{a}\right) \qquad \text{ed è } \equiv 0 \text{ se } x > a$$

$$\tilde{\varphi}_n(x) = \sqrt{\frac{2}{2a}}\ \mathrm{sen}\left(\frac{n\pi x}{2a}\right) = \sqrt{\frac{1}{a}}\ \mathrm{sen}\left(\frac{n\pi x}{2a}\right)$$

Se sviluppo $\varphi_p(x)$ sui $\tilde{\varphi}_n(x)$, ottengo:

$$\varphi_p(x) = \sum_n c_n \tilde{\varphi}_n(x) \quad , \quad \text{dove}$$

$$c_n = \int_0^{2a} \varphi_p(x)\, \tilde{\varphi}_n^*(x)\, dx \overset{\varphi_p(x) \equiv 0 \text{ se } x>a}{=} \int_0^a \varphi_p(x)\, \tilde{\varphi}_n^*(x)\, dx$$

[Formalmente, in notaz. di Dirac: $|\varphi_p\rangle = \sum_n c_n |\tilde{\varphi}_n\rangle = \sum_n |\tilde{\varphi}_n\rangle\langle\tilde{\varphi}_n|\varphi_p\rangle$

$$\Rightarrow c_n = \langle\tilde{\varphi}_n|\varphi_p\rangle = \int dx\, \langle\tilde{\varphi}_n|x\rangle\langle x|\varphi_p\rangle = \int dx\, \varphi_p(x)\, \tilde{\varphi}_n^*(x)$$]

$$\text{Prob}(\psi \mapsto \tilde{\varphi}_2) \overset{\text{def.}}{=} |\langle\tilde{\varphi}_2|\psi\rangle|^2 = |c_2|^2$$

$$c_2 = \int_0^a \sqrt{\frac{2}{a}}\, \text{sen}\left(\frac{\pi x}{a}\right) \sqrt{\frac{1}{a}}\, \text{sen}\left(\frac{\pi x}{a}\right) =$$

$$= \frac{\sqrt{2}}{a} \int_0^a \text{sen}^2\left(\frac{\pi x}{a}\right) dx = \frac{\sqrt{2}}{a} \int_0^a \frac{1 - \cos\left(\frac{2\pi x}{a}\right)}{2}\, dx =$$

$$= \frac{\sqrt{2}}{a}\, \frac{a}{2} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \text{Prob}(\psi \mapsto \tilde{\varphi}_2) = \frac{1}{2}$$

Se volessimo calcolare c_1:

$$c_1 = \int_0^a \sqrt{\frac{2}{a}}\, \text{sen}\left(\frac{\pi x}{a}\right) \sqrt{\frac{1}{a}}\, \text{sen}\left(\frac{\pi x}{2a}\right) =$$

$$= \frac{\sqrt{2}}{a} \int_0^a \frac{1}{2}\left[\cos\frac{\pi x}{2a} - \cos\frac{3\pi x}{2a}\right] =$$

$$= \frac{\sqrt{2}}{2a}\left[\frac{\text{sen}\,\frac{\pi x}{2a}}{\pi/2a}\Bigg|_0^a - \frac{\text{sen}\,\frac{3\pi x}{2a}}{\frac{3\pi}{2a}}\Bigg|_0^a\right] = \frac{\sqrt{2}}{2a}\left[\frac{2a}{\pi} - \frac{2a}{3\pi}(-1)\right] = \frac{4\sqrt{2}}{3\pi}$$

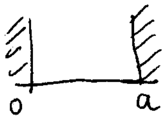
$$\Rightarrow \text{Prob}(\psi \mapsto \tilde{\varphi}_1) = \left(\frac{4\sqrt{2}}{3\pi}\right)^2$$

DA MATHEMATICA:

$$\int_0^a \text{sen}\left(\frac{\pi x}{a}\right) \text{sen}\left(\frac{n\pi x}{2a}\right) = \frac{4a\, \text{sen}\left(\frac{n\pi}{2}\right)}{4\pi - n^2\pi}$$

Es. 4-2

$$\psi(x,t=0) = \frac{1}{\sqrt{2}}\left[\varphi_1(x) + \varphi_2(x)\right]$$



con $\varphi_n(x) = \sqrt{\frac{2}{a}}\ \mathrm{sen}\left(\frac{n\pi x}{a}\right)$, $E_n = n^2 \frac{\pi^2\hbar^2}{2ma^2}$, $\omega_n = \frac{E_n}{\hbar}$, $n=1,2,3,\dots$

a) $$\psi(x,t) = \frac{1}{\sqrt{2}}\left[e^{-iE_1t/\hbar}\varphi_1(x) + e^{-iE_2t/\hbar}\varphi_2(x)\right] = \frac{1}{\sqrt{2}}\left[e^{-i\omega_1 t}\varphi_1(x) + e^{-i\omega_2 t}\varphi_2(x)\right]$$

$$\mathrm{Prob}(H=E_1,t) = \left|\frac{e^{-i\omega_1 t}}{\sqrt{2}}\right|^2 = 1/2 \qquad \mathrm{Prob}(H=E_2,t) = 1/2$$

b) Notaz. di Dirac: $\varphi_n(x) = \langle x|n\rangle \Rightarrow \hat{H}|n\rangle = E_n|n\rangle$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega_1 t}|1\rangle + e^{-i\omega_2 t}|2\rangle\right]$$

$$\langle H\rangle(t) = \langle\psi(t)|H|\psi(t)\rangle =$$

$$= \frac{1}{\sqrt{2}}\left[e^{+i\omega_1 t}\langle 1| + e^{+i\omega_2 t}\langle 2|\right] H \frac{1}{\sqrt{2}}\left[e^{-i\omega_1 t}|1\rangle + e^{-i\omega_2 t}|2\rangle\right] =$$

$$= \frac{1}{2}\left[\qquad\right]\left[E_1 e^{-i\omega_1 t}|1\rangle + E_2 e^{-i\omega_2 t}|2\rangle\right] = \qquad \langle 1|2\rangle = 0$$

$$= \frac{1}{2}(E_1+E_2) = \frac{5}{2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

Definiamo per comodità $\Delta\omega = \omega_2 - \omega_1$.

$$\langle x\rangle(t) = \int_0^a dx\ x\ |\psi(x,t)|^2$$

NB dimostraz. formale dell'identità precedente:

$\langle x\rangle(t) \overset{def}{=} \langle\psi(t)|\hat{x}|\psi(t)\rangle =$ $\qquad \int dx'|x'\rangle\langle x'| = \mathbb{1}$ e inserisco 2 completezze

$$= \int dx'dx'' \underbrace{\langle\psi(t)|x'\rangle}_{\psi(x',t)^*}\underbrace{\langle x'|\hat{x}|x''\rangle}_{x''\delta(x'-x'')}\underbrace{\langle x''|\psi(t)\rangle}_{\psi(x'',t)}$$

$$= \int dx'dx''\ \psi(x',t)^*\ \psi(x'',t)\ x''\ \delta(x'-x'')$$

$$= \int dx'\ x'\ |\psi(x',t)|^2$$

$$= \int_0^a dx\ x\ \frac{1}{2}\left[|\varphi_1(x)|^2 + |\varphi_2(x)|^2 + \varphi_1(x)\varphi_2(x)\left(e^{-i(\omega_1-\omega_2)t} + e^{+i(\omega_1-\omega_2)t}\right)\right] =$$

$$= \frac{1}{2}\int_0^a dx\left\{x|\varphi_1(x)|^2 + x|\varphi_2(x)|^2 + 2\cos(\Delta\omega t)\ x\ \varphi_1(x)\varphi_2(x)\right\} =$$

$$= \frac{1}{2}\left\{ \frac{a}{2} + \frac{a}{2} + 2\cos(\Delta\omega t)\int_0^a dx \; x \left(\frac{2}{a}\right) \operatorname{sen}\left(\frac{\pi x}{a}\right) \operatorname{sen}\left(\frac{2\pi x}{a}\right) \right\} =$$

$\operatorname{sen}\alpha \operatorname{sen}\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

$$= \frac{1}{2}\left\{ a + \frac{4\cos(\Delta\omega t)}{a}\int_0^a dx \; x \; \frac{1}{2}\left[\cos\frac{\pi x}{a} - \cos\frac{3\pi x}{a}\right]\right\}$$

$$= \frac{1}{2}\left\{ a + \frac{2\cos(\Delta\omega t)}{a}\left[\int_0^a dx \; x \cos\frac{\pi x}{a} - \int_0^a dx \; x \cos\frac{3\pi x}{a}\right]\right\}$$

$$\int_0^a dx \; x \cos\frac{\pi x}{a} = \left.\frac{x \operatorname{sen}\frac{\pi x}{a}}{\pi/a}\right|_0^a - \int_0^a \frac{\operatorname{sen}\frac{\pi x}{a}}{\pi/a} =$$

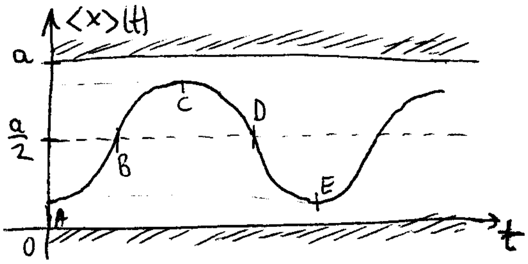
$$= \emptyset + \frac{a}{\pi}\left.\frac{\cos\frac{\pi x}{a}}{\pi/a}\right|_0^a = -2\left(\frac{a}{\pi}\right)^2$$

$$\int_0^a dx \; x \cos\frac{3\pi x}{a} = \text{---} = -2\left(\frac{a}{3\pi}\right)^2$$

$$= \frac{1}{2}\left\{ a + \frac{2\cos(\Delta\omega t)}{a}\left[-2\left(\frac{a}{\pi}\right)^2 + 2\frac{a^2}{9\pi^2}\right]\right\}$$

$$\Rightarrow \langle x\rangle(t) = \frac{a}{2} - \frac{16a}{9\pi^2}\cos(\Delta\omega t)$$

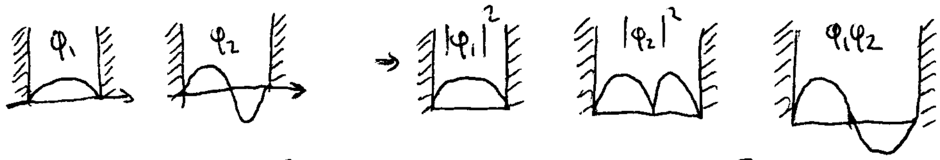
Osservazioni

1) 

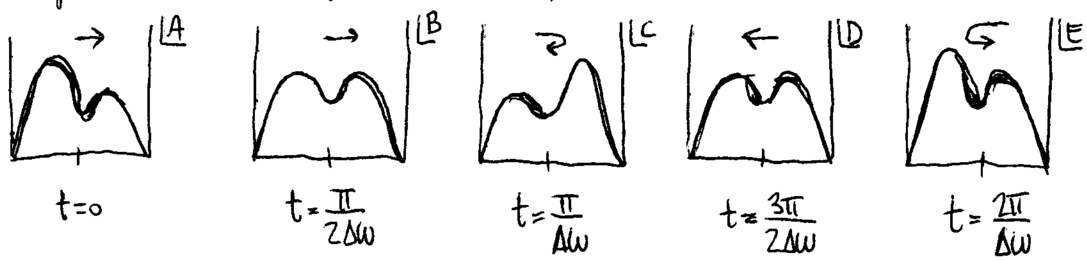
$\Rightarrow$ In MQ il centro del pacchetto NON TOCCA MAI LE PARETI, mentre classicamente sì.

2) Evoluz. temporale del pacchetto, graficamente

$$|\psi(x,t)|^2 = \frac{|\varphi_1(x)|^2}{2} + \frac{|\varphi_2(x)|^2}{2} + \varphi_1(x)\varphi_2(x)\cos(\Delta\omega t)$$



Disegniamo $|\psi(x,t)|^2$ per vari istanti, usando ⤵ :



ES. IV [es. 9] W.S. es. 202

Atomo idrogeno: $\Psi(\vec{x}, t=0) = \frac{1}{\sqrt{10}}\left[2\varphi_{100} + \varphi_{210} + \sqrt{2}\,\varphi_{211} + \sqrt{3}\,\varphi_{21-1}\right]$

$\left[\varphi_{n\ell m} = R_{n\ell} Y_{\ell m} \quad , \; E_n = \frac{E_1}{n^2} , \quad E_1 = -\frac{me^4}{2\hbar^2} = -13.6\text{ eV}\right]$

a) calcolare $\langle E \rangle$

b) calcolare $\Psi(\vec{x},t)$ e Prob $(\ell=1, m=1 \; ; t)$

c) Approx. e ottenere Prob $(r \leq 10^{-10}$ cm $; t=0)$

a) Usiamo formalismo di Dirac:

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{10}}\left[2|100\rangle + |210\rangle + \sqrt{2}|211\rangle + \sqrt{3}|21-1\rangle\right]$$

$$\langle E\rangle = \langle\Psi(t=0)|\hat{H}|\Psi(t=0)\rangle =$$

$$= \frac{1}{10}\left[2\langle100| + \langle210| + \sqrt{2}\langle211| + \sqrt{3}\langle21-1|\right]\left[2E_1|100\rangle + E_2\left(|210\rangle + \sqrt{2}|211\rangle + \sqrt{3}|21-1\rangle\right)\right]$$

$$= \frac{1}{10}\left[4E_1 + E_2\left(1 + (\sqrt{2})^2 + (\sqrt{3})^2\right)\right] =$$

$$= \frac{1}{10}\left[4E_1 + \frac{E_1}{4}(1+2+3)\right] = \frac{E_1}{10}\left(4+\frac{3}{2}\right) = \frac{11}{20}E_1 \simeq -7\,eV$$

[Notare che corrisponde esattamente ad usare la formula $\langle E\rangle = \sum_m |c_m|^2 E_m$ con $|\psi\rangle = \sum_m c_m|\varphi_m\rangle$]

b) $$|\Psi(t)\rangle = \frac{1}{\sqrt{10}}\left[2e^{-iE_1t/\hbar}|100\rangle + e^{-iE_2t/\hbar}\left(|210\rangle + \sqrt{2}|211\rangle + \sqrt{3}|21-1\rangle\right)\right]$$

$$\text{Prob}(\ell=1, m=1; t) = \sum_n |\langle n11|\Psi(t)\rangle|^2 =$$ uso ortonormalità

$$= |\langle211|\Psi(t)\rangle|^2 = \left|\frac{\sqrt{2}}{\sqrt{10}}e^{-iE_2t/\hbar}\right|^2 = \frac{1}{5}$$

c) $$\text{Prob}(r\leq R; t=0) = \int_0^R r^2 dr\, d\Omega\, |\Psi(\vec{x}, t=0)|^2 =$$ $\varphi_{n\ell m} = R_{n\ell}Y_{\ell m}$

$$= \int_0^R r^2 dr\, d\Omega \sum_{n\ell m} c^*_{n\ell m} R^*_{n\ell} Y^*_{\ell m} \sum_{n'\ell'm'} c_{n'\ell'm'} R_{n'\ell'} Y_{\ell'm'}$$

$$= \sum_{n\ell m}\sum_{n'\ell'm'} c^*_{n\ell m} c_{n'\ell'm'} \int_0^R dr\, r^2 R^*_{n\ell} R_{n'\ell'} \int d\Omega\, Y^*_{\ell m} Y_{\ell'm'} =$$

$$= \sum_{n\ell m}\sum_{n'\ell'm'} c^*_{n\ell m} c_{n'\ell'm'} \int_0^R dr\, r^2 R^*_{n\ell} R_{n'\ell'}\, \delta_{\ell\ell'}\delta_{mm'}$$

$$= \sum_{nn'\ell m} c^*_{n\ell m} c_{n'\ell m} \int_0^R dr\, r^2 R^*_{n\ell} R_{n'\ell}$$

Per $\ell=0$, ho solo $n=1$
Per $\ell=1$, ho solo $n=2$

$$\Rightarrow \dots = \int_0^R dr\, r^2 \left[|R_{10}|^2\, |C_{100}|^2 + |R_{21}|^2 \left(|C_{210}|^2 + |C_{211}|^2 + |C_{21-1}|^2 \right) \right]$$

$$= \int_0^R dr\, r^2 \left[\frac{4}{10} |R_{10}|^2 + \frac{6}{10} |R_{21}|^2 \right]$$

$$|R_{10}|^2 = \frac{4}{a_0^3}\, e^{-2r/a_0} \qquad |R_{21}|^2 = \frac{r^2}{24 a_0^5}\, e^{-r/a_0}$$

$$\mathrm{Prob}\,(r \leq R) = \frac{1}{10} \int_0^R dr \left[\frac{16}{a_0} \left(\frac{r}{a_0}\right)^2 e^{-2r/a_0} + \frac{6}{24 a_0} \left(\frac{r}{a_0}\right)^4 e^{-r/a_0} \right]$$

FERMARSI QUA

$$= \frac{1}{10} \int_0^{R/a_0} dx \left[16\, x^2 e^{-2x} + \frac{1}{4} x^4 e^{-x} \right] =$$

$$x < \frac{R}{a_0} \simeq \frac{10^{-10}\ \mathrm{cm}}{0.5 \times 10^{-8}\ \mathrm{cm}} \simeq 2\times$$

$$\Rightarrow e^{-2x} \simeq 1$$
$$e^{-x} \simeq 1$$

$$= \frac{1}{10} \left[16\, \frac{(R/a_0)^3}{3} + \frac{1}{4}\, \frac{(R/a_0)^5}{5} \right]$$

$$\simeq \frac{8}{15} \left(\frac{R}{a_0}\right)^3 \simeq \frac{8}{15} \cdot 8 \cdot 10^{-6} \simeq 4 \cdot 10^{-6}$$

ES. VII [es. 2] C.T. L_{III}, es. 5 (pag. 342)

Particella di massa m in una dimensione, sottoposta a

$V(x) = -fx$ [potenziale gravitazionale o carica in campo elettrico uniforme]

a) scrivere e risolvere le eq. del moto per $\langle x \rangle$ e $\langle p \rangle$

b) calcolare $(\Delta p)^2(t)$ e mostrare che è costante

c) derivare eq. di Schroedinger per $\psi(p)$
dedurne legame tra $\frac{d}{dt}|\psi(p,t)|^2$ e $\frac{d}{dp}|\psi(p,t)|^2$

$$\hat{H} = \frac{p^2}{2m} - fx$$

Risolviamo l'esercizio (a) e b)) usando il Teorema di Ehrenfest sugli operatori; quindi ci mettiamo nella rappresentazione di Heisenberg

$|\psi\rangle_H$ fisso, t.c. $|\psi\rangle_H = |\psi(t=0)\rangle_S$

$O_H(t) = e^{iHt/\hbar}\, O_H(t=0)\, e^{-iHt/\hbar}$, t.c. $O_H(t=0) = O_S$

e vale che $i\hbar \frac{d}{dt} O_H = [O_H, H] + i\hbar \frac{\partial O_H}{\partial t}$

Nel seguito tutti gli operatori sono da pensarsi nella rapp. di Heisenberg ; eq. del moto:

$$\begin{cases} i\hbar \dot{x} = [x, H] \\ i\hbar \dot{p} = [p, H] \end{cases} \quad \begin{cases} i\hbar \dot{x} = i\hbar \frac{p}{m} \\ i\hbar \dot{p} = i\hbar f \end{cases} \quad \begin{cases} \dot{x} = \frac{p}{m} \\ \dot{p} = f \end{cases}$$

Quindi:

$$\begin{cases} \hat{x}(t) = \frac{ft^2}{2m} + \frac{\hat{p}(0)}{m} t + \hat{x}(0) & (i) \\ \hat{p}(t) = ft + \hat{p}(0) & (ii) \end{cases}$$

a) Dalle eq. precedenti, prendendo i valori d'aspettazione si ottiene

$$\begin{cases} \langle x(t) \rangle = \frac{ft^2}{2m} + \frac{\langle p_0 \rangle}{m} t + \langle x_0 \rangle \\ \langle p(t) \rangle = ft + \langle p_0 \rangle \end{cases}$$

dove $\langle p_0 \rangle = \langle \hat{p}(t=0) \rangle$
$\langle x_0 \rangle = \langle \hat{x}(t=0) \rangle$

b) Per trovare $(\Delta p)^2(t)$ mi servono $\langle p^2(t) \rangle$ e $\langle p(t) \rangle^2$:

$$\langle p^2(t) \rangle = \langle \hat{p}(t)\hat{p}(t) \rangle \overset{(ii)}{=} \langle (ft + \hat{p}(0))(ft + \hat{p}(0)) \rangle =$$

$$= \langle f^2 t^2 + (\hat{p}(0))^2 + 2ft\,\hat{p}(0) \rangle =$$

$$= f^2 t^2 + \langle p_0^2 \rangle + 2ft \langle p_0 \rangle$$

dove $\langle p_0^2 \rangle = \langle p^2(t=0) \rangle$

$$\langle p(t) \rangle^2 = (ft + \langle p_0 \rangle)^2 = f^2 t^2 + \langle p_0 \rangle^2 + 2ft \langle p_0 \rangle$$

Quindi:

$$(\Delta p)^2(t) \overset{def}{=} \langle p^2(t) \rangle - \langle p(t) \rangle^2 =$$

$$= (f^2 t^2 + \langle p_0^2 \rangle + 2ft \langle p_0 \rangle) - (f^2 t^2 + \langle p_0 \rangle^2 + 2ft \langle p_0 \rangle) =$$

$$= \langle p_0^2 \rangle - \langle p_0 \rangle^2 = (\Delta p)^2 (t=0)$$

$\Rightarrow (\Delta p)^2(t)$ è costante nel tempo se $V(x) = -fx$

c) Partiamo dall' eq. di S. nello spazio delle coordinate $(\psi = \psi(x))$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - fx\psi = i\hbar\frac{d}{dt}\psi} \qquad (i)$$

Facciamo la F.T. dell' eq. precedente, ovvero:

$$-i\hbar\frac{d}{dx} \rightarrow p$$

$$x \rightarrow i\hbar\frac{d}{dp}$$

$$\psi(x) \rightarrow \psi(p)$$

Otteniamo quindi l'eq. di Schroedinger per $\psi(p)$:

$$\boxed{\frac{p^2}{2m}\psi(p) - i\hbar f\frac{d}{dp}\psi(p) = i\hbar\frac{d}{dt}\psi(p)} \qquad (ii)$$

Possiamo riscriverla come

$$\dot{\psi}(p) = -f\frac{d\psi(p)}{dp} + \frac{p^2}{2mi\hbar}\psi(p) \qquad (iii)$$

e prendendo il complesso coniugato si ha

$$\dot{\psi}^*(p) = -f\frac{d\psi^*(p)}{dp} - \frac{p^2}{2mi\hbar}\psi^*(p) \qquad (iv)$$

Adesso moltiplico (iii) per $\psi^*(p)$ e (iv) per $\psi(p)$. Sommando le due identità si ottiene

$$\frac{d}{dt}|\psi(p,t)|^2 = -f\frac{d}{dp}|\psi(p,t)|^2$$

$$\Rightarrow \boxed{|\psi(p,t)|^2 = F(p-ft)} \qquad (v)$$

dove F è una funzione che dipende ad es. dalle condizioni iniziali. Però il risultato è che $|\psi(p,t)|^2$ dipende da p e t combinati secondo $p-ft$, cioè p e t compaiono in $|\psi(p,t)|^2$ nella combinazione $p-ft$.

Conseguenze: 1) Poiché $\int dp \, |\psi(p,t)|^2 = 1$, allora segue che $\int dp' \, F(p') = 1$.

2) Dalla (V) possiamo ricavare $\langle p(t) \rangle$, infatti:

$$\langle p(t) \rangle = \int dp \; p \, |\psi(p,t)|^2 =$$

$$= \int dp \; p \, F(p - ft) = \int dp \, (p - ft) F(p - ft) + ft \int dp \; F(p - ft)$$

$$= \int dp' \; p' F(p') + ft \int dp' \, F(p') =$$

$$= (\text{cost}) + ft$$

Definendo $(\text{cost}) = \langle p_0 \rangle$, si ottiene

$$\langle p(t) \rangle = \langle p_0 \rangle + ft$$