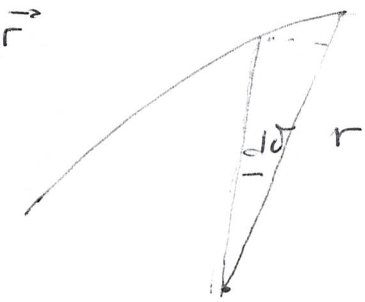
$$\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times m \frac{d\vec{r}}{dt}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{u}_r + r\frac{d\vartheta}{dt}\hat{u}_\vartheta \qquad \text{(polari)}$$

$$L = m r^2 \frac{d\vartheta}{dt}$$

$$\vec{L} \perp \vec{r}\,;\ d\vec{r}\,;\ \vec{r}+d\vec{r}$$



$$dA = \frac{1}{2} r^2 d\vartheta$$

$$\frac{dA}{dt} = \frac{L}{2m}$$

$$\frac{dA}{dt} = \text{cost.} \implies L = \text{cost}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F} = 0 \implies \vec{L} = \text{cost}$$

$$\frac{\vec{L}}{L}\ \text{cost} \implies \text{traiettoria (orbita) piana}$$

$$\left.\begin{array}{l}\text{orbita piana}\\ \dfrac{dA}{dt} = \text{cost}\end{array}\right\} \implies \vec{F}_g \text{ è forza centrale (Newton)}$$

Per $\boxed{\vec{F} \text{ centrale} \implies \vec{L} = \text{cost.}}$

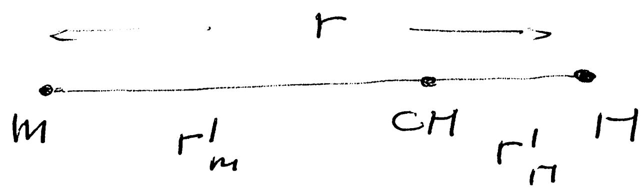
$$\frac{A}{T} \underset{\uparrow}{=} \frac{dA}{dt} = \frac{L}{2m} \implies T = \frac{2mA}{L}$$

$$\frac{dA}{dt} = \text{cost}$$

$$\left.\begin{array}{l}\text{orbite piane (1}^a\text{ Kepl.)}\\ \dfrac{dA}{dt} = \text{cost. (2}^a\text{ Kepl)}\end{array}\right\} \vec{L} = \text{cost}$$

Il momento angolare totale

risp. CM



$$r'_m = r\,\frac{M}{m+M}$$

$$r'_M = r\,\frac{m}{m+M}$$

$$L' = \ell'_m + \ell'_M$$

$$\ell'_m = m\, r'^2_m \left(\frac{d\vartheta}{dt}\right) = m\, r^2 \left(\frac{M}{m+M}\right)^2 \left(\frac{d\vartheta}{dt}\right)$$

$$\ell'_M = M\, r'^2_M \left(\frac{d\vartheta}{dt}\right) = M\, r^2 \left(\frac{m}{m+M}\right)^2 \left(\frac{d\vartheta}{dt}\right)$$

$$L' = r^2\,\frac{m M^2 + M m^2}{(m+M)^2}\left(\frac{d\vartheta}{dt}\right) = r^2\,\frac{m M}{m+M}\left(\frac{d\vartheta}{dt}\right)$$

$$= \mu\, r^2 \left(\frac{d\vartheta}{dt}\right)$$

L' energia risp. CM

- contributo da v'_ϑ

$$E'_{K\vartheta} = \frac{1}{2} L' \left(\frac{d\vartheta}{dt}\right) = \frac{1}{2} \mu r^2 \left(\frac{d\vartheta}{dt}\right)^2 = \frac{1}{2} \mu v_\vartheta^2$$

  - v_ϑ è la comp. trasversa della <u>velocità relativa</u> delle due part.

- contributo da v'_r

$$E'_{K r} = \frac{1}{2} m \left(\frac{d r_m}{dt}\right)^2 + \frac{1}{2} M \left(\frac{d r_M}{dt}\right)^2$$

$$= \frac{1}{2} \frac{m M^2 + M m^2}{(m+M)^2} \left(\frac{dr}{dt}\right)^2$$

$$= \frac{1}{2} \mu v_r^2$$

  - v_r è comp. radiale della <u>velocità relativa</u> delle due part.

$$\Rightarrow \quad E'_K = \frac{1}{2} \mu v^2$$

v : velocità relativa delle due part.

Per un osservatore solidale con una delle due particelle è consistente, come L, con la dinamica 'apparente'

- E'_K e L' ~~son~~ corrispondono a quelli descritti da un oss. solidale con una delle due part. per cui la dinamica della seconda part. è descritta da massa rid. μ

In coord. polari

$$\vec{v} = \frac{dr}{dt}\hat{u}_r + r\frac{d\vartheta}{dt}\hat{u}_\vartheta$$

derivando risp. t ottengo

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2\right]\hat{u}_r + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right)\right]\hat{u}_\vartheta$$

ma per $\vec{F}$ centrale $r^2\frac{d\vartheta}{dt} = \frac{L}{m}$ é cost.; quindi

- $$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2\right]\hat{u}_r$$ (solo comp. radiale ok per $\vec{F}$ centr.)

voglio riscrivere $\vec{a}$ con ϑ var. indip.: $r = r(\vartheta)$

$$\frac{dr}{dt} = \frac{dr}{d\vartheta}\frac{d\vartheta}{dt} = \frac{L}{mr^2}\frac{dr}{d\vartheta} = -\frac{L}{m}\frac{d}{d\vartheta}\left(\frac{1}{r}\right)$$

qui $L = mr^2\frac{d\vartheta}{dt}$

$$\frac{d^2r}{dt^2} = \frac{d}{d\vartheta}\left(\frac{dr}{dt}\right)\frac{d\vartheta}{dt} = \frac{L}{mr^2}\frac{d}{d\vartheta}\left[-\frac{L}{m}\frac{d}{d\vartheta}\left(\frac{1}{r}\right)\right] =$$

$$= -\frac{L^2}{m^2r^2}\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right)$$

- $$\vec{a} = -\frac{L^2}{m^2r^2}\left[\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]\hat{u}_r$$ (Binet)

- introduco dinamica:

$$\vec{F} = -\frac{GmM}{r^2}\hat{u}_r = \mu\vec{a}$$ (usando μ ho dinamica moto relativo)

e uso $L = \mu r^2\frac{d\vartheta}{dt}$

$$\frac{L^2}{\mu r^2}\left[\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right] = \frac{GmM}{r^2}$$

eq. del moto relativo

$$\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = G\frac{\mu mM}{L^2}$$

$$\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = G\frac{\mu m M}{L^2}$$

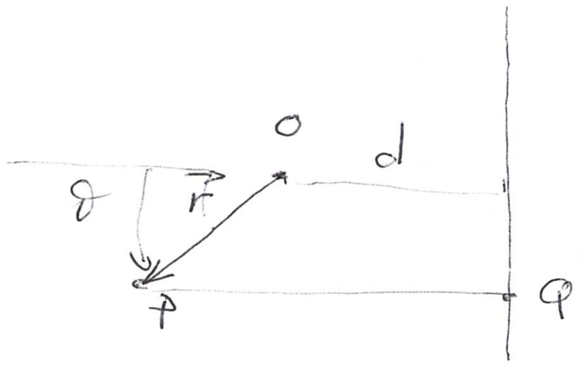
E' eq. di oscillatore armonico con termine noto costante : $G\frac{\mu m M}{L^2}$

funzione incognita : $\frac{1}{r(\vartheta)}$, $\omega = 1$

$$\frac{1}{r} = B\cos(\vartheta + \varphi) + G\frac{\mu m M}{L^2}$$

È eq. di conica : B (e φ che non mi interessa) è det. da condizioni iniziali

Eq. generale di conica in coord. polari, polo nel fuoco



$$\varepsilon = \frac{PO}{PQ} = \frac{r}{d + r\cos\vartheta}$$

$$\frac{1}{r} = \frac{1}{\varepsilon d} - \frac{\cos\vartheta}{d}$$

Ellisse per $\varepsilon < 1$

semiasse magg : $a = \frac{\varepsilon d}{1-\varepsilon^2}$

minore : $b = a\sqrt{1-\varepsilon^2}$

$$\varepsilon^2 = 1 - \frac{b^2}{a^2}$$

$$A = \pi a b = \pi a^2\sqrt{1-\varepsilon^2}$$

$\Rightarrow \boxed{L^2 = G\mu m M \varepsilon d} = \boxed{G\mu m M a(1-\varepsilon^2)}$ per ellisse

Cerco ora relazione tra parametri dell'orbita e E totale

$$E = E_k + \cancel{E_p} V = \frac{1}{2}\mu v^2 - G\frac{mM}{r}$$

$\left(\vec{v} = \frac{d\vec{r}}{dt} \right.$ è vel. relativa dei due corpi $\left. \right)$

scompongo E_K nelle due comp. già discusse

$$= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\mu r^2\left(\frac{d\vartheta}{dt}\right)^2 - G\frac{mM}{r}$$

- da eq. conica $\left(\frac{1}{r} = \frac{1}{\varepsilon d} - \frac{\cos\vartheta}{d} \right)$ ottengo:

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{1}{r}\right) = \frac{\operatorname{sen}\vartheta}{d}\frac{d\vartheta}{dt} \\ \text{inoltre} \\ \frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{dt} \end{array}\right\} \Rightarrow \frac{dr}{dt} = -r^2\frac{\operatorname{sen}\vartheta}{d}\frac{d\vartheta}{dt}$$

quindi, tenendo conto di $L = \mu r^2 \frac{d\vartheta}{dt}$:

$$E_{K_r}: \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 = +\frac{1}{2}\mu r^4\frac{\operatorname{sen}^2\vartheta}{d^2}\left(\frac{d\vartheta}{dt}\right)^2 = \frac{1}{2}\frac{\operatorname{sen}^2\vartheta}{d^2}\frac{L^2}{\mu}$$

inoltre

$$E_{K\vartheta}: \frac{1}{2}\mu r^2\left(\frac{d\vartheta}{dt}\right)^2 = \frac{1}{2}\frac{L^2}{\mu r^2}$$

$$V\,\cancel{E_p}: -\frac{GMm}{r} = -\frac{L^2}{r\mu\varepsilon d}$$

sost. $L^2 = G\mu mM\varepsilon d$

$$E_T = \frac{1}{2}\frac{\operatorname{sen}^2\vartheta}{d^2}\frac{L^2}{\mu} + \frac{1}{2}\frac{1}{r^2}\frac{L^2}{\mu} - \frac{1}{r}\frac{1}{\varepsilon d}\frac{L^2}{\mu}$$

eq. conica $\frac{1}{r} = \frac{1}{\varepsilon d} - \frac{1}{d}\cos\vartheta$

$$\frac{1}{r^2} = \frac{1}{(\varepsilon d)^2} + \frac{1}{d^2}\cos^2\vartheta - \frac{2}{\varepsilon d^2}\cos\vartheta$$

$$E_T = \frac{L^2}{\mu}\left[\frac{1}{2}\frac{\operatorname{sen}^2\vartheta}{d^2} + \frac{1}{2}\frac{1}{(\varepsilon d)^2} + \frac{1}{2}\frac{\cos^2\vartheta}{d^2} - \cancel{\frac{1}{\varepsilon d^2}\cos\vartheta} - \frac{1}{(\varepsilon d)^2} + \cancel{\frac{1}{\varepsilon d^2}\cos\vartheta}\right]$$

$$E_T = \frac{L^2}{2\mu}\left(\frac{1}{d^2} - \frac{1}{(\varepsilon d)^2}\right) = \frac{-L^2}{2\mu\varepsilon^2 d^2}(1-\varepsilon^2)$$

quindi per $\varepsilon < 1$ $E_T < 0$ come deve essere per sistema legato (ellisse) con la scelta fatta della costante arbit. di V

~~per ellisse $\varepsilon d/2$~~ uso ancora $L^2 = G\mu m M \varepsilon d$

$$\boxed{E_T = -\frac{GmM}{2\varepsilon d}(1-\varepsilon^2)}$$

per ellisse $\varepsilon d = a(1-\varepsilon^2)$ quindi

$$\left.\begin{aligned} &\boxed{E_T = -\frac{GmM}{2a}} \\ &L^2 = G\mu m M a(1-\varepsilon^2) \end{aligned}\right\} \text{ellisse}$$

per orbita ellittica E_T dipende solo da a, L da a ed ε

<u>Orbita circolare</u> $r = a$

$$E_T = -\frac{GmM}{2r} = \frac{1}{2}V = E_k + V$$

$$\Rightarrow \left.\begin{aligned} E_K &= -\frac{1}{2}V \\ E_T &= -E_K \end{aligned}\right\} \text{orbita circolare}$$

Si dimostra che le relazioni trovate valgono per valori medi di orbite qualunque, inoltre valgono per valori medi anche in sistema di N corpi tra cui agisca solo $F \propto 1/r^2$ attr.

$$\Rightarrow \langle E_K \rangle = -\frac{1}{2}\langle V \rangle$$

$$\boxed{E_T = -\langle E_K \rangle = \frac{1}{2}\langle V \rangle}$$

<u>commentare</u>

nota: per OAS $E_T = 2\langle E_K \rangle = 2\langle V \rangle$ $F \propto r$ attr.