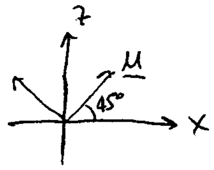
Es. VI [es. 7] C.T, F_{VI}, es. 6 (pag. 768)

Sistema con $\ell=1$ $(\vec{L}^2=2\hbar^2)$ ha la seguente H:



$$H=\frac{\omega_0}{\hbar}\left(L_u^2-L_v^2\right) \qquad L_u=\vec{u}\cdot\vec{L} \quad , L_v=\vec{v}\cdot\vec{L}$$

(cioè $\underline{u}=\frac{\vec{e}_x+\vec{e}_z}{\sqrt{2}}$ $\quad \underline{v}=\frac{-\vec{e}_x+\vec{e}_z}{\sqrt{2}}$)

a) Trovare spettro e autostati

b) Trovare $|\psi(t)\rangle$ e $Prob(L_z;t)$ se $|\psi(0)\rangle=\frac{|+\rangle-|-\rangle}{\sqrt{2}}$

c) $\langle\vec{L}\rangle(t)$

d) Esiste t t.c. se misuro L_z^2 c'è un solo valore possibile (cioè Prob =1)

a) Devo diagonalizzare H:

$|+\rangle=\begin{vmatrix}1\\0\\0\end{vmatrix}$ $\quad |-\rangle=\begin{vmatrix}0\\0\\1\end{vmatrix}$ $\quad |0\rangle=\begin{vmatrix}0\\1\\0\end{vmatrix}$ $\qquad L_z=\hbar\begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix}$ $\quad L_x=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$ $\quad L_y=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$

Calcoliamo H:

$$L_u=\vec{u}\cdot\vec{L}=\frac{1}{\sqrt{2}}(L_x+L_z) \qquad L_v=\vec{v}\cdot\vec{L}=\frac{1}{\sqrt{2}}(-L_x+L_z)$$

$$L_u^2-L_v^2=\frac{1}{2}\left[(L_x+L_z)(L_x+L_z)-(L_z-L_x)(L_z-L_x)\right]=$$

$$=\frac{1}{2}\left[\cancel{L_x^2}+\cancel{L_z^2}+L_xL_z+L_zL_x-\left(\cancel{L_z^2}+\cancel{L_x^2}-L_xL_z-L_zL_x\right)\right]=$$

$$=L_xL_z+L_zL_x=$$

$$=\frac{\hbar^2}{\sqrt{2}}\left[\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}+\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\right]=\frac{\hbar^2}{\sqrt{2}}\left[\begin{pmatrix}0&0&0\\1&0&-1\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&0\\0&-1&0\end{pmatrix}\right]$$

$$=\frac{\hbar^2}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&-1\\0&-1&0\end{pmatrix}$$

$$H=\frac{\omega_0\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&-1\\0&-1&0\end{pmatrix}$$

Spettro, cioè autovalori: $\det\begin{pmatrix}-\lambda&1&0\\1&-\lambda&-1\\0&-1&-\lambda\end{pmatrix}=0 \rightarrow -\lambda(\lambda^2-1)-1(-\lambda)=-\lambda(\lambda^2-2)=0$

$$\rightarrow \lambda=0,\pm\sqrt{2} \qquad \rightarrow E=0,\pm\hbar\omega_0$$

Autostati: $\begin{pmatrix}0&1&0\\1&0&-1\\0&-1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\lambda\begin{pmatrix}a\\b\\c\end{pmatrix} \rightarrow \begin{pmatrix}b\\a-c\\-b\end{pmatrix}=\lambda\begin{pmatrix}a\\b\\c\end{pmatrix} \rightarrow \begin{cases}b=\lambda a\\a-c=\lambda b\\-b=\lambda c\end{cases}$

$\lambda = 0 \ (E_0 = 0) \rightarrow \begin{cases} b = 0 \\ a - c = 0 \end{cases} \rightarrow |E_0\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$ $\quad [E_0 = 0]$

$\lambda = \sqrt{2} \ (E_+ = \hbar\omega_0) \rightarrow \begin{cases} b = \sqrt{2}\, a \\ a - c = \sqrt{2}\, b \\ -b = \sqrt{2}\, c \end{cases} \rightarrow |E_+\rangle = \frac{1}{2} \begin{vmatrix} 1 \\ \sqrt{2} \\ -1 \end{vmatrix}$ $\quad [E_+ = \hbar\omega_0]$

Z

$\lambda = -\sqrt{2} \ (E_- = -\hbar\omega_0) \rightarrow \begin{cases} b = -\sqrt{2}\, a \\ a - c = -\sqrt{2}\, b \\ -b = -\sqrt{2}\, c \end{cases} \rightarrow |E_-\rangle = \frac{1}{2} \begin{vmatrix} 1 \\ -\sqrt{2} \\ -1 \end{vmatrix}$ $\quad [E_- = -\hbar\omega_0]$

b) $|\psi(t=0)\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} = \frac{|E_+\rangle + |E_-\rangle}{\sqrt{2}}$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-iE_+t/\hbar} |E_+\rangle + e^{-iE_-t/\hbar} |E_-\rangle \right) =$$

$$= \frac{1}{\sqrt{2}} \left(e^{-i\omega_0 t} |E_+\rangle + e^{i\omega_0 t} |E_-\rangle \right)$$

Per calcolare Prob(L_z) devo sviluppare $|\psi(t)\rangle$ su autostati di L_z. Questo è banale x gli autostati sono $|+\rangle = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$ $|0\rangle = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$ $|-\rangle = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$.

Quindi:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left[\frac{e^{-i\omega_0 t}}{2} \begin{vmatrix} 1 \\ \sqrt{2} \\ -1 \end{vmatrix} + \frac{e^{i\omega_0 t}}{2} \begin{vmatrix} 1 \\ -\sqrt{2} \\ -1 \end{vmatrix} \right] = \frac{1}{\sqrt{2}} \begin{vmatrix} \cos\omega_0 t \\ -i\sqrt{2}\, \mathrm{sen}\,\omega_0 t \\ -\cos\omega_0 t \end{vmatrix} =$$

$$= \frac{1}{\sqrt{2}} \left[\cos\omega_0 t\, |+\rangle - i\sqrt{2}\, \mathrm{sen}\,\omega_0 t\, |0\rangle + -\cos\omega_0 t\, |-\rangle \right]$$

$$\mathrm{Prob}(L_z = \hbar\,; t) = \frac{\cos^2\omega_0 t}{2}$$

$$\mathrm{Prob}(L_z = -\hbar\,; t) = \frac{\cos^2\omega_0 t}{2}$$

$$\mathrm{Prob}(L_z = 0\,; t) = \mathrm{sen}^2\,\omega_0 t$$

c) $\langle L_z\rangle(t) = \mathrm{Prob}(L_z = \hbar)\,\hbar + \mathrm{Prob}(L_z = 0)\,0 + \mathrm{Prob}(L_z = -\hbar)\,(-\hbar) =$

$$= \hbar\,\frac{\cos^2\omega_0 t}{2} - \hbar\,\frac{\cos^2\omega_0 t}{2} = 0$$

$$\langle L_x\rangle(t) = \langle\psi(t)|L_x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\cos\omega_0 t,\ i\sqrt{2}\,\mathrm{sen}\,\omega_0 t,\ -\cos\omega_0 t\right) \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{vmatrix} \cos\omega_0 t \\ -i\sqrt{2}\,\mathrm{sen}\,\omega_0 t \\ -\cos\omega_0 t \end{vmatrix} =$$

$$= \frac{\hbar}{2\sqrt{2}} \left(\cos,\ i\sqrt{2}\,\mathrm{sen},\ -\cos\right) \begin{pmatrix} -i\sqrt{2}\,\mathrm{sen} \\ 0 \\ -i\sqrt{2}\,\mathrm{sen} \end{pmatrix} = 0$$

$$\langle L_y\rangle(t)=\frac{\hbar}{2\sqrt2}(\cos,\ i\sqrt2\,\mathrm{sen},\ -\cos)\begin{pmatrix}0&-i&0\\ i&0&-i\\ 0&i&0\end{pmatrix}\begin{pmatrix}\cos\\ -i\sqrt2\,\mathrm{sen}\\ -\cos\end{pmatrix}=$$

$$=\frac{\hbar}{2\sqrt2}(\cos,\ i\sqrt2\,\mathrm{sen},\ -\cos)\begin{pmatrix}-\sqrt2\,\mathrm{sen}\\ 2i\cos\\ \sqrt2\,\mathrm{sen}\end{pmatrix}=\frac{\hbar}{2\sqrt2}\left(-\sqrt2\,\mathrm{sen}\cos-2\sqrt2\,\mathrm{sen}\cos-\sqrt2\,\mathrm{sen}\cos\right)$$

$$=-\hbar 2\,\mathrm{sen}\,\omega_0 t\cos\omega_0 t=-\hbar\,\mathrm{sen}(2\omega_0 t)$$

d) Probabilità per L_z^2:

$$\mathrm{Prob}(L_z^2=\hbar^2;t)=\mathrm{Prob}(L_z=\hbar;t)+\mathrm{Prob}(L_z=-\hbar;t)=$$

$$=\frac{\cos^2\omega_0 t}{2}+\frac{\cos^2\omega_0 t}{2}=\cos^2\omega_0 t$$

$$\mathrm{Prob}(L_z^2=0;t)=\mathrm{sen}^2\omega_0 t$$

Quindi ho un risultato certo per L_z^2 se $\omega_0 t=m\frac{\pi}{2}$

$$\Rightarrow t=\frac{m\pi}{2\omega_0}$$

Più in particolare: Se $\omega_0 t=m\pi$, cioè $t=\frac{m\pi}{\omega_0}$ $\Rightarrow L_z^2=\hbar^2$

Se $\omega_0 t=(2m+1)\frac{\pi}{2}$, cioè $t=\frac{\pi}{\omega_0}(m+\frac12)$ $\Rightarrow L_z^2=0$

Es. III [es.12] C.T, J_{IV}, es. 1 (pag. 476)

Particella di spin $\frac12$, $\vec\mu=\gamma\vec S$, t.c. $|\psi(t=0)\rangle=|+\rangle$

{ Fare commento su $\vec\mu$ e sua relazione con $\vec J$
$\vec\mu=\vec\mu_S+\vec\mu_L$ $\vec\mu_S=\frac{q}{2mc}g\vec S$ $\vec\mu_L=\frac{q}{2mc}\vec L$
Per elettrone $q=-|e|$ $g=2$
$H=-\vec\mu\cdot\vec B$

a) $\mathrm{Prob}(S_x;t=0)$

b) $H=-\vec\mu\cdot\vec B$, $\vec B=B_0\vec e_y$ Calcolare $|\psi(t)\rangle$.

c) $\mathrm{Prob}(S_x,S_y,S_z;t)$. Relazione tra B_0 e t per avere un risultato certo in una di queste misure.

$S_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $S_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $S_y=\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$|+\rangle=|+,\hat z\rangle=\begin{vmatrix}1\\0\end{vmatrix}$ $|+,\hat x\rangle=\frac{1}{\sqrt2}\begin{vmatrix}1\\1\end{vmatrix}$ $|+,\hat y\rangle=\frac{1}{\sqrt2}\begin{vmatrix}1\\i\end{vmatrix}$

$|-\rangle=|-,\hat z\rangle=\begin{vmatrix}0\\1\end{vmatrix}$ $|-,\hat x\rangle=\frac{1}{\sqrt2}\begin{vmatrix}1\\-1\end{vmatrix}$ $|-,\hat y\rangle=\frac{1}{\sqrt2}\begin{vmatrix}1\\-i\end{vmatrix}$

a) $|\psi(t=0)\rangle=|+\rangle=\begin{vmatrix}1\\0\end{vmatrix}=\frac{1}{\sqrt2}(|+,\hat x\rangle+|-,\hat x\rangle)$ $\Rightarrow \mathrm{Prob}(S_x=\frac{\hbar}{2})=\frac12$

$\mathrm{Prob}(S_x=-\frac{\hbar}{2})=\frac12$

b) $H = -\vec{\mu}\cdot\vec{B} = -\gamma\,\vec{S}\cdot\vec{B} = -\gamma B_0 S_y$

$$H|+,\hat{y}\rangle = -\gamma B_0 \frac{\hbar}{2} |+,\hat{y}\rangle = E_- |+,\hat{y}\rangle \qquad E_- = -\gamma B_0 \frac{\hbar}{2}$$

$$H|-,\hat{y}\rangle = +\gamma B_0 \frac{\hbar}{2} |-,\hat{y}\rangle = E_+ |-,\hat{y}\rangle \qquad E_+ = +\gamma B_0 \frac{\hbar}{2}$$

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(t=0)\rangle =$$

$$= e^{-iHt/\hbar} \frac{1}{\sqrt{2}}\left(|+,\hat{y}\rangle + |-,\hat{y}\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{-iE_-t/\hbar} |+,\hat{y}\rangle + e^{-iE_+t/\hbar} |-,\hat{y}\rangle\right)$$

$$= \frac{1}{2} e^{i\gamma B_0 t/2} \begin{vmatrix} 1 \\ i \end{vmatrix} + \frac{1}{2} e^{-i\gamma B_0 t/2} \begin{vmatrix} 1 \\ -i \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\left(\frac{\gamma B_0 t}{2}\right) \\ -\operatorname{sen}\left(\frac{\gamma B_0 t}{2}\right) \end{vmatrix} \qquad \left(= \cos\left(\frac{\gamma B_0 t}{2}\right)|+\rangle - \operatorname{sen}\left(\frac{\gamma B_0 t}{2}\right)|-\rangle \right)$$

c) $\mathrm{Prob}\left(S_z = \frac{\hbar}{2}; t\right) = \cos^2\left(\frac{\gamma B_0 t}{2}\right) \qquad \mathrm{Prob}\left(S_z = -\frac{\hbar}{2}; t\right) = \operatorname{sen}^2\left(\frac{\gamma B_0 t}{2}\right)$

$$\mathrm{Prob}\left(S_y = \frac{\hbar}{2}; t\right) = \left|\frac{e^{-iE_-t/\hbar}}{\sqrt{2}}\right|^2 = \frac{1}{2}. \qquad \mathrm{Prob}\left(S_y = -\frac{\hbar}{2}; t\right) = \frac{1}{2}$$

$$\mathrm{Prob}\left(S_x = \frac{\hbar}{2}; t\right) = \left|\langle +,\hat{x}|\psi(t)\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(1,1)\begin{vmatrix} \cos(\;) \\ -\operatorname{sen}(\;) \end{vmatrix}\right|^2 =$$

$$= \frac{1}{2}\left(\cos\frac{\gamma B_0 t}{2} - \operatorname{sen}\frac{\gamma B_0 t}{2}\right)^2 = \frac{1}{2}\left(\cos^2 + \operatorname{sen}^2 - 2\operatorname{sen}(\;)\cos(\;)\right) =$$

$$= \frac{1}{2}\left(1 - \operatorname{sen}(\gamma B_0 t)\right)$$

$$\mathrm{Prob}\left(S_x = -\frac{\hbar}{2}; t\right) = 1 - \mathrm{Prob}\left(S_x = \frac{\hbar}{2}; t\right) = \frac{1}{2}\left(1 + \operatorname{sen}(\gamma B_0 t)\right)$$

S_y NON PUÒ ASSUMERE UN VALORE CON CERTEZZA, avendo prob $\equiv \frac{1}{2}$, fisse nel tempo $[H, S_y] = 0$

Se voglio misura certa di S_z, $\frac{\gamma B_0 t}{2} = n\frac{\pi}{2}$, cioè $t = \frac{n\pi}{\gamma B_0}$ ← Z

Se voglio misura certa di S_x, $\gamma B_0 t = (2n+1)\frac{\pi}{2}$, cioè $t = \frac{(2n+1)\pi}{2\gamma B_0}$

ES. VIII [es.15] C.T, J_{IV}, es. 5 (pag. 478)

Particella di spin $\frac{1}{2}$, $H = -\vec{\omega}\cdot\vec{S}$; con $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ $|\vec{\omega}| = \omega_0$

Se $|\psi(t=0)\rangle = |+\rangle$, trovare $|\psi(t)\rangle$ e $\text{Prob}(S_z = \frac{\hbar}{2}; t)$

Ricordiamo che $e^{i\alpha\vec{m}\cdot\vec{\sigma}} = (\cos\alpha)\mathbb{1}_{2\times2} + i(\vec{m}\cdot\vec{\sigma})(\operatorname{sen}\alpha)$ $\alpha\in\mathbb{R}$, $|\vec{m}|=1$

Poiché ci servirà, scriviamo $\frac{-iHt}{\hbar}$ nella forma $i\alpha\vec{m}\cdot\vec{\sigma}$,

cioè troviamo α e $\vec{m}$ t.c. $i\alpha\vec{m}\cdot\vec{\sigma} = -i\frac{Ht}{\hbar}$

$$\alpha\vec{m}\cdot\vec{\sigma} \overset{\text{deve}}{=} -\frac{Ht}{\hbar} = -\frac{t}{\hbar}(-\vec{\omega}\cdot\vec{S}) = \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$= t\,\vec{\omega}\cdot\frac{\vec{\sigma}}{2} = \frac{\omega_0 t}{2}\left(\frac{\vec{\omega}}{\omega_0}\cdot\vec{\sigma}\right)$$

$$\Rightarrow \alpha = \frac{\omega_0 t}{2}, \quad \vec{m} = \frac{\vec{\omega}}{\omega_0}$$

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = e^{i\alpha\vec{m}\cdot\vec{\sigma}}\begin{vmatrix}1\\0\end{vmatrix} =$$

$$\vec{m}\cdot\vec{\sigma} = m_x\sigma_x + m_y\sigma_y + m_z\sigma_z = \begin{pmatrix} m_z & m_x - \\ m_x + im_y & - \end{pmatrix}$$

$$= \left[\cos\alpha\,\mathbb{1}_{2\times2} + i(\vec{m}\cdot\vec{\sigma})\operatorname{sen}\alpha\right]\begin{vmatrix}1\\0\end{vmatrix} =$$

$$= \begin{pmatrix} \cos\alpha + im_z\operatorname{sen}\alpha & i\operatorname{sen}\alpha(m_x - im_y) \\ i\operatorname{sen}\alpha(m_x + im_y) & \cos\alpha - im_z\operatorname{sen}\alpha \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} \cos\alpha + im_z\operatorname{sen}\alpha \\ i\operatorname{sen}\alpha(m_x + im_y) \end{pmatrix}$$

e esplicitare α e $\vec{m}$

$$\text{Prob}(S_z = \tfrac{\hbar}{2}; t) = |\langle +|\psi(t)\rangle|^2 = |\cos\alpha + im_z\operatorname{sen}\alpha|^2 =$$

$$= \cos^2\alpha + m_z^2\operatorname{sen}^2\alpha\,(+\operatorname{sen}^2\alpha - \operatorname{sen}^2\alpha) =$$

$$= 1 + \operatorname{sen}^2\alpha\,(m_z^2 - 1) =$$

$$= 1 + \operatorname{sen}^2\frac{\omega_0 t}{2}\left[\left(\frac{\omega_z}{\omega_0}\right)^2 - 1\right] = 1 - \operatorname{sen}^2\frac{\omega_0 t}{2}\left(\frac{\omega_x^2 + \omega_y^2}{\omega_0^2}\right)$$

ES. IX

Particella di spin $\frac{1}{2}$. Una misura per lo spin lungo $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$ dà $+\frac{\hbar}{2}$.

Calcolare, subito dopo, $\text{Prob}\left(S_z = \frac{\hbar}{2}\right)$

Per trovare $|\psi\rangle$ devo trovare $|\psi\rangle$ t.c. $\vec{n}\cdot\vec{S}|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$, cioè:

$$|\psi\rangle = |+,\hat{n}\rangle$$

Devo cioè diagonalizzare $\vec{n}\cdot\vec{S} = \frac{\hbar}{2}\vec{n}\cdot\vec{\sigma} = \frac{\hbar}{2}\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$

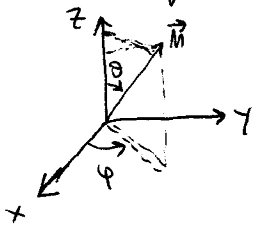
Se $\vec{n} = (\text{sen}\theta\cos\varphi, \text{sen}\theta\,\text{sen}\varphi, \cos\theta) \Rightarrow \vec{n}\cdot\vec{S} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \text{sen}\theta\, e^{-i\varphi} \\ \text{sen}\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$

Avete già visto che gli stati t.c.

$$\vec{n}\cdot\vec{S}\,|\pm,\hat{n}\rangle = \pm\frac{\hbar}{2}|\pm,\hat{n}\rangle$$

sono $|+,\hat{n}\rangle = \begin{vmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \text{sen}\frac{\theta}{2}\, e^{i\varphi/2} \end{vmatrix}$ $\qquad |-,\hat{n}\rangle = \begin{vmatrix} -\text{sen}\frac{\theta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\theta}{2}\, e^{i\varphi/2} \end{vmatrix}$

NB $|\pm,\hat{n}\rangle$ si possono anche ricavare applicando su $|+,\hat{z}\rangle$ gli op. unitari che rappresentano una rotazione. Cioè si usa $\vec{S}$ come generatore delle rotazioni (in qs. caso rotazione dello spin)



$$|+,\hat{n}\rangle = U(R(\varphi,\vec{e}_z))\,U(R(\theta,\vec{e}_y))\,|+,\hat{z}\rangle$$

$$U(R(\alpha,\vec{n})) = e^{-i\alpha\vec{n}\cdot\vec{S}/\hbar}$$

$$\Rightarrow U(R(\theta,\vec{e}_y)) = e^{-i\theta\vec{e}_y\cdot\frac{\vec{\sigma}}{2}} = \left(\cos\frac{\theta}{2}\right) - i\,\text{sen}\left(\frac{\theta}{2}\right)\sigma_y = \begin{pmatrix} \cos\theta/2 & -\text{sen}\frac{\theta}{2} \\ \text{sen}\frac{\theta}{2} & \cos\theta/2 \end{pmatrix}$$

$$U(R(\varphi,\vec{e}_z)) = e^{-i\varphi\frac{\sigma_z}{2}} = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$$

$$|+,\vec{n}\rangle = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} \cos\theta/2 & -\text{sen}\theta/2 \\ \text{sen}\frac{\theta}{2} & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2\; e^{-i\varphi/2} \\ \text{sen}\theta/2\; e^{i\varphi/2} \end{pmatrix}$$

Quindi $|\psi\rangle = |+,\vec{n}\rangle$ $\qquad \vec{n} = \frac{1}{\sqrt{3}}(1,1,1) = (\text{sen}\theta\cos\varphi, \text{sen}\theta\,\text{sen}\varphi, \cos\theta) \Rightarrow \cos\theta = 1/\sqrt{3}$

φ per ora non lo specifichiamo

$$\text{Prob}\left(S_z = \frac{\hbar}{2}\right) = |\langle +|\psi\rangle|^2 = \left|(1,0)\begin{pmatrix} \cos\theta/2\; e^{-i\varphi/2} \\ \text{sen}\frac{\theta}{2}\; e^{i\varphi/2} \end{pmatrix}\right|^2 = \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} = \frac{1+\frac{\sqrt{3}}{3}}{2}$$

Es. X

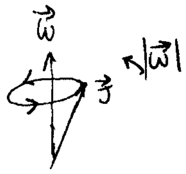
Vedi es_L6.pdf

ES. XI [es. 31] "precessione di Larmor"

In meccanica classica è noto che, dato $\vec{\mu} = \gamma \vec{J}$ immerso in un campo magnetico $\vec{B}$, il moto di $\vec{\mu}$ è un moto di precessione attorno a $\vec{B}$ con frequenza $\omega_0 = -\gamma |\vec{B}|$.

Questo va sotto il nome di precessione di Larmor

schema del ragionamento: Fisica I, Meccanica razionale: $\dfrac{d\vec{J}}{dt} = \vec{\omega} \wedge \vec{J}$ ha come soluzione $\vec{J}$ che precede attorno a $\dfrac{\vec{\omega}}{\omega}$ con frequenza ω



Fisica II: momento magnetico in campo $\vec{B}$ subisce un momento torcente $\vec{\tau} = \vec{\mu} \wedge \vec{B}$

Poiché $\vec{\tau} = \dfrac{d\vec{J}}{dt}$, abbiamo $\dfrac{d\vec{J}}{dt} = \vec{\mu} \wedge \vec{B}$

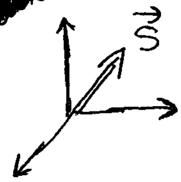
Adesso usando $\vec{J} = \dfrac{1}{\gamma}\vec{\mu}$, si ha: $\dfrac{d\vec{\mu}}{dt} = -\gamma \vec{B} \wedge \vec{\mu}$

cioè $\vec{\mu}$ precede attorno a $-\vec{B}$ con frequenza $\gamma|\vec{B}|$, ossia precede attorno a $\vec{B}$ con frequenza $-\gamma|\vec{B}| = \omega_0$

Studiamo l'analogo quantistico, supponendo x es. di avere solo gradi di libertà di spin (no $\vec{L}$, cioè $\vec{J} = \vec{S}$). Quindi:

spin $\frac{1}{2}$ in $\vec{B} = B\vec{e}_z$, con $\vec{\mu} = \dfrac{gq}{2mc}\vec{S} = \gamma\vec{S}$ $\left[\begin{array}{l}\text{per elettrone } g=2 \\ q=-|e|\end{array}\right]$

$|\psi(t=0)\rangle$ t.c. $\vec{n}\cdot\vec{S}\,|\psi(t=0)\rangle = \dfrac{\hbar}{2}|\psi(t=0)\rangle$ — $\vec{S}$ diretto lungo $\vec{n}$



$\vec{n} = \vec{n}(\theta,\varphi) = (\operatorname{sen}\theta\cos\varphi, \operatorname{sen}\theta\operatorname{sen}\varphi, \cos\theta)$

a) Verificare che al tempo t lo spin è lungo $\vec{n}(t) = \vec{n}(\theta, \varphi + \omega_0 t)$

b) Mostrare che $\langle\vec{S}\rangle(t) = \dfrac{\hbar}{2}\vec{n}(t)$

a) $|\psi(t=0)\rangle = |+,\hat{n}\rangle = \begin{vmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \operatorname{sen}\frac{\theta}{2}\, e^{i\varphi/2} \end{vmatrix}$

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{B}\cdot\vec{S} = -\gamma B S_z = \omega_0 S_z$$

($\omega_0 = -\gamma B$, come è stato definito all'inizio)

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(t=0)\rangle = e^{-i\omega_0 S_z t/\hbar}\left(\cos\tfrac{\theta}{2}\, e^{-i\varphi/2}|+\rangle + \operatorname{sen}\tfrac{\theta}{2}\, e^{i\varphi/2}|-\rangle\right) =$$

$$= \cos\frac{\theta}{2} e^{-i\varphi/2} e^{-i\omega_0 t/2} |+\rangle + \mathrm{sen}\frac{\theta}{2} e^{i\varphi/2} e^{+i\omega_0 t/2} |-\rangle$$

$$= \begin{vmatrix} \cos\frac{\theta}{2} \; e^{-\frac{i}{2}(\varphi+\omega_0 t)} \\ \mathrm{sen}\frac{\theta}{2} \; e^{\frac{i}{2}(\varphi+\omega_0 t)} \end{vmatrix}$$

Ricordando come è fatto $|+,\vec{n}\rangle$, concludiamo che $|\psi(t)\rangle$ è un autostato di $\vec{n}(t)\cdot\vec{S}$ con $\vec{n}(t) = \vec{n}(\theta, \varphi+\omega_0 t) = \vec{n}(\theta, \varphi(t))$ con $\varphi(t) = \varphi+\omega_0 t$

Cioè $\vec{S}$ è diretto lungo $\vec{n}(t)$.

b) $\langle S_x\rangle(t) = \langle\psi(t)|S_x|\psi(t)\rangle = \left(\cos\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)}, \mathrm{sen}\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)}\right)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)} \\ \mathrm{sen}\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)}\end{pmatrix}$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)}, \mathrm{sen}\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)}\right)\begin{pmatrix}\mathrm{sen}\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)} \\ \cos\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)}\end{pmatrix} = \frac{\hbar}{2} 2\,\mathrm{sen}\frac{\theta}{2}\cos\frac{\theta}{2}\left(\frac{e^{i\varphi(t)}+e^{-i\varphi(t)}}{2}\right)$$

$$= \frac{\hbar}{2}\,\mathrm{sen}\theta\cos\varphi(t)$$

$\langle S_y\rangle(t) =$ ... farlo x esercizio .. $= \frac{\hbar}{2}\,\mathrm{sen}\theta\,\mathrm{sen}\,\varphi(t)$

$$\cancel{\langle S_z\rangle(t) = \frac{\hbar}{2}\left(\cos\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)}, \mathrm{sen}\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\mathrm{sen}\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)} \\ \cos\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)}\end{pmatrix} = \frac{\hbar}{2}}$$

$$\langle S_z\rangle(t) = \left(\cos\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)}, \mathrm{sen}\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)}\right)\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} e^{-\frac{i}{2}\varphi(t)} \\ \mathrm{sen}\frac{\theta}{2} e^{\frac{i}{2}\varphi(t)}\end{pmatrix} =$$

$$= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \mathrm{sen}^2\frac{\theta}{2}\right) = \frac{\hbar}{2}\cos\theta$$

$\Rightarrow$ I valori d'aspettazione di $\vec{S}$, cioè $\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle$, evolvono nel tempo come un momento angolare classico di modulo $\frac{\hbar}{2}$ che subisce precessione di Larmor: $\langle\vec{S}\rangle(t)$ è un vettore che precede attorno a $\vec{B}$ con frequenza ω_0 -

Quindi spiega anche in MQ la precessione di Larmor di $\vec{\mu}$.

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Particella di spin $\frac{1}{2}$ t.c. a $t=0$ misurando S_y ottengo $\frac{\hbar}{2}$.

a) vettore di stato $|\psi(t=0)\rangle$

b) $H(t) = -\vec{\mu}\cdot\vec{B}(t)$, $\vec{B}(t)$ cresce linearmente con t ed è diretto lungo $\vec{e}_z$ ed è uniforme. È acceso tra 0 e T, cioè $\vec{B}(t) = 0$ se $t<0$ o $t>T$.

$$\Rightarrow H(t) = \mu\,\frac{\omega_0 t}{T}\,S_z$$

Trovare $|\psi(t)\rangle$

c) Se misuro S_y a $t'>T$, discutere probabilità per S_y. Relazione tra ω_0 e T per avere risultato certo.

a) Postulato: dopo una misura, sono nell'autostato corrispondente.

$$\Rightarrow |\psi(t=0)\rangle = |+,\hat{y}\rangle = \frac{1}{\sqrt{2}}\begin{vmatrix}1\\ i\end{vmatrix}$$

b) $$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{vmatrix}a(t)\\ b(t)\end{vmatrix}$$

Schroedinger: $$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$\frac{i\cancel{\hbar}}{\cancel{\sqrt{2}}}\begin{vmatrix}\dot{a}\\ \dot{b}\end{vmatrix} = \mu\,\frac{\omega_0 t}{T}\,\frac{\cancel{\hbar}}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{\cancel{\sqrt{2}}}\begin{vmatrix}a\\ b\end{vmatrix}$$

$$\begin{cases} i\dot{a} = \dfrac{\mu\omega_0 t}{2T}\,a \\[2mm] i\dot{b} = -\dfrac{\mu\omega_0 t}{2T}\,b \end{cases}\qquad \begin{cases} \dfrac{d}{dt}\log a = -i\,\dfrac{\mu\omega_0}{2T}\,t \\[2mm] \dfrac{d}{dt}\log b = +i\,\dfrac{\mu\omega_0}{2T}\,t \end{cases}$$

$$\begin{cases} \log a(t) - \log a(0) = -i\,\dfrac{\mu\omega_0}{4T}\,t^2 \\ \quad \dots \end{cases}\qquad \begin{cases} a(t) = a(0)\,e^{-i\frac{\mu\omega_0}{4T}t^2} \\[2mm] b(t) = b(0)\,e^{i\frac{\mu\omega_0}{4T}t^2} \end{cases}$$

Quindi, guardando solo $t>0$ e inserendo $a(0)=1$, $b(0)=i$

$$|\psi(t)\rangle = \begin{cases} \dfrac{1}{\sqrt{2}}\begin{vmatrix} e^{-i\frac{\mu\omega_0}{4T}t^2} \\ i\,e^{i\frac{\mu\omega_0}{4T}t^2}\end{vmatrix} & 0\le t\le T \\[4mm] |\psi(T)\rangle & t>T \qquad (\vec{B}=0 \Rightarrow H=0) \end{cases}$$

c) Al solito, S_y può valere $\pm \frac{\hbar}{2}$.

$$\text{Prob}\left(S_y = \frac{\hbar}{2}\,;\, t'\right) = |\langle +, \hat{y} | \psi(t') \rangle|^2 = \qquad \forall\, t' > T$$

$$= |\langle +, \hat{y} | \psi(T) \rangle|^2 =$$

$$= \left| \frac{1}{\sqrt{2}}(1, -i) \frac{1}{\sqrt{2}} \begin{vmatrix} e^{-i \frac{\mu\omega_0}{4T} T^2} \\ ie^{i \frac{\mu\omega_0}{4T} T^2} \end{vmatrix} \right|^2 =$$

$$= \frac{1}{4} \left| e^{-i \frac{\mu\omega_0 T}{4}} + e^{+i \frac{\mu\omega_0}{4} T} \right|^2 =$$

$$= \frac{1}{4} \left(2\cos \frac{\mu\omega_0 T}{4} \right)^2 = \cos^2\left(\frac{\mu\omega_0 T}{4} \right)$$

$$\text{Prob}\left(S_y = -\frac{\hbar}{2}\,;\, t'\right) = \ldots \text{ conti } =$$

$$= 1 - \text{Prob}\left(S_y = +\frac{\hbar}{2}\,;\, t'\right) =$$

$$= \text{sen}^2\left(\frac{\mu\omega_0 T}{4} \right)$$

Ho un risultato certo se $\frac{\mu\omega_0 T}{4} = m\frac{\pi}{2}$ $\left(m \text{ pari} \Rightarrow S_y = \frac{\hbar}{2} \quad m \text{ dispari} \Rightarrow S_y = -\frac{\hbar}{2} \right)$

ES. XIII [es. 11] Destri 23/07/2007

Particella di spin $\frac{1}{2}$, ad un certo t è t.c.

$$\psi(r, \theta, \varphi) = A e^{-r/r_0} \begin{pmatrix} 1 \\ \text{sen}\varphi \end{pmatrix} = A e^{-r/r_0} \left[|+\rangle + \text{sen}\varphi\, |-\rangle \right]$$

(scelto z come asse di quantizzazione per lo spin)

Trovare $\text{Prob}(J_z)$, con $\vec{J} = \vec{L} + \vec{S}$.

$$\psi = f(r)\, \tilde{\psi}(\theta, \varphi), \quad \text{con } \tilde{\psi}(\theta,\varphi) = \begin{pmatrix} 1 \\ \text{sen}\varphi \end{pmatrix} = |+\rangle + \text{sen}\theta\, |-\rangle$$

cioè: $\tilde{\psi}(\theta,\varphi) = |+\rangle + \frac{e^{i\varphi} - e^{-i\varphi}}{2i} |-\rangle$

ATTENZIONE: NON TENTARE DI SVILUPPARE senφ SULLE $Y_{\ell m}$, SAREBBERO ∞. QUI MI SERVE SOLO L_z.

$\hat{L}_z = -i\hbar \frac{d}{d\varphi}$, autofunzioni normalizzate: $\Phi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$ $m = 0, \pm 1, \pm 2, \ldots$, $L_z = m\hbar$

Quindi:

$$\tilde{\psi}(\theta,\varphi) = \sqrt{2\pi}\left[\Phi_0(\varphi)|+\rangle + \frac{1}{2i}\Phi_1(\varphi)|-\rangle - \frac{1}{2i}\Phi_{-1}(\varphi)|-\rangle \right]$$

Per evitare confusione: $|+\rangle \mapsto |\frac{1}{2}\rangle$
$|-\rangle \mapsto |-\frac{1}{2}\rangle$

Poi, in notazione di Dirac: $\Phi_m(\varphi) = \langle\varphi|m\rangle$, quindi:

$$|\tilde{\psi}\rangle = \sqrt{2\pi}\left[|0,\tfrac{1}{2}\rangle + \frac{1}{2i}|1,-\tfrac{1}{2}\rangle - \frac{1}{2i}|-1,-\tfrac{1}{2}\rangle \right]$$

con $|\ \rangle = |m_\ell, m_s\rangle$

<u>NB</u> Formalmente sto facendo composizione di due momenti angolari, quindi:
$|m_\ell, m_s\rangle = |m_\ell\rangle \otimes |m_s\rangle$. Lo rivediamo meglio nella discussione singoletto-tripletto.

Adesso è ovvio capire a che valore di J_z corrisponde ciascun termine:

$$J_z|0,\tfrac{1}{2}\rangle = (L_z + S_z)|0,\tfrac{1}{2}\rangle = 0|0,\tfrac{1}{2}\rangle + \tfrac{\hbar}{2}|0,\tfrac{1}{2}\rangle = \tfrac{\hbar}{2}|0,\tfrac{1}{2}\rangle$$

$$J_z|1,-\tfrac{1}{2}\rangle = (L_z + S_z)|1,-\tfrac{1}{2}\rangle = \hbar|1,-\tfrac{1}{2}\rangle - \tfrac{\hbar}{2}|1,-\tfrac{1}{2}\rangle = \tfrac{\hbar}{2}|1,-\tfrac{1}{2}\rangle$$

$$J_z|-1,-\tfrac{1}{2}\rangle = -\tfrac{3}{2}\hbar|-1,-\tfrac{1}{2}\rangle$$

cioè, in generale: $J_z|m_\ell, m_s\rangle = (m_\ell + m_s)\hbar|m_\ell, m_s\rangle$

$$\mathrm{Prob}\left(J_z = \frac{\hbar}{2}\right) = \frac{1 + \left|\frac{1}{2i}\right|^2}{1 + \left|\frac{1}{2i}\right|^2 + \left|\frac{-1}{2i}\right|^2} = \frac{5/4}{6/4} = \frac{5}{6}$$

$$\mathrm{Prob}\left(J_z = -\frac{3}{2}\hbar\right) = \frac{\left|\frac{-1}{2i}\right|^2}{1 + |\ |^2 + |\ |^2} = \frac{1/4}{6/4} = \frac{1}{6}$$