An Introduction to CP Violation

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1 The $K^0 - \overline{K}^0$ meson system

1.1. General features

We are interested in the time evolution of a K^0 or a $\overline{K}^0$ meson in its rest frame[1]. Since K^0 mesons are produced by strong interaction processes (like for example $\pi^- + p \to K^0 + \Lambda^0$), at the initial time $t = 0$ the quantum state describing the system has a definite value of the strangeness quantum number S ($S = 1$ for K^0 and $S = -1$ for $\overline{K}^0$). K^0 mesons are stable with respect to strong interactions, and can decay only because of weak interactions; furthermore, the transition $K^0 \to \overline{K}^0$ is *a priori* allowed, since strangeness is not conserved by the weak hamiltonian. Therefore, we introduce a two-dimensional space spanned by the two states $|K^0\rangle$ and $|\overline{K}^0\rangle$:

$$|K^0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\overline{K}^0\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.1}$$

and we study the Schrödinger equation for a generic state

$$|\Psi(t)\rangle = \Psi_1(t)|K^0\rangle + \Psi_2(t)|\overline{K}^0\rangle = \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \end{bmatrix}. \tag{1.2}$$

The Schrödinger equation has the form

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H^{\rm eff}|\Psi(t)\rangle, \tag{1.3}$$

where the effective hamiltonian $H^{\rm eff}$ is a generic 2×2 matrix, in general non-hermitian in order to account for decays. The stationary states (that is, states with definite mass and lifetime) will be the two linear combinations of $|K^0\rangle$ and $|\overline{K}^0\rangle$ that diagonalize the matrix $H^{\rm eff}$. It is always possible to decompose the effective hamiltonian as

$$H^{\rm eff} = M - \frac{i}{2}\Gamma, \tag{1.4}$$

where M and Γ are both hermitian matrices:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}; \quad \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{22} \end{bmatrix}. \tag{1.5}$$

The meaning of the effective hamiltonian $H^{\rm eff}$ is discussed in Appendix A. Some of its properties, however, can be deduced on general grounds.

First, we observe that, if weak interactions were switched off, then $|K^0\rangle$ and $|\overline{K}^0\rangle$ would be stationary, degenerate states, and we would have

$$M_{11} = M_{22} = m_K; \quad M_{12} = 0; \quad \Gamma = 0. \tag{1.6}$$

This is not true in general, but it is easy to prove that the equality between the diagonal entries of M and Γ,

$$M_{11} = M_{22}; \quad \Gamma_{11} = \Gamma_{22} \tag{1.7}$$

holds even in the presence of weak interactions, due to CPT invariance (which we will always assume). The proof goes as follows[2]: if we define[1] the CP operator by

$$CP|K^0\rangle = |\overline{K}^0\rangle \quad CP|\overline{K}^0\rangle = |K^0\rangle, \tag{1.8}$$

then, in our two-dimensional space, CP is represented by a matrix U_{CP} that transforms the vectors $(1,0)$ and $(0,1)$ into each other:

$$U_{CP} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad U_{CP}^{-1} = U_{CP}. \tag{1.9}$$

The transformation laws for M and Γ under CP are then

$$CP \ : \ M \to U_{CP} M U_{CP}^{-1} = \begin{bmatrix} M_{22} & M_{12}^* \\ M_{12} & M_{11} \end{bmatrix}. \tag{1.10}$$

$$CP \ : \ \Gamma \to U_{CP} \Gamma U_{CP}^{-1} = \begin{bmatrix} \Gamma_{22} & \Gamma_{12}^* \\ \Gamma_{12} & \Gamma_{11} \end{bmatrix}. \tag{1.11}$$

From ordinary quantum mechanics we know that the time-reversal operation T acts as complex conjugation on observables:

$$T \ : \ M \to M^* \tag{1.12}$$

$$T \ : \ \Gamma \to \Gamma^* \tag{1.13}$$

Using eqs. (1.10-1.13) we conclude that

$$CPT : M \to \begin{bmatrix} M_{22} & M_{12} \\ M_{12}^* & M_{11} \end{bmatrix}; \ \Gamma \to \begin{bmatrix} \Gamma_{22} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{11} \end{bmatrix}. \tag{1.14}$$

Clearly, CPT invariance is achieved only if

$$M_{11} = M_{22} \equiv m_K; \quad \Gamma_{11} = \Gamma_{22} \equiv \gamma, \tag{1.15}$$

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[1] In general, one should define the operator CP by $CP|K^0\rangle = e^{i\alpha}|\overline{K}^0\rangle$, where α is an arbitrary phase. Here we choose $\alpha = 0$ for simplicity.

which is the announced result.

We now diagonalize the matrix H^{eff}. The two eigenvalues are

$$M_{S,L} - \frac{i}{2}\Gamma_{S,L} = m_K - \frac{i}{2}\gamma \pm R, \tag{1.16}$$

where

$$R = \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)}. \tag{1.17}$$

Therefore, mass and lifetime differences between the two eigenstates are

$$\Delta M = M_S - M_L = 2\,\mathrm{Re}\,R \tag{1.18}$$

$$\Delta\Gamma = \Gamma_S - \Gamma_L = -4\,\mathrm{Im}\,R. \tag{1.19}$$

The measured values for masses and lifetimes are

$$m_K = \frac{M_S + M_L}{2} = 497.676 \pm 0.030 \text{ MeV} \tag{1.20}$$

$$\Delta M = (-0.5351 \pm 0.0024) \times 10^{10}\ s^{-1} \simeq -3.57 \times 10^{-6}\mathrm{eV} \tag{1.21}$$

$$\Gamma_L = 1.93 \times 10^{7}\ s^{-1} \tag{1.22}$$

$$\Gamma_S = 1.12 \times 10^{10}\ s^{-1}. \tag{1.23}$$

Notice that $\Gamma_L \ll \Gamma_S$, and that

$$\Delta M \simeq -\frac{1}{2}\Delta\Gamma. \tag{1.24}$$

The two eigenvectors are given by[2]

$$|K_S\rangle = p|K^0\rangle + q|\overline{K}^0\rangle \tag{1.25}$$

$$|K_L\rangle = p|K^0\rangle - q|\overline{K}^0\rangle, \tag{1.26}$$

where

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}}; \quad |p|^2 + |q|^2 = 1. \tag{1.27}$$

Equations (1.27) fix q and p up to a common phase. The indices L (for *long*) and S (for *short*) reflect the fact that the two eigenstates have very different lifetimes. Notice that q and p are not physical quantities. In fact, we have the freedom of performing a strangeness rotation on $|K^0\rangle$ and $|\overline{K}^0\rangle$:

$$|K^0\rangle \to |K^0\rangle' = e^{i\sigma}|K^0\rangle; \quad |\overline{K}^0\rangle \to |\overline{K}^0\rangle' = e^{-i\sigma}|\overline{K}^0\rangle. \tag{1.28}$$

[2] Almost everywhere in the literature the notation $q/p = (1-\epsilon)/(1+\epsilon)$ is adopted. We will not use it here, however.

Equation (1.28) has also the effect of transforming the off-diagonal entries of M and Γ according to

$$
\begin{aligned}
M_{12} &= \langle K^0|M|\overline{K}^0\rangle \to e^{-2i\sigma} M_{12} \\
\Gamma_{12} &= \langle K^0|\Gamma|\overline{K}^0\rangle \to e^{-2i\sigma} \Gamma_{12},
\end{aligned}
\tag{1.29}
$$

and therefore

$$
\frac{q}{p} \to e^{2i\sigma}\,\frac{q}{p}. \tag{1.30}
$$

One can check that measurable quantities are not affected by this arbitrary phase definition. This is the case, for example, of the eigenvalues in eq. (1.16). They depend on M_{12} and Γ_{12} only through the combination R, which is invariant under the transformation in eq. (1.29).

1.2. Oscillations

We are now ready to solve the problem of time evolution of K^0 meson states. Integrating eq. (1.3), one immediately obtains

$$
|K_S(t)\rangle = e^{-i\left(M_S - \frac{i}{2}\Gamma_S\right)t}|K_S(0)\rangle \tag{1.31}
$$

$$
|K_L(t)\rangle = e^{-i\left(M_L - \frac{i}{2}\Gamma_L\right)t}|K_L(0)\rangle. \tag{1.32}
$$

The time evolution of a generic state $|\Psi(t)\rangle$ is obtained by expanding $|\Psi(0)\rangle$ on the $|K_S\rangle$, $|K_L\rangle$ basis and then using eqs. (1.31-1.32). For example, consider a beam composed of $|K^0\rangle$ mesons only, produced at $t = 0$. From eqs. (1.25-1.26),

$$
|\Psi(0)\rangle = |K^0\rangle = \frac{1}{2p}\left(|K_S(0)\rangle + |K_L(0)\rangle\right), \tag{1.33}
$$

which in turn implies

$$
\begin{aligned}
|\Psi(t)\rangle &= \frac{1}{2p}\left[|K_S(t)\rangle + |K_L(t)\rangle\right] \\
&= \frac{1}{2p}\left[e^{-i\left(M_S - \frac{i}{2}\Gamma_S\right)t}\left(p|K^0\rangle + q|\overline{K}^0\rangle\right) + e^{-i\left(M_L - \frac{i}{2}\Gamma_L\right)t}\left(p|K^0\rangle - q|\overline{K}^0\rangle\right)\right] \\
&= f_+(t)|K^0\rangle + \frac{q}{p} f_-(t)|\overline{K}^0\rangle,
\end{aligned}
\tag{1.34}
$$

where

$$
f_\pm(t) = \frac{1}{2}\left[e^{-i\left(M_S - \frac{i}{2}\Gamma_S\right)t} \pm e^{-i\left(M_L - \frac{i}{2}\Gamma_L\right)t}\right]. \tag{1.35}
$$

The probability of finding a K^0 in the beam after a time t is proportional to

$$|\langle K^0|\Psi(t)\rangle|^2 = |f_+(t)|^2 = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\cos(\Delta M t)\right] \quad (1.36)$$

and analogously the probability of finding a $\overline{K}^0$ is proportional to

$$|\langle \overline{K}^0|\Psi(t)\rangle|^2 = \left|\frac{q}{p}\right|^2 |f_-(t)|^2 = \left|\frac{q}{p}\right|^2 \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\cos(\Delta M t)\right]. \quad (1.37)$$

Also in this case, one can check that the quantities in eqs. (1.36-1.37), which are measurable quantities, are left unchanged by the phase transformation defined in eq. (1.28).

It will be useful to derive formulae for the time-integrated fractions of K^0 and $\overline{K}^0$ in the beam. We find

$$\frac{\int_0^{+\infty} dt\, |f_+(t)|^2}{\int_0^{+\infty} dt\left[|f_+(t)|^2 + |f_-(t)|^2\right]} = \frac{1}{2}\frac{2+x^2-y^2}{1+x^2} \quad (1.38)$$

$$\frac{\int_0^{+\infty} dt\, |f_-(t)|^2}{\int_0^{+\infty} dt\left[|f_+(t)|^2 + |f_-(t)|^2\right]} = \frac{1}{2}\frac{x^2+y^2}{1+x^2}, \quad (1.39)$$

where

$$x = \frac{\Delta M}{\Gamma}; \qquad y = \frac{\Delta\Gamma}{2\Gamma} \quad (1.40)$$

and $\Gamma = (\Gamma_S + \Gamma_L)/2$. Both quantities in eqs. (1.38,1.39) are approximately equal to $1/2$ in the case of K^0 mesons.

1.3. Measuring ΔM

Semileptonic decays of K^0 and $\overline{K}^0$ are interesting from the experimental point of view, because they allow a determination of the mass difference ΔM, one of the most precise measurements in the whole field of particle physics. We have seen that a beam initially containing only K^0 mesons, after a time t will contain both K^0 and $\overline{K}^0$ with probabilities given by eqs. (1.36) and (1.37). Neglecting for the moment CP-violation effects (which amounts to assuming $|q/p| = 1$) we find

$$\frac{N(K^0) - N(\overline{K}^0)}{N(K^0) + N(\overline{K}^0)} = \frac{2\cos(\Delta M t)}{e^{-\frac{\Delta\Gamma t}{2}} + e^{\frac{\Delta\Gamma t}{2}}}, \quad (1.41)$$

where $N(K^0)$ ($N(\overline{K}^0)$) denotes the number of K^0 ($\overline{K}^0$) mesons in the beam at the time t. These numbers can be determined experimentally by studying the semileptonic decays of K^0 mesons. In fact, semileptonic decays obey the $\Delta S = \Delta Q$ rule, that states that the variation of strangeness in the process equals the charge difference between hadrons in the initial and final states. Because of this selection rule (which is easily understood in the context of the standard model), the following decay modes

$$K^0 \rightarrow \pi^- e^+ \nu_e \tag{1.42}$$

$$\overline{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e, \tag{1.43}$$

are allowed, while the same processes with opposite sign of the lepton charges are forbidden. As a consequence, $N(K^0)$ is proportional to the number of observed *positrons*, and $N(\overline{K}^0)$ to the number of observed *electrons*, and the quantity in eq. (1.41) can be experimentally determined as a function of time. It is worthwhile noticing that K^0 meson systems provides a significant example of the superposition principle in quantun mechanics: if strangeness is observed, via semileptonic decays, than there is no possibility of measuring masses and lifetimes, since the strangeness observable does not commute with the full hamiltonian. Strangeness eigenstates, like K^0 or $\overline{K}^0$, are linear combinations of states with definite masses and lifetimes. Conversely, if we measure masses and lifetimes, we force the system to take definite values of these quantities, and at the same time it becomes impossible to measure strangeness: mass eigenstates are superpositions of states with assigned strangeness quantum number.

1.4. CP violation

We now turn to the problem of CP violation. Let us define the operator CP according to

$$CP|K^0\rangle = e^{i\alpha}|\overline{K}^0\rangle \quad CP|\overline{K}^0\rangle = e^{-i\alpha}|K^0\rangle\,, \tag{1.44}$$

where α is an arbitrary phase.[3] We begin by proving the following, important result:

$$|K_S\rangle \textit{ and } |K_L\rangle \textit{ are } CP \textit{ eigenstates} \Leftrightarrow \mathrm{Im}(M_{12}\Gamma^*_{12}) = 0.$$

The proof is straightforward: let us assume that $|K_S\rangle$ and $|K_L\rangle$ are eigenstates of the CP operator, defined as in eq. (1.44), with eigenvalues $+1$ and -1

[3] We observe that under the transformation in eq (1.28), also the CP operator is modified. In fact, the original definition eq. (1.44) is changed into $CP|K^0\rangle' = e^{i(\alpha+2\sigma)}|\overline{K}^0\rangle'$, $CP|\overline{K}^0\rangle' = e^{-i(\alpha+2\sigma)}|K^0\rangle'$.

respectively:

$$CP|K_S\rangle = pe^{i\alpha}|\overline{K}^0\rangle + qe^{-i\alpha}|K^0\rangle = p|K^0\rangle + q|\overline{K}^0\rangle \tag{1.45}$$

$$CP|K_L\rangle = pe^{i\alpha}|\overline{K}^0\rangle - qe^{-i\alpha}|K^0\rangle = -p|K^0\rangle + q|\overline{K}^0\rangle\,. \tag{1.46}$$

Equations (1.45,1.46) give $pe^{i\alpha} = q$, or

$$\frac{q}{p} = e^{i\alpha}\,. \tag{1.47}$$

On the other hand, using eq. (1.27) we find

$$\left|\frac{q}{p}\right|^2 = \sqrt{\frac{|M_{12}|^2 + |\Gamma_{12}|^2/4 + \mathrm{Im}(M_{12}\Gamma_{12}^*)}{|M_{12}|^2 + |\Gamma_{12}|^2/4 - \mathrm{Im}(M_{12}\Gamma_{12}^*)}}\,. \tag{1.48}$$

The condition $|q/p|^2 = 1$ is therefore equivalent to

$$\mathrm{Im}(M_{12}\Gamma_{12}^*) = 0\,, \tag{1.49}$$

the announced result. Obviously, the argument is also valid the other way around: if $\mathrm{Im}(M_{12}\Gamma_{12}^*) = 0$, then $|K_S\rangle$ and $|K_L\rangle$ are CP eigenstates, provided the CP operator is defined by $CP|K^0\rangle = \exp(i\alpha)|\overline{K}^0\rangle$, with $\exp(i\alpha) = q/p$. Notice that, because of eq. (1.30), this result is independent of phase conventions.

Is it possible to decide experimentally whether the eigenstates of the effective hamiltonian $H^{\rm eff}$ are also CP eigenstates? To answer this question, we must consider the decays of K^0 mesons into two-pion states, $|\pi^0\pi^0\rangle$ or $|\pi^+\pi^-\rangle$, and three-pion states, $|\pi^0\pi^0\pi^0\rangle$ or $|\pi^+\pi^-\pi^0\rangle$. This is because two-pion states are CP eigenstates with eigenvalue $+1$, while three-pion states are predominantly in a CP eigenstate with eigenvalue -1. As a consequence, if K_L and K_S are also CP eigenstates, we expect K_S to decay mainly into two pions, and K_L only into three pions. Studying these processes, we will see that another mechanism of CP violation, independent of $K^0-\overline{K}^0$ mixing, can take place.

The study of two-pion decays of K^0 mesons is most conveniently carried on in terms of states with definite isotopic spin I. A two-pion state can have $I = 0, 1, 2$, but $I = 1$ is forbidden by Bose statistic. The familiar Clebsch-Gordan decomposition gives

$$\begin{aligned} |\pi^+\pi^-\rangle &= \sqrt{\frac{2}{3}}\,|\pi\pi, I=0\rangle + \sqrt{\frac{1}{3}}\,|\pi\pi, I=2\rangle \\ |\pi^0\pi^0\rangle &= -\sqrt{\frac{1}{3}}\,|\pi\pi, I=0\rangle + \sqrt{\frac{2}{3}}\,|\pi\pi, I=2\rangle. \end{aligned} \tag{1.50}$$

From now on, we will use the simplified notation

$$|0\rangle \equiv |\pi\pi, I=0\rangle; \quad |2\rangle \equiv |\pi\pi, I=2\rangle. \tag{1.51}$$

A well-established experimental fact of hadron physics (at least, for low-lying states) is that weak amplitudes corresponding to processes with isospin variation $\Delta I = 1/2$ are strongly enhanced with respect to those with higher ΔI (the so-called $\Delta I = 1/2$ selection rule). Let us consider the ratio

$$\frac{\langle \pi^+\pi^-|T|K_S\rangle}{\langle \pi^0\pi^0|T|K_S\rangle} = \frac{\sqrt{2}\langle 0|T|K_S\rangle + \langle 2|T|K_S\rangle}{-\langle 0|T|K_S\rangle + \sqrt{2}\langle 2|T|K_S\rangle}, \tag{1.52}$$

where T is the weak transition matrix. Since K^0 mesons belong to $I = 1/2$ multiplets, amplitudes with the state $|2\rangle$ should be suppressed with respect to amplitudes with the state $|0\rangle$, according to the $\Delta I = 1/2$ rule. If this rule were exact, eq. (1.52) would imply

$$\frac{\Gamma(K_S \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^0\pi^0)} = 2. \tag{1.53}$$

The observed value is 2.18 ± 0.03, thus confirming that the $\Delta I = 1/2$ rule is approximately valid in this context.

We introduce the two quantities

$$\eta_\pm = \frac{\langle \pi^+\pi^-|T|K_L\rangle}{\langle \pi^+\pi^-|T|K_S\rangle} \tag{1.54}$$

$$\eta_{00} = \frac{\langle \pi^0\pi^0|T|K_L\rangle}{\langle \pi^0\pi^0|T|K_S\rangle}. \tag{1.55}$$

Clearly, both $\eta_\pm$ and η_{00} vanish if CP is conserved, since in that case K_L would be a CP eigenstate with eigenvalue -1, and could not decay into two pions[4]. So, $\eta_\pm$ and η_{00} are the appropriate quantities to investigate CP violation. Decomposing the two-pion states as in eq. (1.50) we find

$$\eta_\pm = \frac{\epsilon_0 + \frac{\omega}{\sqrt{2}}\epsilon_2}{1 + \frac{\omega}{\sqrt{2}}} \tag{1.56}$$

$$\eta_{00} = \frac{\epsilon_0 - \sqrt{2}\omega\epsilon_2}{1 - \sqrt{2}\omega}, \tag{1.57}$$

[4]Observe that the reversed statement for three pion states is not true: K_S can decay into three pions even if CP is an exact symmetry, since three pion states contain a small component with $CP = +1$.

where

$$\epsilon_I = \frac{\langle I|T|K_L\rangle}{\langle I|T|K_S\rangle} = \frac{1 - \frac{q}{p}\frac{\langle I|T|\overline{K}^0\rangle}{\langle I|T|K^0\rangle}}{1 + \frac{q}{p}\frac{\langle I|T|\overline{K}^0\rangle}{\langle I|T|K^0\rangle}}, \quad I = 0, 2 \tag{1.58}$$

$$\omega = \frac{\langle 2|T|K_S\rangle}{\langle 0|T|K_S\rangle} = \frac{1 + \frac{q}{p}\frac{\langle 2|T|\overline{K}^0\rangle}{\langle 2|T|K^0\rangle}}{1 + \frac{q}{p}\frac{\langle 0|T|\overline{K}^0\rangle}{\langle 0|T|K^0\rangle}} \frac{\langle 2|T|K^0\rangle}{\langle 0|T|K^0\rangle}. \tag{1.59}$$

Observe that ϵ_I and ω are manifestly invariant under the phase redefinition defined in eqs. (1.28-1.30).

1.5. Superweak CP violation

The weak hamiltonian H_W can be decomposed as a sum of two terms:

$$H_W = H_W^{\Delta S=2} + H_W^{\Delta S=1}, \tag{1.60}$$

where the first term is responsible for $K^0 - \overline{K}^0$ mixing, a process where strangeness changes by two units, and the second is the one giving rise to K^0 meson decays. Correspondingly, at first order in weak interaction, the transition matrix T can be decomposed in an analogous way:

$$T = T^{\Delta S=2} + T^{\Delta S=1}. \tag{1.61}$$

The amplitudes $\langle I|T|K^0\rangle$ and $\langle I|T|\overline{K}^0\rangle$ receive contributions only from $T^{\Delta S=1}$, while $T^{\Delta S=2}$ affects ϵ_I and ω indirectly through q/p.

Let us assume for the moment that $H_W^{\Delta S=1}$ is CP-conserving, and that all CP violating effects are contained in $H_W^{\Delta S=2}$, that is in the mixing process. This means

$$(CP)T^{\Delta S=1}(CP)^{-1} = T^{\Delta S=1}, \tag{1.62}$$

and exploiting the fact that $\pi\pi$ states are CP eigenstate with $+1$ eigenvalue, we find

$$\langle I|T^{\Delta S=1}|K^0\rangle = \langle I|(CP)^{-1}T^{\Delta S=1}(CP)|K^0\rangle = \langle I|T^{\Delta S=1}|\overline{K}^0\rangle e^{i\alpha} \tag{1.63}$$

for both $I = 0$ and $I = 2$ (the phase factor $\exp(i\alpha)$ originates from the the definition of the CP operator, eq. (1.44)). From eq. (1.58) we obtain

$$\epsilon_2 = \epsilon_0, \tag{1.64}$$

and therefore

$$\eta_{\pm} = \eta_{00} = \epsilon_0. \tag{1.65}$$

This situation, in which CP violating effects are confined in $K^0-\overline{K}^0$ mixing, while K^0 decays are CP-conserving, is usually called *superweak CP violation.* Present experimental data are consistent with this picture of CP violation; in fact, the measured values of $\eta_\pm$ and η_{00} are

$$|\eta_\pm| = \sqrt{\frac{\Gamma(K_L \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^+\pi^-)}} = (2.285 \pm 0.019) \times 10^{-3} \tag{1.66}$$

$$|\eta_{00}| = \sqrt{\frac{\Gamma(K_L \to \pi^0\pi^0)}{\Gamma(K_S \to \pi^0\pi^0)}} = (2.275 \pm 0.019) \times 10^{-3}, \tag{1.67}$$

that is, $\eta_\pm$ and η_{00} are equal within one standard deviation.

What is the relationship between the quantity ϵ_0, or equivalently $\eta_\pm$ and η_{00}, and the effective hamiltonian $M - i\Gamma/2$ defined in the previous section? To see this, we define

$$M_{12} = |M_{12}|\, e^{i(\phi+\delta)} \qquad \Gamma_{12} = |\Gamma_{12}|\, e^{i\phi}. \tag{1.68}$$

We know that the phase difference δ (which is a phase-convention independent quantity) is responsible for CP violation; it must be close to 0 or π, since eqs. (1.66) and (1.67) tell us that CP is violated by a small amount. To make this statement more precise, we use eqs. (1.27) and (1.58) to obtain

$$\left|\frac{1-\epsilon_0}{1+\epsilon_0}\right|^2 = \left|\frac{q}{p}\right|^2 = \sqrt{\frac{|M_{12}|^2 + |\Gamma_{12}|^2/4 + |M_{12}|\,|\Gamma_{12}|\sin\delta}{|M_{12}|^2 + |\Gamma_{12}|^2/4 - |M_{12}|\,|\Gamma_{12}|\sin\delta}}. \tag{1.69}$$

From eqs. (1.65-1.67) we know that the above quantity is equal to 1 up to corrections of the order of a few 10^{-3}. Therefore, $\sin\delta$ must also be of the order of a few 10^{-3}. Expanding in $\sin\delta$ we obtain

$$\frac{q}{p} = e^{-i\phi}\left(1 - \frac{i\,|M_{12}|\sin\delta}{|M_{12}|\cos\delta - \frac{i}{2}|\Gamma_{12}|}\right) + \mathcal{O}(\sin^2\delta), \tag{1.70}$$

and

$$R = \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)} = |M_{12}|\cos\delta - \frac{i}{2}|\Gamma_{12}| + \mathcal{O}(\sin^2\delta). \tag{1.71}$$

Clearly, up to terms of order $\sin^2\delta$, $\cos\delta = +1$ if δ is close to 0, and $\cos\delta = -1$ if δ is close to π. This ambiguity is resolved recalling eqs. (1.18,1.19), which give

$$\Delta M = 2\,|M_{12}|\cos\delta + \mathcal{O}(\sin^2\delta), \qquad \Delta\Gamma = 2\,|\Gamma_{12}| + \mathcal{O}(\sin^2\delta). \tag{1.72}$$

Since experimentally $\Delta M < 0$, $\Delta\Gamma > 0$, $\cos\delta$ must be negative, and δ close to π. We may therefore replace everywhere $\delta \to \pi + \delta$ and expand around $\delta = 0$: $\sin(\pi+\delta) = -\sin\delta \simeq -\delta$, $\cos(\pi+\delta) \simeq -1$. Exploiting the fact that the ratio $\langle 0|T|\overline{K}^0\rangle/\langle 0|T|K^0\rangle$ is approximately equal to the phase $\exp(i\phi)$ of Γ_{12} (see eq. (A.18) in Appendix A) we obtain, from eq. (1.58),

$$\epsilon_0 = \frac{1-(q/p)e^{i\phi}}{1+(q/p)e^{i\phi}} = \frac{i}{2}\frac{|M_{12}|\,\delta}{|M_{12}| + \frac{i}{2}|\Gamma_{12}|} + \mathcal{O}(\delta^2). \tag{1.73}$$

Finally, the experimental fact that $\Delta M \simeq -\Delta\Gamma/2$, together with eq. (1.72) allow us to obtain the simple and important result

$$\epsilon_0 = \frac{i\delta}{1+i} + \mathcal{O}(\delta^2) = \frac{\delta}{\sqrt{2}}\,e^{i\pi/4} + \mathcal{O}(\delta^2). \tag{1.74}$$

Notice that the value of $\pi/4$ we have obtained for the phase of ϵ_0 (and therefore, approximately, of $\eta_\pm$ and η_{00}) is an *experimental* value; it follows from the fact that CP violation effects are so small, which allowed us to neglect terms of order δ^2, and from the experimental observation that $\Delta M \simeq -\Delta\Gamma/2$ for kaons. This is not the case, for example, for B^0 mesons, for which $\Delta M \gg \Delta\Gamma$.

1.6. Direct CP violation

It is interesting to consider also the possibility that the CP symmetry be violated in K^0 decays, that is, by the $\Delta S = 1$ term of the hamiltonian. In this case, in general,

$$\frac{\langle 2|T|\overline{K}^0\rangle}{\langle 2|T|K^0\rangle} \neq \frac{\langle 0|T|\overline{K}^0\rangle}{\langle 0|T|K^0\rangle}, \tag{1.75}$$

and eqs. (1.64), (1.65) are no longer true. In view of the experimental results in eqs. (1.66-1.67), however, it seems natural to write both $\eta_\pm$ and η_{00} as ϵ_0 plus some quantities which vanish in the superweak limit. We obtain

$$\eta_\pm = \epsilon_0 + \frac{1}{1+\omega/\sqrt{2}}\frac{\omega}{\sqrt{2}}(\epsilon_2 - \epsilon_0) \tag{1.76}$$

$$\eta_{00} = \epsilon_0 - \frac{2}{1-\omega\sqrt{2}}\frac{\omega}{\sqrt{2}}(\epsilon_2 - \epsilon_0). \tag{1.77}$$

The following notation is universally adopted:

$$\epsilon \equiv \epsilon_0; \quad \epsilon' \equiv \frac{\omega}{\sqrt{2}}(\epsilon_2 - \epsilon_0). \tag{1.78}$$

Neglecting terms of order ω^2, because of the $\Delta I = 1/2$ rule, eqs. (1.76-1.77) become

$$\eta_{\pm} = \epsilon + \epsilon' \tag{1.79}$$

$$\eta_{00} = \epsilon - 2\epsilon'. \tag{1.80}$$

The quantity ϵ' signals a new kind of CP violation, independent of $K^0 - \overline{K}^0$ mixing, which is called *direct* CP violation. We know from eqs. (1.66) and (1.67) that it must be very small, of order $10^{-3}\epsilon$, but apart from this the experimental information about this important parameter is not yet conclusive. Two collaborations, the NA31 Collaboration at CERN[3] and the E731 Collaboration at FNAL[4], have measured the quantity

$$D = \frac{B(K_L \to \pi^+\pi^-)/B(K_S \to \pi^+\pi^-)}{B(K_L \to \pi^0\pi^0)/B(K_S \to \pi^0\pi^0)} = \left|\frac{\eta_{\pm}}{\eta_{00}}\right|^2 \simeq 1 + 6\,\mathrm{Re}\,\frac{\epsilon'}{\epsilon}. \tag{1.81}$$

from which a measurement of $\mathrm{Re}(\epsilon'/\epsilon)$ can be extracted. Their results are as follows:

$$\text{NA31}: \quad \mathrm{Re}\,\frac{\epsilon'}{\epsilon} = (2.3 \pm 0.7) \times 10^{-3} \tag{1.82}$$

$$\text{E731}: \quad \mathrm{Re}\,\frac{\epsilon'}{\epsilon} = (0.74 \pm 0.59) \times 10^{-3}. \tag{1.83}$$

More recently, the KTEV Collaboration at Fermilab has announced the result

$$\text{KTEV}: \mathrm{Re}\,\frac{\epsilon'}{\epsilon} = (2.8 \pm 0.3(\text{stat.}) \pm 0.3(\text{syst.}) \pm 0.1(\text{MC})) \times 10^{-3} \tag{1.84}$$

while the NA48 Collaboration at CERN has presented the preliminary result

$$\text{NA48}: \mathrm{Re}\,\frac{\epsilon'}{\epsilon} = (1.85 \pm 0.45(\text{stat.}) \pm 0.58(\text{syst.})) \times 10^{-3}\,. \tag{1.85}$$

A precise determination of ϵ'/ϵ is of great importance, in order to test the predictions of the standard model of electroweak interactions.

1.7. $K^0 - \overline{K}^0$ mixing in the standard model

The effective lagrangian for $\Delta S = 2$ transitions in the standard model is given by

$$\mathcal{L}_{\text{eff}}^{\Delta s=2} \;=\; \frac{G_F^2}{16\pi^2}\left[\bar{s}\gamma^\mu(1-\gamma_5)d\right]^2 \times \Big[\xi_u^2 m_u^2 + \xi_c^2 m_c^2 + \xi_t^2 m_t^2$$

$$
\begin{aligned}
&+ \quad 2\xi_u\xi_c \frac{m_u^2 m_c^2}{m_c^2 - m_u^2} \log \frac{m_c^2}{m_u^2} + 2\xi_u\xi_t \frac{m_u^2 m_t^2}{m_t^2 - m_u^2} \log \frac{m_t^2}{m_u^2} \\
&+ \quad 2\xi_c\xi_t \frac{m_c^2 m_t^2}{m_t^2 - m_c^2} \log \frac{m_t^2}{m_c^2} \Bigg] + \text{h.c.},
\end{aligned}
\tag{1.86}
$$

where

$$
\xi_k = V_{ks} V_{kd}^*, \quad k = u, c, t \tag{1.87}
$$

and V_{ij} are the elements of the CKM matrix. Equation (1.86) holds in the limit $m_{u,c,t} \ll m_W$ (the exact computation is given in Appendix B); this approximation is perfectly adequate for m_u and m_c, but it is obvioulsly wrong for m_t. However, the contributions proportional to m_t can be dropped in eq. (1.86). This can be easily seen using the Wolfenstein parametrization of the CKM matrix,

$$
V = \begin{bmatrix} 1 - \lambda^2/2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda - A^2\lambda^5(\rho + i\eta) & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{bmatrix}, \tag{1.88}
$$

where $\lambda = \sin\theta_C = 0.219 \sim 0.224$. This parametrization is valid up to terms of order λ^4, and it is particularly useful because it displays explicitly the hierarchy of the mixing among different generations. We find

$$
\xi_u = V_{su} V_{ud}^* \sim \lambda \tag{1.89}
$$

$$
\xi_c = V_{sc} V_{cd}^* \sim -\lambda \tag{1.90}
$$

$$
\xi_t = V_{st} V_{td}^* \sim -A^2\lambda^5. \tag{1.91}
$$

Therefore, the ratio between the top and the charm contributions is approximately given by

$$
\frac{m_t^2 \xi_t^2}{m_c^2 \xi_c^2} \simeq \frac{m_t^2 \lambda^8}{m_c^2} \simeq \left(\frac{175}{1.5}\right)^2 \times (0.2)^8 \simeq 0.03, \tag{1.92}
$$

since the parameter A is of order 1. We can conclude that the top quark contributions can be neglected in eq. (1.86). Neglecting also the up quark mass, we find

$$
\mathcal{L}_{\text{eff}}^{\Delta s=2} \simeq \frac{G_F^2}{16\pi^2} \xi_c^2 m_c^2 \left[\bar{s}\gamma^\mu(1-\gamma_5)d\right]^2 + \text{h.c.}, \tag{1.93}
$$

an excellent approximation as far as mixing is concerned. We are now ready to compute the $K_L K_S$ mass difference

$$
\Delta M \simeq 2\,|M_{12}|\,. \tag{1.94}
$$

Using the standard Lorentz-invariant state normalization

$$\langle K^0|K^0\rangle = (2\pi)^3\delta^3(0)2m_K = \mathcal{V}2m_K\ , \tag{1.95}$$

we find

$$M_{12} = \frac{1}{2m_K\mathcal{V}}\int d^3x\langle K^0|\left(-\mathcal{L}_{\rm eff}^{\Delta s=2}(x)\right)|\overline{K}^0\rangle\ . \tag{1.96}$$

It is easy to show that the integrand is constant, and thus a factor of $\mathcal{V}$ is generated that cancels the $\mathcal{V}$ in the denominator. We get

$$M_{12} = -\frac{1}{2m_K}\langle K^0|\mathcal{L}_{\rm eff}^{\Delta s=2}|\overline{K}^0\rangle\,, \tag{1.97}$$

and therefore

$$\Delta M \simeq 2\,|M_{12}| = 2\frac{G_F^2}{16\pi^2}\xi_c^2 m_c^2\ \frac{1}{2m_K}\langle K^0|\left[\bar{s}\gamma^\mu(1-\gamma_5)d\right]^2|\overline{K}^0\rangle\,. \tag{1.98}$$

The matrix element that appears in eq. (1.98) is determined by the strong interaction dynamics at energies of the order of the K^0 mass, and therefore it cannot be computed in a perturbative framework. In practice, it is usually parametrized as

$$\langle K^0|\left[\bar{s}\gamma^\mu(1-\gamma_5)d\right]^2|\overline{K}^0\rangle = \frac{8}{3}f_K^2 m_K^2 B_K\,, \tag{1.99}$$

where $f_K \simeq 1.23 f_\pi \simeq 114$ MeV is the K decay constant, extracted from the measured decay rate for the process $K^+ \to \mu^+\nu_\mu$, and B_K is a parameter which is expected to be of order 1 on the basis of flavor symmetry considerations and lattice calculations. We have therefore

$$\Delta M \simeq \frac{G_F^2}{6\pi^2}\,\xi_c^2 m_c^2\, f_K^2 m_K B_K\,. \tag{1.100}$$

Equation (1.100) allows an estimate of the charm quark mass (we recall that the $K_L K_S$ mass difference was measured before the charm quark discovery). For $B_K = 1$ one finds

$$m_c \sim 1.6 \text{ GeV}, \tag{1.101}$$

which is remarkably close to current estimates.

The superweak CP-violating quantity ϵ_0 can also be estimated in the standard model. We have

$$\epsilon_0 = \frac{\delta}{\sqrt{2}}\,e^{i\pi/4} + \mathcal{O}(\delta^2), \tag{1.102}$$

where δ is the difference between the complex phases of M_{12} and Γ_{12}. Since, as we have seen, ϵ_0 is phase-convention independent, we may choose a definition of the K^0 state such that Γ_{12} is real. It is easy to show that this is the case in the Wolfenstein parametrization of the CKM matrix. In this way,

$$\delta \simeq \frac{\mathrm{Im} M_{12}}{|M_{12}|} \,. \tag{1.103}$$

We have therefore

$$|\epsilon_0| \simeq \frac{\delta}{\sqrt{2}} \simeq \frac{1}{\sqrt{2}} \left| \frac{\mathrm{Im} M_{12}}{M_{12}} \right| \simeq \frac{1}{\sqrt{2}} \left| \frac{\mathrm{Im} M_{12}}{\mathrm{Re} M_{12}} \right| \,. \tag{1.104}$$

In this case, the approximation eq. (1.93) is not accurate enough, and the full expression of $\mathcal{L}_{\mathrm{eff}}^{\Delta s=2}$ must be used (see Appendix B):

$$\mathcal{L}_{\mathrm{eff}}^{\Delta s=2} = \frac{G_F^2 m_W^2}{16\pi^2} \left[\bar{s}\gamma^\mu (1-\gamma_5) d \right]^2 \sum_{a,b} \xi_a \xi_b \frac{x_a x_b}{x_a - x_b} \left[f(x_a) - f(x_b) \right] \,, \tag{1.105}$$

where $x_a = m_a^2/m_W^2$, and

$$f(x) = \frac{x^2 - 8x + 4}{(x-1)^2} \log x + \frac{3}{x-1} \,. \tag{1.106}$$

We find

$$|\epsilon_0| \simeq \frac{1}{2\sqrt{2}} \frac{\mathrm{Im}\Phi}{\mathrm{Re}\Phi} \,, \tag{1.107}$$

where

$$\Phi = \sum_{a,b} \xi_a \xi_b \frac{x_a x_b}{x_a - x_b} \left[f(x_a) - f(x_b) \right] \,. \tag{1.108}$$

Note that in eq. (1.107) the dependence on the matrix element

$$\langle K^0 | \left[\bar{s}\gamma^\mu (1-\gamma_5) d \right]^2 | \overline{K}^0 \rangle \tag{1.109}$$

cancels out.

The fact that ϵ_0 is nonzero in the standard model can be understood on the basis of physical considerations. In fact, what makes ϵ_0 nonzero is the phase mismatch δ between M_{12} and Γ_{12}. The complex phase of M_{12} is essentially the phase of ξ_c, as we have shown. On the other hand, Γ_{12} gets a nonzero contribution only when the internal quark in the box loop is an up quark; therefore, it is proportional to ξ_u, which has a different complex phase from ξ_c (in the Wolfenstein parametrization ξ_u is real, while ξ_c is not).

1.8. More on direct CP violation

Let us examine in more detail the issue of direct CP violation. We will establish the conditions under which it can indeed take place, in the context of a formalism[5] that automatically takes into account the constraints imposed by unitarity of the scattering matrix and CPT invariance, and that can be easily extended to systems other than the $K^0 - \overline{K}^0$ one.

Let us consider the full scattering matrix $\mathcal{S}$ at all orders in strong interactions, but at first order in weak interactions:

$$\mathcal{S} = S + iT, \tag{1.110}$$

where S is the contribution of strong interactions, and T is the weak interaction transition matrix. The unitarity condition at first order in T gives

$$\mathcal{S}^\dagger \mathcal{S} = S^\dagger S + i\left(S^\dagger T - T^\dagger S\right) + \ldots = I, \tag{1.111}$$

or

$$S^\dagger S = I \tag{1.112}$$

$$S^\dagger T - T^\dagger S = 0. \tag{1.113}$$

Eq. (1.113) can be written as

$$T^\dagger = S^\dagger T S^\dagger. \tag{1.114}$$

The weak interaction hamiltonian can be written as the sum of a CP-conserving term and a CP-violating one. Correspondingly, the transition matrix T can be decomposed as

$$T = T_+ + T_-, \tag{1.115}$$

where

$$\mathrm{CP} \;:\; T_+ \rightarrow \;\; T_+ \tag{1.116}$$

$$\mathrm{CP} \;:\; T_- \rightarrow -T_- \tag{1.117}$$

It can be proved that, because of CPT invariance, T_+ is symmetric and T_- antisymmetric (we have seen a proof of this fact in a particular case in section 1). Therefore, using eqs. (1.115), (1.116) and (1.117), eq. (1.114) becomes

$$T_\pm^* = \pm S^\dagger T_\pm S^\dagger. \tag{1.118}$$

We consider now states $|P\rangle$ and $|I\rangle$, and their CP-conjugates $|\overline{P}\rangle = CP|P\rangle$ and $|\overline{I}\rangle = CP|I\rangle$. Equation (1.118) relates the two amplitudes $\langle\overline{I}|T|\overline{P}\rangle$ and $\langle I|T|P\rangle$:

$$\begin{aligned}
\langle\overline{I}|T|\overline{P}\rangle^* &= \langle\overline{I}|T_+|\overline{P}\rangle^* + \langle\overline{I}|T_-|\overline{P}\rangle^* \\
&= \langle I|T_+|P\rangle^* - \langle I|T_-|P\rangle^* \\
&= \langle I|S^\dagger T_+ S^\dagger|P\rangle + \langle I|S^\dagger T_- S^\dagger|P\rangle \\
&= \langle I|S^\dagger T S^\dagger|P\rangle \\
&= \sum_{J,J'}\langle I|S^\dagger|J\rangle\langle J|T|J'\rangle\langle J'|S^\dagger|P\rangle. \qquad (1.119)
\end{aligned}$$

Equation (1.119) further simplifies if $|P\rangle$ is a weakly-decaying particle (like K^0 or $\overline{K}^0$). In this case, $\langle J'|S^\dagger|P\rangle = \delta_{J'P}$ and therefore

$$\langle\overline{I}|T|\overline{P}\rangle^* = \sum_J\langle I|S^\dagger|J\rangle\langle J|T|P\rangle. \qquad (1.120)$$

We are interested in CP asymmetries, that is, in the differences

$$\Delta_I = \left|\langle\overline{I}|T|\overline{P}\rangle\right|^2 - |\langle I|T|P\rangle|^2, \qquad (1.121)$$

since the observation of $\Delta_I \neq 0$ for some channel I would be a signal of CP violation. From eq. (1.120), using the unitarity of S, we derive immediately that

$$\sum_I \left|\langle\overline{I}|T|\overline{P}\rangle\right|^2 = \sum_I |\langle I|T|P\rangle|^2, \qquad (1.122)$$

or

$$\sum_I \Delta_I = 0. \qquad (1.123)$$

The result in eq. (1.123) is the well-known result that a particle has the same lifetime as its antiparticle, which follows from CPT invariance. It is automatically taken into account by eq. (1.120).

The constraint in eq. (1.120) will also help us to establish the conditions under which some of the asymmetries Δ_I can be non-zero. Let us first consider the case when the states $|I\rangle$ are strong interaction eigenstates, that is, the matrix S is diagonal in the basis of the $|I\rangle$'s. In this case, eq. (1.120) becomes

$$\langle\overline{I}|T|\overline{P}\rangle^* = e^{-2i\delta_I}\langle I|T|P\rangle \qquad (1.124)$$

for each $|I\rangle$ separately. Here δ_I is the strong interaction phase shift corresponding to the state $|I\rangle$. From eq. (1.124) it follows immediately that in this case

$$\Delta_I = 0 \qquad (1.125)$$

for any state $|I\rangle$. We conclude that, even if CP is not conserved by the scattering matrix, the asymmetries are not necessarily non-zero; for some Δ_I to be nonzero it is necessary that CP is violated, *and* that two or more channels are coupled by strong interactions (or, equivalently, that the matrix S is not diagonal in the basis of the observed final states). Equations (1.124) are satisfied if we define amplitudes $\mathcal{A}_I$ by

$$\langle I|T|P\rangle = \mathcal{A}_I e^{i\delta_I} \tag{1.126}$$

$$\langle \overline{I}|T|\overline{P}\rangle = \mathcal{A}_I^* e^{i\delta_I}. \tag{1.127}$$

In eqs. (1.126) and (1.127), the phases due to final state rescattering have been factorized, while the truly CP violating phases are those of the amplitudes $\mathcal{A}_I$. In fact, no direct CP violation takes place if $\mathcal{A}_I = \mathcal{A}_I^*$, as apparent from eqs. (1.126,1.127).

Let us assume then that S is not diagonal on the basis $|I\rangle$, but that its non-diagonal entries can be treated as a perturbation with respect to diagonal entries. Time-reversal invariance forces S to be symmetric, and the unitarity constraint allow us to write S in the following form:

$$S_{IJ} \equiv \langle I|S|J\rangle = e^{2i\delta_I}\delta_{IJ} + 2it_{IJ}e^{i(\delta_I+\delta_J)} + \mathcal{O}(t^2), \tag{1.128}$$

with t_{IJ} real and symmetric. The analogous of eqs. (1.126) and (1.127) are now

$$\langle I|T|P\rangle = e^{i\delta_I}\left(\mathcal{A}_I + i\sum_J t_{IJ}\mathcal{A}_J\right) \tag{1.129}$$

$$\langle \overline{I}|T|\overline{P}\rangle = e^{i\delta_I}\left(\mathcal{A}_I^* + i\sum_J t_{IJ}\mathcal{A}_J^*\right). \tag{1.130}$$

It can be checked by direct substitution that eqs. (1.129) and (1.130) do indeed satisfy eq. (1.120) with S given by eq. (1.128), up to terms of order t^2. We are now able to compute the asymmetries Δ_I. We find

$$\Delta_I = 4\sum_J t_{IJ}\,\mathrm{Im}\,\mathcal{A}_I^*\mathcal{A}_J. \tag{1.131}$$

Some important features of eq. (1.131) must be stressed. First, as noticed above, it authomatically incorporates the CPT constraint $\sum_I \Delta_I = 0$: in fact, t_{IJ} is symmetric and $\mathrm{Im}\,\mathcal{A}_I^*\mathcal{A}_J$ is antisymmetric in I and J. Secondly, it does not depend on the strong interaction phase shifts δ_I. Finally, it exhibits the fact that asymmetries vanish in a strong interaction basis, where $t_{IJ} = 0$.

The case when there are only two states in the set $\{|I\rangle\}$ deserves a detailed treatment, for two different reasons. First, if S is a 2×2 matrix, the problem of computing asymmetries can be solved exactly, without making an expansion in the non-diagonal entries of S. Secondly, the only physical system in which CP violation has ever been observed, namely the $K^0 - \overline{K}^0$ system, is precisely a two-dimensional case in this respect, since two-pion states can take two values of the isotopic spin (or equivalently they can be found either in the $\pi^+\pi^-$ or in the $\pi^0\pi^0$ state). We will refer explicitly to K^0 mesons, but our formulae apply to any two-state system.

In order to simplify notations, we define

$$\mathcal{M}_I = \langle I|T|K^0\rangle \tag{1.132}$$

$$\overline{\mathcal{M}}_I = \langle \overline{I}|T|\overline{K}^0\rangle, \tag{1.133}$$

where $|I\rangle$ are precisely the two-pion states with definite isospin $I = 0, 2$ defined in section 2 (incidentally, we notice that in this case $|I\rangle = |\overline{I}\rangle$ since $\pi\pi$ states are CP eigenstates). Equation (1.120) becomes, in matrix notation,

$$\overline{\mathcal{M}}^* = S^\dagger \mathcal{M}, \tag{1.134}$$

where now $S = e^{2i\delta}$, $\delta = \mathrm{diag}(\delta_0, \delta_2)$. The solutions of eq. (1.134) are, as before,

$$\mathcal{M}_I = e^{i\delta_I}\mathcal{A}_I \tag{1.135}$$

$$\overline{\mathcal{M}}_I = e^{i\delta_I}\mathcal{A}_I^*. \tag{1.136}$$

The states $|\pi^+\pi^-\rangle$ and $|\pi^0\pi^0\rangle$ can be expressed in terms of the states $|I\rangle$ by eq. (1.50), that we rewrite here in matrix notation:

$$\begin{bmatrix} |\pi^+\pi^-\rangle \\ |\pi^0\pi^0\rangle \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} |0\rangle \\ |2\rangle \end{bmatrix} \equiv V \begin{bmatrix} |0\rangle \\ |2\rangle \end{bmatrix}. \tag{1.137}$$

If we define

$$\mathcal{M}' = \begin{bmatrix} \mathcal{M}'_\pm \\ \mathcal{M}'_{00} \end{bmatrix} = \begin{bmatrix} \langle \pi^+\pi^-|T|K^0\rangle \\ \langle \pi^0\pi^0|T|K^0\rangle \end{bmatrix} \tag{1.138}$$

$$\overline{\mathcal{M}}' = \begin{bmatrix} \overline{\mathcal{M}}'_\pm \\ \overline{\mathcal{M}}'_{00} \end{bmatrix} = \begin{bmatrix} \langle \pi^+\pi^-|T|\overline{K}^0\rangle \\ \langle \pi^0\pi^0|T|\overline{K}^0\rangle \end{bmatrix}, \tag{1.139}$$

we have, from eq. (1.137),

$$\mathcal{M}' = V\mathcal{M} = Ve^{i\delta}\mathcal{A} \tag{1.140}$$

$$\overline{\mathcal{M}}' = V\overline{\mathcal{M}} = Ve^{i\delta}\mathcal{A}^*. \tag{1.141}$$

We are now able to compute the asymmetries:

$$\begin{aligned}
\Delta_\pm &= \left|\overline{\mathcal{M}}'_\pm\right|^2 - \left|\mathcal{M}'_\pm\right|^2 = \left|V_{00}e^{i\delta_0}\mathcal{A}_0^* + V_{02}e^{i\delta_2}\mathcal{A}_2^*\right|^2 - \left|V_{00}e^{i\delta_0}\mathcal{A}_0 + V_{02}e^{i\delta_2}\mathcal{A}_2\right|^2 \\
&= 4V_{00}V_{02}\sin(\delta_0-\delta_2)\,\mathrm{Im}\,(\mathcal{A}_0\mathcal{A}_2^*) \\
&= 4\frac{\sqrt{2}}{3}\sin(\delta_0-\delta_2)\,\mathrm{Im}\,(\mathcal{A}_0\mathcal{A}_2^*)\,. \qquad (1.142)
\end{aligned}$$

$$\begin{aligned}
\Delta_{00} &= \left|\overline{\mathcal{M}}'_{00}\right|^2 - \left|\mathcal{M}'_{00}\right|^2 = \left|V_{20}e^{i\delta_0}\mathcal{A}_0^* + V_{22}e^{i\delta_2}\mathcal{A}_2^*\right|^2 - \left|V_{20}e^{i\delta_0}\mathcal{A}_0 + V_{22}e^{i\delta_2}\mathcal{A}_2\right|^2 \\
&= 4V_{20}V_{22}\sin(\delta_0-\delta_2)\,\mathrm{Im}\,(\mathcal{A}_0\mathcal{A}_2^*) \\
&= -4\frac{\sqrt{2}}{3}\sin(\delta_0-\delta_2)\,\mathrm{Im}\,(\mathcal{A}_0\mathcal{A}_2^*)\,. \qquad (1.143)
\end{aligned}$$

As expected, $\Delta_\pm + \Delta_{00} = 0$.

In this notation, the quantity ϵ' in eq. (1.78) becomes

$$\epsilon' = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\mathrm{Re}\,\mathcal{A}_2}{\mathrm{Re}\,\mathcal{A}_0}\left[\frac{\mathrm{Im}\,\mathcal{A}_2}{\mathrm{Re}\,\mathcal{A}_2} - \frac{\mathrm{Im}\,\mathcal{A}_0}{\mathrm{Re}\,\mathcal{A}_0}\right] \qquad (1.144)$$

(terms of order ϵ have been neglected). It can be checked that the asymmetries in eqs. (1.142) and (1.143) are proportional to Re ϵ', as expected since all these quantities are connected with direct CP violation.

2 Mixing and CP violation in the B^0–$\overline{B}^0$ meson system

In this section, we discuss mixing CP violation in B^0 meson systems in the context of the standard model. The general formalism used for the $B^0-\overline{B}^0$ is quite similar to that of the $K^0-\overline{K}^0$ system. From the phenomenological point of view, however, the case of the B^0 system differs drastically from that of the K^0. This is mainly due to the fact that CP violation in the K^0 system has to go through higher generations, and therefore is strongly suppressed by off-diagonal elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

In analogy with what we did in the K^0 case, we define

$$B^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overline{B}^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2.1}$$

and we define an effective hamiltonian for the $B^0\overline{B}^0$ system

$$H^{\rm eff} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{2.2}$$

where $H_{22} = H_{11}$ by CPT invariance. The eigenvalues of the hamiltonian are given by

$$\alpha_\pm = H_{11} \pm \sqrt{H_{12}H_{21}} \tag{2.3}$$

and the eigenvectors are

$$B_\pm = \frac{(1+\epsilon_B)B^0 \pm (1-\epsilon_B)\overline{B}^0}{\sqrt{2(1+|\epsilon_B|^2)}} \equiv p_B B^0 \pm q_B \overline{B}^0\,, \tag{2.4}$$

where

$$\frac{q_B}{p_B} = \frac{1-\epsilon_B}{1+\epsilon_B} = \sqrt{\frac{H_{21}}{H_{12}}}\,. \tag{2.5}$$

Let us now consider the effective lagrangian for $\Delta b = 2$ transitions. The full standard model computation is illustrated in Appendix B; the result is

$$\begin{aligned} \mathcal{L}_{\rm eff}^{\Delta b=2} &= \sum_{l=d,s} \frac{G_F^2}{16\pi^2}\left[\bar b\gamma^\mu(1-\gamma_5)l\right]^2 \times \Bigg[\xi_u^2 m_u^2 + \xi_c^2 m_c^2 + \xi_t^2 m_t^2 \\ &+ 2\xi_u\xi_c \frac{m_u^2 m_c^2}{m_c^2 - m_u^2}\log\frac{m_c^2}{m_u^2} + 2\xi_u\xi_t\frac{m_u^2 m_t^2}{m_t^2 - m_u^2}\log\frac{m_t^2}{m_u^2} \\ &+ 2\xi_c\xi_t\frac{m_c^2 m_t^2}{m_t^2 - m_c^2}\log\frac{m_t^2}{m_c^2}\Bigg] + \text{h.c.}, \end{aligned} \tag{2.6}$$

where

$$\xi_k = V_{kb} V^*_{kl}, \quad k = u, c, t \tag{2.7}$$

and V_{ij} are the elements of the CKM matrix. Equation (2.6) holds in the approximation that all the up quark masses are much smaller than m_W. This is not the case for the top quark, but the qualitative conclusions do not change using the correct expression, which is given in Appendix B. In the Wolfenstein scheme, the CKM matrix is parametrized as

$$V = \begin{bmatrix} 1-\lambda^2/2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda - A^2\lambda^5(\rho + i\eta) & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{bmatrix}, \tag{2.8}$$

where $\lambda = \sin\theta_C = 0.219 \sim 0.224$. For the B_d meson (composed by a b antiquark and a d quark) the relevant combinations of CKM elements are all of the same order of magnitude:

$$V_{bu}V^*_{ud} = \lambda^3 A(\rho - i\eta) \quad V_{bc}V^*_{cd} = -\lambda^3 A \quad V_{bt}V^*_{td} = \lambda^3 A(1-\rho-i\eta)\,. \tag{2.9}$$

For this reason, M_{12} is dominated by the term proportional to m_t^2. Thus, there is a large complex phase in the transition matrix element, which enters directly M_{12} and M_{21} through the combination $(V_{bt}V^*_{td})^2$, giving potentially large values of ϵ_B. This phase, however, is very close to the phase of Γ_{12}. In fact, Γ_{12} is given by the diagram shown in fig. 1. In the intermediate state,

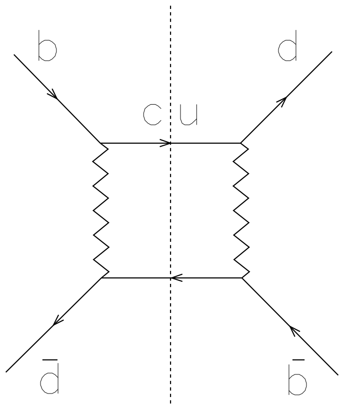

Figure 1: *The diagram contributing to* Γ_{12}.

only c and u quarks can be on the mass shell, and they may be considered light with respect to the total incoming energy, which is the mass of the b (about 5 GeV versus 1.5 GeV for charm). Thus, the graph is proportional to the sum $(V_{bu}V^*_{ud} + V_{bc}V^*_{cd})^2$, which by unitarity is equal to $(V_{bt}V^*_{td})^2$. Thus, since $H_{12} = M_{12} - i\Gamma_{12}/2$, the left hand side of eq. (2.5) is close to a pure phase. In fact, we get

$$\frac{1-\epsilon_B}{1+\epsilon_B} = \sqrt{\frac{H_{21}}{H_{12}}} = \sqrt{\frac{M_{12}^* - i\Gamma_{12}^*/2}{M_{12} - i\Gamma_{12}/2}} = e^{-2i\beta} \,, \tag{2.10}$$

where β is the phase of $(1-\rho-i\eta)$. Thus, ϵ_B is purely imaginary ($\epsilon_B = i\tan\beta$), but not small. A residual phase mismatch in M_{12} and Γ_{12} could give rise to CP violating effects; there is however a further suppression, due to the fact that while M_{12} is proportional to m_t^2, Γ_{12} is proportional to m_b^2. Thus, the suppression factor is as large as $m_b^2/m_t^2 \times m_c^2/m_b^2$, of the order of 10^{-4}.

For the B_s, formed by an s quark and a b antiquark, things are slightly different. Here we have

$$V_{ub}V_{us}^* = \lambda^4 A(\rho - i\eta) \quad V_{cb}V_{cs}^* = \lambda^2 A \quad V_{tb}V_{ts}^* = -\lambda^2 A \,. \tag{2.11}$$

The last term dominates M_{12} because of the large m_t^2 factor. Furthermore, Γ_{12} is dominated by the charm component. So, ϵ_B is very small in this case, of the order of $m_b^2/m_t^2 \approx 10^{-3}$.

2.1. Mixing

The eigenvalues specify the mixing properties of the B^0 system. We easily find that, under the assumption that M_{12} and Γ_{12} have the same phase,

$$\alpha_\pm = H_{11} \pm \sqrt{H_{12}H_{21}} = M \pm |M_{12}| - \frac{i}{2}\left(\Gamma \pm |\Gamma_{12}|\right) \,. \tag{2.12}$$

Thus

$$\Delta m = 2\,|M_{12}| \;, \quad \Delta\Gamma = 2\,|\Gamma_{12}| \;. \tag{2.13}$$

It is clear from the above discussion that $\Delta\Gamma$ is very small compared to ΔM. Thus the two CP eigenstates have nearly the same lifetime, unlike the case of the K^0 system. We are now in the position to estimate the mass splitting. M_{12} is given by the expectation value of the effective hamiltonian over normalized B-meson states. Since the standard Lorentz-invariant state normalization in a space volume $\mathcal{V}$ is

$$\langle B^0|B^0\rangle = (2\pi)^3\delta^3(0)2m_B = \mathcal{V}2m_B \;, \tag{2.14}$$

we have

$$M_{12} = \frac{1}{2m_B\mathcal{V}}\int d^3x \langle B^0|\left(-\mathcal{L}_{\rm eff}^{\Delta b=2}(x)\right)|\overline{B}^0\rangle \;. \tag{2.15}$$

It is easy to show that the integrand is constant, and thus a factor of $\mathcal{V}$ is generated that cancels the $\mathcal{V}$ in the denominator. We get

$$M_{12} = -\frac{1}{2m_B}\langle B^0|\mathcal{L}_{\text{eff}}^{\Delta b=2}|\overline{B}^0\rangle \,. \tag{2.16}$$

Therefore,

$$\Delta m = 2\,|M_{12}| = 2\frac{G_F^2}{16\pi^2}\,\frac{1}{2m_B}\langle B^0|\left[\bar{b}\gamma^\mu(1-\gamma_5)l\right]^2|\overline{B}^0\rangle|V_{tb}V_{tl}|^2 m_t^2 \,. \tag{2.17}$$

The matrix element divided by $2m_B$ is a strong interaction quantity, of the dimension of the cube of a mass. Let us denote its value by C. It will be given by the cube of a typical hadronic scale:

$$\Delta m = \frac{G_F^2}{8\pi^2}C|V_{tb}V_{tl}|^2 m_t^2 \,. \tag{2.18}$$

On the other hand, the B lifetime can be estimated to be

$$\Gamma \approx \frac{G_F^2}{192\pi^3}m_b^5|V_{bc}|^2 \tag{2.19}$$

up to a factor of order 1. So

$$\frac{\Delta m}{\Gamma} \approx 24\pi\frac{Cm_t^2}{m_b^5}\frac{|V_{tl}|^2}{|V_{bc}|^2} \,. \tag{2.20}$$

For the B_d the ratio of CKM matrix elements is of order λ^2; with $C =$ 300 MeV we get $\Delta m/\Gamma \approx 0.8$, with possibly large uncertainties.

Had we found $\Delta m \ll \Gamma$, it would have been very difficult to observe oscillations, since the B mesons would decay much earlier than the time needed for one oscillation. The large mixing in the case of B_d mesons is therefore a combined effect of the large top quark mass, and of the fact that B decays are Cabibbo suppressed. Observe that this is not the case for the D system. There, in fact, the quarks circulating in the box loop are down quarks, so that Δm is proportional to m_b^2, instead of m_t^2. Furthermore, the c quark is heavier the s quark, which is its partner in the weak doublet; therefore, charmed meson decays are not suppressed by off-diagonal CKM matrix elements. As a result, $\Delta m/\Gamma \ll 1$ for D mesons, and the mixing is consequently very small.

Mixing of B mesons can be observed in the following way. Suppose we produce a $b\bar{b}$ pair in an experiment. Suppose that the $\bar{b}$ hadronizes into a charged B meson, and the b hadronizes instead in a B^0. The B^0 will be a pure B^0 at the time of production, but after some time it would oscillate into a $\overline{B}^0$. Then we could look, for example, at a semileptonic decay of the B^0. We would observe a negatively charged lepton if the state that was a B^0 at the initial time decays as a B^0 at subsequent time, and a positively charged lepton otherwise. More specifically, the time evolution of a B meson will be given by

$$|\Psi(t)\rangle = \frac{1}{2p_B}\left[e^{-i(M_1-i\Gamma_1/2)t}|B_+\rangle + e^{-i(M_2-i\Gamma_2/2)t}|B_-\rangle\right], \tag{2.21}$$

so that at $t=0$ we have a pure B^0 state. We have

$$|\Psi(t)\rangle = A_{BB}(t)|B^0\rangle + A_{B\bar{B}}(t)|\overline{B}^0\rangle, \tag{2.22}$$

where, for $\Gamma_1 = \Gamma_2 = \Gamma$,

$$A_{BB}(t) = \frac{e^{-\Gamma t/2}}{2}\left(e^{-iM_1 t} + e^{-iM_2 t}\right) = e^{-\Gamma t/2}e^{-iMt}\cos\frac{t\Delta m}{2} \tag{2.23}$$

$$A_{B\bar{B}}(t) = \frac{q_B}{p_B}\frac{e^{-\Gamma t/2}}{2}\left(e^{-iM_1 t} - e^{-iM_2 t}\right) = i\,\frac{q_B}{p_B}e^{-\Gamma t/2}\sin\frac{t\Delta m}{2}\,. \tag{2.24}$$

Thus, the probability that an initial B^0 decays as a $\overline{B}^0$ is given by

$$\chi = \frac{\int_0^\infty |A_{B\bar{B}}|^2 dt}{\int_0^\infty \left(|A_{B\bar{B}}|^2 + |A_{BB}|^2\right) dt} = \frac{x^2}{2(x^2+1)} \tag{2.25}$$

where $x = \Delta M/\Gamma$. A typical experiment will measure

$$\chi = \frac{\Gamma(B\to\mu^- + X)}{\Gamma(B\to\mu^\pm + X)} \tag{2.26}$$

Experiments that produce B's in the reaction $e^+e^- \to \Upsilon_{4s} \to B_d\bar{B}_d$, like the CLEO and ARGUS experiments, and like the future e^+e^- B-factories, produce pure samples of B_d. In this case, they measure $\chi_d = 0.172 \pm 0.010$, which gives $x_d = \Delta m_B/\Gamma_B \simeq 0.723 \pm 0.032$, a value of order 1, as expected. No direct access to B_s^0 is possible. At LEP, the $b\bar{b}$ system is produced much above threshold. One can, in this case, tag the initial flavour of one b by looking at the decay of a charged B meson on the other side of the event. Once the flavour is known in this way, one measures $\chi_B = f_d\chi_d + f_s\chi_s = 0.118 \pm 0.006$. The sum $f_d + f_s$ is not equal to 1, since in some cases charged B hadrons are produced. However, both f_d and f_s are measured at LEP, and one extracts a limit $\chi_s > 0.497$, which implies $x_s > 14$. The values given here are taken from the 1998 particle data booklet. Updates of these numbers can be found in `http://pdg.lbl.gov`.

2.2. CP violation in B decays

In order to observe CP violation, several conditions have to be met. First of all, complex couplings must be present in the Lagrangian. If these couplings cannot be removed by a phase redefinition of the fields, then CP is violated in the fundamental lagrangian. Even in this case, one can see that one cannot get CP violating observables so easily. Let us consider the example of a charged scalar field $\phi^\pm$ of mass M coupled, through a complex coupling C, to a charged field $\chi^\pm$ of mass m and to a neutral field χ_0 of mass m_0, with $m + m_0 < M$. The amplitude for the decay process $\phi^+ \to \chi^+\chi_0$ is proportional to the complex coupling C, but its square contains CC^*, thus the process proceeds at the same rate for ϕ^+ and for ϕ^-. In order to get a non-zero difference, the phase must be exposed by an interference effect. This can never be achieved through tree diagrams. In fact, even if the same process mentioned above could also proceed via another trilinear coupling C', we still would get $(C+C') \times (C^*+C'^*)$, the same for the particle and the antiparticle. Thus, in order to get any effect, we should have two paths to the same final state that differ by some phase of different origin, i.e. virtual effects. Such a phase does not change sign when going from the particle to the antiparticle. To make an example, suppose that we have vertices $C\phi^+\chi^-\chi_0 + C\phi^+\chi'^-\chi'_0$ with C real, and a complex coupling H for the process $\chi^+\chi_0 \to \chi'^+\chi'_0$. The amplitude for $\phi^+ \to \chi'^+\chi'_0$ is given by the diagrams of fig. 2. It has the coupling structure $C + CAH$, where A is the loop amplitude corresponding to the second diagram of fig. 2, while the amplitude for the conjugate process $\phi^- \to \chi'^-\chi'_0$ would be $C + CAH^*$. Since A is in general a complex number, the rate for the two processes would differ, thus exposing a CP violation effect. In this example the phase is provided by the loop in the Feynman

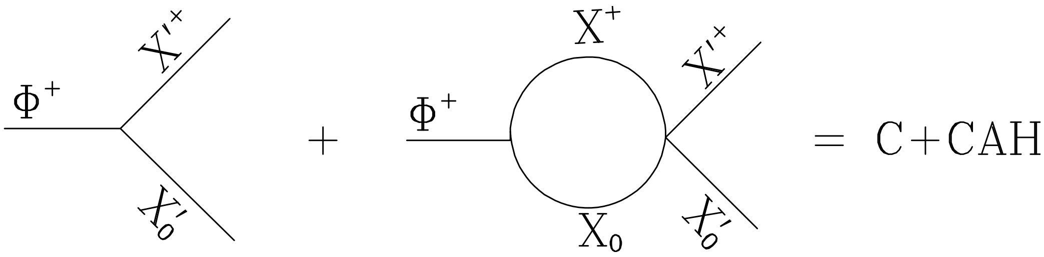

Figure 2: *An example of diagrams contributing to a CP-violating decay of a fictitious charged scalar.*

diagram.

In general, one parametrizes the amplitude for the decay of a meson B

into the final state f as

$$\langle f|B\rangle = \sum_i A_i e^{i(\alpha_i+\phi_i)}, \tag{2.27}$$

where the ϕ_i phases come from CP-violating couplings, and the α_i come from loop diagrams. In the case of weak interactions, the α_i phases are generally produced by strong interactions (since weak interaction loops are much more suppressed). For this reason, the phases α_i are usually referred to as the strong phases, and the ϕ as the weak phases. The CP-conjugate process proceeds with the amplitude

$$\langle \bar{f}|\bar{B}\rangle = \sum_i A_i e^{i(\alpha_i-\phi_i)}; \tag{2.28}$$

thus, in the case when at least two amplitudes with different strong and weak phases are present, it is possible to have different decay rates for the two CP-conjugated processes. Typically, one amplitude will go through the tree-level electroweak interaction, while the other, in order to have a different phase, will have to proceed through a so-called penguin diagram. Observe that in all cases the total decay rates (summed over all channels) must be the same for the particle and the antiparticle, by the CPT theorem. However, the rate into specific channels may differ if CP is violated.

It is possible, in principle, to have CP violation in the decay of charged B mesons, according to the above mechanism. However, the rates are too small to be useful for the determination of the CP-violating phases of the CKM matrix. They can be used, however, to put limits on different CP violating couplings, not arising from the CKM matrix.

The easiest way to produce a CP-violating signal in the B system is to look for a final state with definite CP, into which both the B^0 and the $\overline{B}^0$ can decay. If CP is violated, the amplitude for the two processes will have different weak phases

$$\langle f|B^0\rangle = e^{2i\phi}\langle f|\overline{B}^0\rangle\,. \tag{2.29}$$

Thus, the transition amplitude as a function of time for a state which is produced at time $t=0$ as a pure B^0 state is

$$\begin{aligned}\langle f|\psi(t)\rangle &= A_{BB}(t)\langle f|B^0\rangle + A_{B\bar{B}}\langle f|\overline{B}^0\rangle \\ &= A_{BB}(t)\langle f|B^0\rangle + A_{B\bar{B}}e^{-2i\phi}\langle f|B^0\rangle,\end{aligned} \tag{2.30}$$

so that

$$\Gamma_t(B^0\to f) \propto \left|A_{BB}(t) + A_{B\bar{B}}e^{-2i\phi}\right|^2\,. \tag{2.31}$$

As shown in the previous section, one easily obtains the result

$$\Gamma_t(B^0\to f) \approx e^{-\Gamma t}\left[1+\sin 2(\phi+\beta)\sin t\,\Delta m\right]\,. \tag{2.32}$$

The extra β phase comes from eq. (2.10). This gives rise to a time-integrated asymmetry

$$\frac{\Gamma(B^0 \to f) - \Gamma(\overline{B}^0 \to f)}{\Gamma(B^0 \to f) + \Gamma(\overline{B}^0 \to f)} = \frac{2x}{1+x^2} \sin 2(\phi + \beta)\,. \tag{2.33}$$

Thus, large asymmetries are possible in B^0 decays to CP eigenstates.

The potential of CP violation measurements in the B^0 system is better understood by considering the so-called unitarity triangle. Unitarity of the CKM matrix implies that the scalar product of any two rows is zero. Thus, the three complex numbers $V_{ud}V^*_{ub}$, $V_{cd}V^*_{cb}$, and $V_{td}V^*_{tb}$ add up to zero, which implies that if we associate to each complex number a two dimensional vector, the three vectors above form a triangle. This is shown in fig. 3, where the sides of the triangle are given using the Wolfenstein parametrization. For each

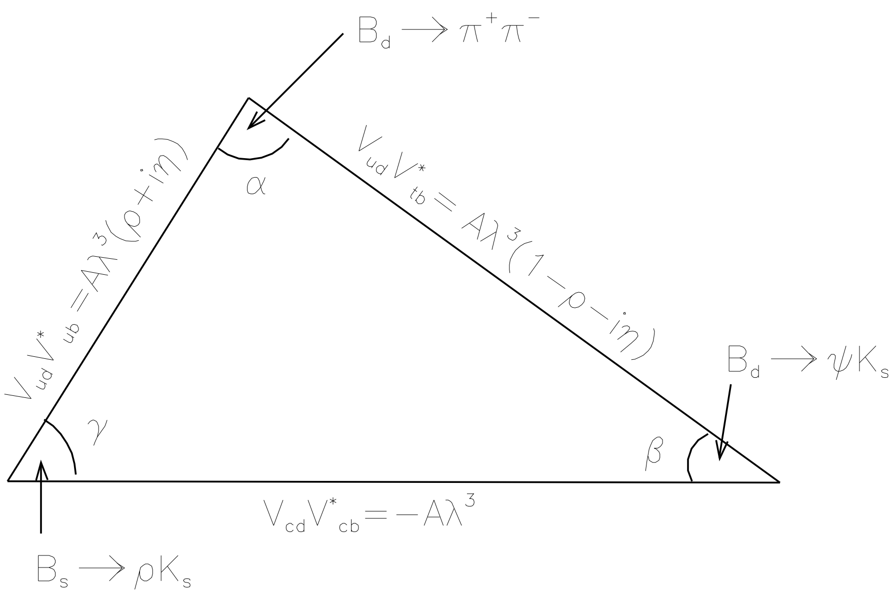

Figure 3: *The unitarity triangle, and some of the processes relevant to its measurement.*

angle, one example is given of a decay that can be used for its determination. This is illustrated in fig. 4.

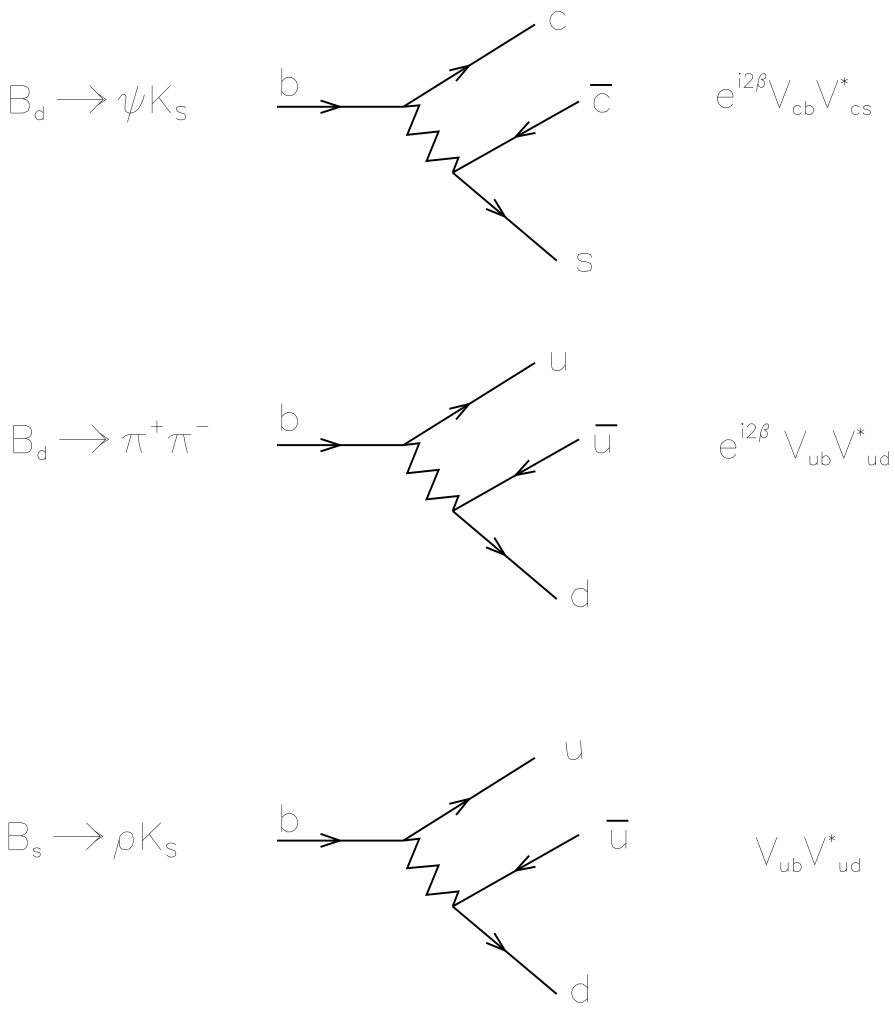

Figure 4: *The phases measured in each sample decay. Since $V_{cb}V_{cs}^*$ is real, the first process measures β. The second process is sensitive to the phase $\gamma+\beta$, and therefore measures α. The third process measures γ directly.*

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A Effective hamiltonian for the $K^0 - \overline{K}^0$ system

In this Appendix we derive an expression of the weak effective hamiltonian $H^{\rm eff}$ in the framework of old-fashioned perturbation theory. For a more rigorous treatment of the subject, see ref. [1]. We take as our unperturbed hamiltonian H the sum of the strong and electromagnetic interacion hamiltonians, and consider the weak hamiltonian H_W as a perturbation. Therefore, the states $|\alpha\rangle$, with $\alpha = K^0, \overline{K}^0$ can be taken as our unperturbed eigenstates. We are interested in computing the elements of the S-matrix:

$$S_{\beta\alpha} = \langle\beta|T \exp\left[-i\int_{-\infty}^{+\infty} H_W(t)dt\right]|\alpha\rangle, \tag{A.1}$$

where T is the time-ordering operator, $H_W(t) = e^{iHt}H_W e^{-iHt}$ is the weak hamiltonian in the interaction picture, and β, α can be either K^0 and $\overline{K}^0$. At the second order in H_W we find

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta|\alpha\rangle - i\langle\beta|\int_{-\infty}^{+\infty} e^{iHt}H_W e^{-iHt}dt|\alpha\rangle \\ &- \frac{1}{2}\langle\beta|\int_{-\infty}^{+\infty} e^{iHt}H_W e^{-iHt}dt\int_{-\infty}^{+\infty} e^{iHt'}H_W e^{-iHt'}dt'\theta(t-t')|\alpha\rangle \\ &- \frac{1}{2}\langle\beta|\int_{-\infty}^{+\infty} e^{iHt'}H_W e^{-iHt'}dt'\int_{-\infty}^{+\infty} e^{iHt}H_W e^{-iHt}dt\theta(t'-t)|\alpha\rangle \\ &\equiv S^{(0)}_{\beta\alpha} + S^{(1)}_{\beta\alpha} + S^{(2)}_{\beta\alpha}, \end{aligned} \tag{A.2}$$

where θ is the usual step function. Let us consider the three terms separately. Clearly,

$$S^{(0)}_{\beta\alpha} = \delta_{\beta\alpha} \tag{A.3}$$

since the unperturbed eigenstates are orthogonal and normalized. Also the first-order term is easily computed:

$$S^{(1)}_{\beta\alpha} = -i\int_{-\infty}^{+\infty} e^{i(E_\beta - E_\alpha)t}dt\langle\beta|H_W|\alpha\rangle = -2\pi i\delta(E_\beta - E_\alpha)\langle\beta|H_W|\alpha\rangle, \tag{A.4}$$

where E_β and E_α are the unperturbed energy eigenvalues of $|\beta\rangle$ and $|\alpha\rangle$.

The second-order term can be computed observing that the two terms are equal, as apparent after exchanging t and t' in the second one, and inserting the identity operator

$$I = \sum_\lambda |\lambda\rangle\langle\lambda|, \tag{A.5}$$

where $\{|\lambda\rangle\}$ is an orthonormal basis for the unperturbed hamiltonian. We get

$$\begin{aligned}
S^{(2)}_{\beta\alpha} &= -\sum_\lambda \langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle \int_{-\infty}^{+\infty} e^{i(E_\beta-E_\lambda)t}dt \int_{-\infty}^{+\infty} e^{i(E_\lambda-E_\alpha)t'}dt'\theta(t-t') \\
&= -\sum_\lambda \langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle \int_{-\infty}^{+\infty} e^{i(E_\beta-E_\lambda)\tau}d\tau\theta(\tau) \int_{-\infty}^{+\infty} e^{i(E_\beta-E_\alpha)t'}dt' \\
&= -2\pi\delta(E_\beta-E_\alpha)\sum_\lambda \langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle \int_0^{+\infty} e^{i(E_\beta-E_\lambda)\tau}d\tau. \qquad (A.6)
\end{aligned}$$

The last integral can be evaluated by replacing

$$E_\beta - E_\lambda \to E_\beta - E_\lambda + i\eta, \quad \eta > 0 \qquad (A.7)$$

and then taking the limit $\eta \to 0$:

$$\int_0^{+\infty} e^{i(E_\beta-E_\lambda+i\eta)\tau}d\tau = \frac{i}{E_\beta - E_\lambda + i\eta}. \qquad (A.8)$$

We get finally

$$S^{(2)}_{\beta\alpha} = -2\pi i\delta(E_\beta-E_\alpha)\sum_\lambda \frac{\langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle}{E_\beta - E_\lambda + i\eta}. \qquad (A.9)$$

Summing the three terms, we get, at the second order in H_W,

$$S_{\beta\alpha} = \delta_{\beta\alpha} - 2\pi i\delta(E_\beta-E_\alpha)\left[\langle\beta|H_W|\alpha\rangle + \sum_\lambda \frac{\langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle}{E_\beta - E_\lambda + i\eta}\right]. \qquad (A.10)$$

We can now define an effective weak hamiltonian $H_W^{\rm eff}$ in such a way that the S-matrix computed with $H_W^{\rm eff}$ at the first order has the same matrix elements of eq. (A.10). This condition gives

$$H^{\rm eff}_{W\ \beta\alpha} = \langle\beta|H_W|\alpha\rangle + \sum_\lambda \frac{\langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle}{E_\beta - E_\lambda + i\eta}. \qquad (A.11)$$

If K^0, $\overline{K}^0$ are in their rest frame, then $E_\beta = E_\alpha = m_K$, so that a full effective hamiltonian can be defined by

$$H^{\rm eff}_{\beta\alpha} = m_K\delta_{\beta\alpha} + H^{\rm eff}_{W\ \beta\alpha}. \qquad (A.12)$$

Notice that $H^{\rm eff}$ is not hermitian, as expected. The following identity holds:

$$\frac{1}{E_\beta - E_\lambda + i\eta} = P\frac{1}{E_\beta - E_\lambda} - i\pi\delta(E_\beta - E_\lambda), \qquad (A.13)$$

where P denotes the principal value. We can conclude that

$$H^{\rm eff}_{\beta\alpha} = M_{\beta\alpha} - \frac{i}{2}\Gamma_{\beta\alpha}, \tag{A.14}$$

where

$$M_{\beta\alpha} = m_K\delta_{\beta\alpha} + \langle\beta|H_W|\alpha\rangle + P\sum_\lambda \frac{\langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle}{m_K - E_\lambda} \tag{A.15}$$

$$\Gamma_{\beta\alpha} = 2\pi\sum_\lambda \langle\beta|H_W|\lambda\rangle\langle\lambda|H_W|\alpha\rangle\delta(m_K - E_\lambda). \tag{A.16}$$

The sum in eq. (A.16) is extended to on-shell amplitudes, and is therefore dominated by the contribution of the state

$$|\lambda\rangle = |\pi\pi, I = 0\rangle = |0\rangle, \tag{A.17}$$

and we have

$$\begin{aligned}\Gamma_{12} &\simeq 2\pi\langle K^0|H_W|0\rangle\langle 0|H_W|\overline{K}^0\rangle\delta(m_K - E_0) \\ &= 2\pi\left|\langle 0|H_W|K^0\rangle\right|^2 e^{i\phi}\delta(m_K - E_0),\end{aligned} \tag{A.18}$$

where

$$e^{i\phi} = \frac{\langle 0|H_W|\overline{K}^0\rangle}{\langle 0|H_W|K^0\rangle}. \tag{A.19}$$

B Calculation of the box diagram

The diagram giving rise to mixing is the box diagram represented in fig. 5. One assumes that the external momenta are negligible due to the large

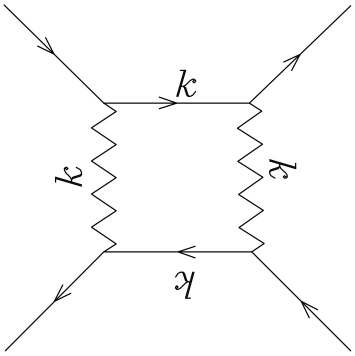


Figure 5: Box diagram.

momenta flowing inside the graph, which is then computed at zero external momenta. The fact that in this approximation we obtain an infrared finite result can be interpreted exactly in these terms, which is to say that the graph is dominated by momenta of the order of the masses involved in the graph. In the internal loop there is thus a single momentum k flowing in each line. The W propagator is given by

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}}{k^2 - m_W^2}$$

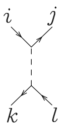
and the vertices are given by

$$\gamma_\mu \left(1 - \gamma_5\right) .$$

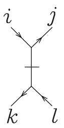
The masses in the numerators of the fermion lines never contribute, since

$$\gamma_\mu \left(1 - \gamma_5\right) \left(\not{k} - m\right) \gamma_\nu \left(1 - \gamma_5\right) = \gamma_\mu \left(1 - \gamma_5\right) \not{k}\, \gamma_\nu \left(1 - \gamma_5\right) .$$

We introduce the notation



$$= \quad \bar{\psi}_j \gamma^\mu \left(1 - \gamma_5\right) \psi_i \;\; \bar{\psi}_k \gamma_\mu \left(1 - \gamma_5\right) \psi_l ,$$



$$= \quad \bar{\psi}_j \not{k} \left(1 - \gamma_5\right) \psi_i \;\; \bar{\psi}_k \not{k} \left(1 - \gamma_5\right) \psi_l . \qquad \text{(B.1)}$$

For each vertex we include a factor

$$i\frac{g}{2\sqrt{2}}$$

and we use the identity

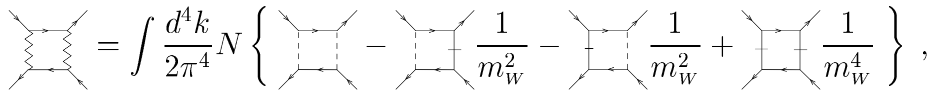
$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}\,.$$

Furthermore, we must include a factor $\xi_a\xi_b$ to account for CKM factor on each vertex. The indices a and b label the two quark lines inside the loop. The product of all the denominators and couplings is then given by

$$N = \xi_a\xi_b\left(\frac{G_F}{\sqrt{2}}\right)^2 m_W^4\,\frac{1}{k^2-m_a^2}\,\frac{1}{k^2-m_b^2}\,\frac{1}{(k^2-m_W^2)^2}\,, \tag{B.2}$$

where m_a and m_b are the quark masses in the internal upper and lower lines, respectively. A sum over $a, b = u, c, t$ is understood.

We have



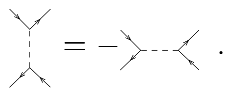
(B.3)

where in the diagrams in parentesis the internal fermion lines represent only the numerator structure, since the denominators are already included in N.

We will use the Fierz identity

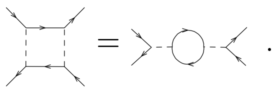
$$\bar\psi_j\gamma^\mu(1-\gamma_5)\psi_i\;\;\bar\psi_k\gamma_\mu(1-\gamma_5)\psi_l \;=\; \bar\psi_j\gamma^\mu(1-\gamma_5)\psi_l\;\;\bar\psi_k\gamma_\mu(1-\gamma_5)\psi_i\,, \tag{B.4}$$

which can be expressed graphically as



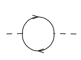
(B.5)

Using the Fierz identity we find immediately



(B.6)

Thus

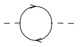


$$= \mathrm{Tr}\left[\gamma^\alpha(1-\gamma_5)\not{k}\gamma^\beta(1-\gamma_5)\not{k}\right] = 2\,\mathrm{Tr}\left[\gamma^\alpha\not{k}\gamma^\beta\not{k}\right] = 8(2k^\alpha k^\beta - k^2 g^{\alpha\beta})\,. \tag{B.7}$$

Since we will perform an angular integration over d^4k, and since the integrand is fully rotational invariant, we can replace immediately

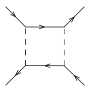
$$k^\alpha k^\beta \to \frac{k^2\,g^{\alpha\beta}}{4}\,, \tag{B.8}$$

and thus

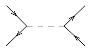


$$= -4k^2\, g^{\alpha\beta}\ . \tag{B.9}$$

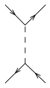
So



$$=$$



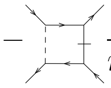
$$\left(-4k^2\right) =$$



$$4k^2\ . \tag{B.10}$$

Now we turn to

$$-$$



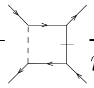
$$\frac{1}{m_W^2}\ . \tag{B.11}$$

In the upper line we have

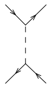
$$\not{k}(1-\gamma_5)\not{k}\gamma^\mu(1-\gamma_5) = 2k^2\,\gamma^\mu(1-\gamma_5)\,, \tag{B.12}$$

and likewise for the lower line. We thus have immediately

$$-$$



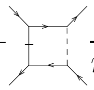
$$\frac{1}{m_W^2} =$$



$$\frac{-4k^4}{m_W^2}\ . \tag{B.13}$$

Also, the diagram

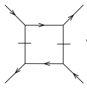
$$-$$



$$\frac{1}{m_W^2} \tag{B.14}$$

gives the same contribution.

Finally, let us consider



$$\frac{1}{m_W^4}\ . \tag{B.15}$$

In the upper line we have

$$\not{k}(1-\gamma_5)\not{k}\not{k}(1-\gamma_5) = 2k^2\,k_\alpha\,\gamma^\alpha(1-\gamma_5)\,, \tag{B.16}$$

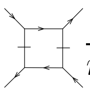
and in the lower line

$$\not{k}(1-\gamma_5)\not{k}\not{k}(1-\gamma_5) = 2k^2\,k_\beta\,\gamma^\beta(1-\gamma_5)\,. \tag{B.17}$$

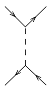
Now, as we have seen before, we can replace

$$k_\alpha k_\beta \to g_{\alpha\beta}\frac{k^2}{4}\,, \tag{B.18}$$

so that

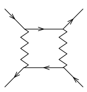


$$\frac{1}{m_W^4} =$$

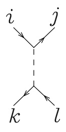


$$\frac{k^6}{m_W^4}\ . \tag{B.19}$$

Collecting all results, we get



$$= \sum_{a,b}$$



$$\xi_a\xi_b\left(\frac{G_F m_W}{\sqrt{2}}\right)^2 m_W^2 \int \frac{d^4k}{(2\pi)^4}\frac{4k^2 + k^6/m_W^4 - 8k^4/m_W^2}{(k^2-m_a^2)(k^2-m_b^2)(k^2-m_W^2)^2}. \tag{B.20}$$

The loop integral can be computed by means of usual techniques. First, we perform a rotation to Euclidean four-momenta $\bar{k}$, with $\bar{k}_i = k_i$ for $i = 1, \ldots, 3$ and $\bar{k}_4 = ik_0$. We have therefore

$$\begin{aligned} I &\equiv m_W^2 \sum_{a,b} \xi_a \xi_b \int \frac{d^4k}{(2\pi)^4} \frac{4k^2 + k^6/m_W^4 - 8k^4/m_W^2}{(k^2 - m_a^2)(k^2 - m_b^2)(k^2 - m_W^2)^2} \\ &= i\, m_W^2 \sum_{a,b} \xi_a \xi_b \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{4\bar{k}^2 + \bar{k}^6/m_W^4 + 8\bar{k}^4/m_W^2}{(\bar{k}^2 + m_a^2)(\bar{k}^2 + m_b^2)(\bar{k}^2 + m_W^2)^2}. \end{aligned} \tag{B.21}$$

The angular integration can be performed immediately, since the integrand only depends on $\bar{k}^2$, giving

$$I = \frac{im_W^2}{16\pi^2} \sum_{a,b} \xi_a \xi_b \int_0^\infty d\bar{k}^2\, \bar{k}^2\, \frac{4\bar{k}^2 + \bar{k}^6/m_W^4 + 8\bar{k}^4/m_W^2}{(\bar{k}^2 + m_a^2)(\bar{k}^2 + m_b^2)(\bar{k}^2 + m_W^2)^2}, \tag{B.22}$$

or, defining $x_{a,b} = m_{a,b}^2/m_W^2$ and changing integration variable,

$$I = \frac{i}{16\pi^2} \sum_{a,b} \xi_a \xi_b \int_0^\infty dx\, \frac{4x^2 + x^4 + 8x^3}{(x + x_a)(x + x_b)(x + 1)^2}\,. \tag{B.23}$$

The x integration is divergent in the large-x (ultraviolet) region; however, the divergence cancels among the different contributions. In fact, one can make the replacements

$$\frac{1}{x + x_a} \to \frac{1}{x + x_a} - \frac{1}{x}; \quad \frac{1}{x + x_b} \to \frac{1}{x + x_b} - \frac{1}{x} \tag{B.24}$$

without changing the result, since $\sum_a \xi_a = 0$ due to the unitarity of the CKM matrix. We have therefore

$$\begin{aligned} I &= \frac{i}{16\pi^2} \sum_{a,b} \xi_a \xi_b \int_0^\infty dx\, \frac{4x^2 + x^4 + 8x^3}{(x+1)^2} \left[\frac{1}{x + x_a} - \frac{1}{x}\right] \left[\frac{1}{x + x_b} - \frac{1}{x}\right] \\ &= \frac{i}{16\pi^2} \sum_{a,b} \xi_a \xi_b\, x_a x_b \int_0^\infty dx\, \frac{4 + x^2 + 8x}{(x + x_a)(x + x_b)(x + 1)^2} \end{aligned} \tag{B.25}$$

which is manifestly convergent. The result is

$$\begin{aligned} I = \frac{i}{16\pi^2} \sum_{a,b} \xi_a \xi_b \frac{x_a x_b}{x_a - x_b} \Bigg[&\frac{(x_a^2 - 8x_a + 4)\log x_a + 3(x_a - 1)}{(x_a - 1)^2} \\ &- \frac{(x_b^2 - 8x_b + 4)\log x_b + 3(x_b - 1)}{(x_b - 1)^2} \Bigg] \end{aligned} \tag{B.26}$$

In the limit $x_a, x_b \ll 1$ (quark masses much smaller than the W mass), the above expression simplifies considerably; we find

$$I \simeq \frac{i}{4\pi^2} \sum_{a,b} \xi_a \xi_b \frac{x_a x_b}{x_a - x_b} \log \frac{x_a}{x_b} , \tag{B.27}$$

which is easily recognized to be equivalent to eq. (2.6).