YITP-SB-05-27

Quantum Chromodynamics[1]

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1 Introduction

Quantum chromodynamics, or QCD, as it is normally known in high energy physics, is the quantum field theory that describes the strong interactions. It is the $SU(3)$ gauge theory of the current Standard Model for elementary particles and forces, $SU(3) \times SU(2)_L \times U(1)$, which encompasses the strong, electromagnetic and weak interactions. The symmetry group of QCD, with its eight conserved charges, is referred to as color $SU(3)$. As is characteristic of quantum field theories, each field may be described in terms of quantum waves or particles.

Because it is a gauge field theory, the fields that carry the forces of QCD transform as vectors under the Lorentz group. Corresponding to these vector fields are the particles called *gluons*, which carry an intrinsic angular momentum, or spin, of one in units of $\hbar$. The strong interactions are understood as the cumulative effects of gluons, interacting among themselves and with the quarks, the spin-1/2 particles of the Dirac quark fields.

There are six quark fields of varying masses in QCD. Of these, three are called *light* quarks, in a sense to be defined below, and three *heavy*. The light quarks are the up (u), down (d) and strange (s), while the heavy quarks are the charm (c), bottom (b) and top (t). Their electric charge are famously $e_f = 2e/3\,(u,c,t)$ and $e_f = -e/3\,(d,s,b)$, with e the positron

[1] Contribution to appear in the Elsevier *Encyclopedia of Mathematical Physics*.

charge. The gluons interact with each quark field in an identical fashion, and the relatively light masses of three of the quarks provide the theory with a number of approximate global symmetries that profoundly influence the manner in which QCD manifests itself in the Standard Model.

These quark and gluon fields and their corresponding particles are enumerated with complete confidence by the community of high energy physicists. Yet none of these particles has ever been observed in isolation, as one might observe a photon or an electron. Rather, all known strongly interacting particles are colorless; most are *mesons*, combinations with the quantum numbers of a quark q and a antiquark $\bar{q}'$, or *baryons* with the quantum numbers of (possibly distinct) combinations of three quarks $qq'q''$. This feature of QCD, that its underlying fields never appear as asymptotic states, is called *confinement*. The very existence of confinement required new ways of thinking about field theory, and only with these was the discovery and development of QCD possible.

2 The Background of QCD

The strong interactions have been recognized as a separate force of nature since the discovery of the neutron as a constituent of atomic nuclei, along with the proton. Neutrons and protons (collectively, nucleons) possess a force, attractive at intermediate distances and so strong that it overcomes the electric repulsion of the protons, each with charge e. A sense of the relative strengths of the electromagnetic and strong interactions may be inferred from the typical distance between mutually-repulsive electrons in an atom, about 10^{-8} cm, and the typical distance between protons in a nucleus, of order 10^{-13} cm.

The history that led up to the discovery of QCD is a fascinating one, beginning with Yukawa's 1935 theory of pion exchange as the source of the forces that bind nuclei, still a useful tool for low energy scattering. Other turning points include the creation of nonabelian gauge theories by Yang and Mills in 1954, the discovery of the quantum number known as strangeness, the consequent development of the quark model and then the proposal of color as a global symmetry. The role of point-like constituents in hadrons was foreshadowed by the identification of electromagnetic and weak currents and the analysis of their quantum-mechanical algebras. Finally, the observation of *scaling* in deep-inelastic scattering, which we will describe below, made quantum chromodynamics, with color as a local symmetry, the unique explanation of the strong interactions, through its property of asymptotic freedom.

3 The Lagrangian and its Symmetries

The QCD Lagrangian may be written as

$$\mathcal{L} = \sum_{f=1}^{n_f} \bar{q}_f \, (i \not{D}[A] - m_f\,)\, q_f - \frac{1}{2}\,\mathrm{Tr}[F^2_{\mu\nu}(A)] - \frac{\lambda}{2}\,(B_a(A))^2 + \bar{c}_b \left[\frac{\delta B_b(A)}{\delta \alpha_a} \right] c_a \;, \tag{1}$$

with $\not{D}[A] = \gamma \cdot \partial + i g_s \gamma \cdot A$ the covariant derivative in QCD. The γ^μ are the Dirac matrices, satisfying the anticommutation relations, $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$. The $SU(3)$ gluon fields are $A^\mu = \sum_{a=1}^{8} A^\mu_a T_a$, where T_a are the generators of $SU(3)$ in the fundamental representation. The field strengths $F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_s [A_\mu, A_\nu]$ specify the three- and four-point gluon couplings of nonabelian gauge theory. In QCD, there are $n_f = 6$ flavors of quark fields, q_f, with conjugate $\bar{q}_f = q_f^\dagger \gamma^0$.

The first two terms in the expression (1) make up the classical Lagrangian, followed by the gauge fixing term, specified by a (usually, but not necessarily linear) function $B_a(A)$, and the ghost Lagrangian. The ghost (antighost) fields c_a ($\bar{c}_a$) carry the same adjoint index as the gauge fields.

The classical QCD Lagrangian before gauge fixing is invariant under the local gauge transformations

$$\begin{aligned}
A'_\mu(x) &= \frac{i}{g_s} \partial_\mu \Omega(x)\, \Omega^{-1}(x) + \Omega(x)\, A'_\mu(x)\, \Omega^{-1}(x) \\
&= A_\mu(x) - \partial_\mu \delta\alpha(x) + i g_s\, [\delta\alpha(x), A_\mu(x)] + \dots \\
\psi'_i(x) &= \Omega(x)_{ij}\, \psi_j(x) = \psi_i(x) + i g_s\, \delta\alpha(x)_{ij}\, \psi_j(x) + \dots \\
\delta\alpha(x) &= \sum_{a=1}^{8} \delta\alpha_a(x)\, T_a\,.
\end{aligned} \tag{2}$$

The full QCD action including gauge fixing and ghost terms is also invariant under the BRST transformations, with $\delta\xi$ an anticommuting variable,

$$\begin{aligned}
\delta A_{\mu,a} &= (\delta_{ab}\partial_\mu + g A_{\mu c} f_{abc})\; c_b\, \delta\xi \\
\delta c_a = -\frac{1}{2}\, g C_{abc}\, c_b\, c_c\, \delta\xi \qquad & \delta\bar{c} = \lambda B_a\, \delta\xi \\
\delta\psi_i &= i g\, [T_b]_{ij}\, c_b \psi_j \;,
\end{aligned} \tag{3}$$

and with f_{abc} the $SU(3)$ structure constants. The Jacobean of these transformation is unity.

In addition, neglecting masses of the light quarks, u, d and s, the QCD Lagranian has a class of global flavor and chiral symmetries, the latter connecting left- and right-handed components of the quark fields, $\psi_{L,R} \equiv (1/2)(1 \mp \gamma_5)\,\psi$,

$$\psi'(x) = \mathrm{e}^{i\alpha\gamma_5^P}\,\psi(x)\,, \quad P = 0, 1\,. \tag{4}$$

Here power $P = 0$ describes phase, and $P = 1$ chiral, transformations. Both transformations can be extended to transformations among the light flavors, by letting ψ become a vector, and α an element in the Lie Algebra of $SU(M)$, with $M = 2$ if we take only the u and d quarks, and $M = 3$ if we include the somewhat heavier strange quark. These symmetries, not to be confused with the local symmetries of the Standard Model, are strong isospin and its extension to the "eightfold way", which evolved into the (3-)quark model of Gell-Mann and Zweig. The many successes of these formalisms are automatically incorporated into QCD.

4 Green Functions, Phases and Gauge Invariance

In large part, the business of quantum field theory is to calculate Green functions,

$$G_n\,(x_1 \dots x_n) = \langle 0 \mid T\ (\Phi_1(x_1) \dots \Phi_i(x_i) \dots \Phi_n(x_n)\ ) \,|\, 0 \rangle\ , \tag{5}$$

where T denotes time ordering. The $\Phi_i(x)$ are elementary fields, such as A or q_f, or composite fields, such as currents like $J^\mu = \bar{q}_f \gamma^\mu q_f$. Such a Green function generates amplitudes for the scattering of particles of definite momenta and spin, when in the limit of large times the x_i-dependence of the Green function is that of a plane wave. For example, we may have in the limit $x_i^0 \to \infty$,

$$\begin{aligned} &G_n\,(x_1 \dots x_n) \to \\ &\phi_i(p,\lambda)\,\mathrm{e}^{ip\cdot x_i}\ \langle (p,\lambda) \mid T\ (\Phi_1(x_1) \dots \Phi_{i-1}(x_{i-1})\,\Phi_{i+1}(x_{i+1}) \dots \Phi_n(x_n)\ ) \,|\, 0 \rangle\ , \end{aligned} \tag{6}$$

where $\phi_i(p,\lambda)$ is a solution to the free-field equation for field Φ_i, characterized by momentum p and spin λ. (An inegral over possible momenta p is understood.) When this happens for field i, the vacuum state is replaced by $|(p,\lambda)\rangle$, a particle state with precisely this momentum and spin; when it occurs for all fields, we derive a scattering (S) matrix amplitude. In essence, the statement of confinement is that Green functions with fields $q_f(x)$ never behave as plane waves at large times in the past or future. Only Green functions of color singlet composite fields, invariant under gauge transformations, are associated with plane wave behavior at large times.

Green functions remain invariant under the BRST transformations (3), and this invariance implies a set of Ward identities

$$\frac{\delta}{\delta\xi(z)} \sum_{i=1}^{n} \langle 0 \mid T \ \left(\Phi_1(x_1)\ldots\delta_{\rm BRS}\Phi_i(x_i)\ldots\Phi_n(x_n)\ \right) | \, 0\rangle = 0\,. \tag{7}$$

The variation of the antighost as in (3) is equivalent to an infinitesimal change in the gauge fixing term; variations in the remaining fields all cancel single-particle plane wave behavior in the corresponding Green functions. These identities then ensure the gauge-invariance of the perturbative S-matrix, a result that turns out to be useful despite confinement.

To go beyond a purely perturbative description of QCD, it is useful to introduce a set of nonlocal operators that are variously called nonabelian phases, ordered exponentials and Wilson lines,

$$U_C(z,y) = P\exp\left[-ig_s\int_y^z dx^\mu\, A_\mu(x)\right]\,, \tag{8}$$

where C is some self-avoiding curve between y and z. The U's transform at each end linearly in nonabelian gauge transformations $\Omega(x)$ at that point,

$$U'_C(z,y) = \Omega(z)\, U_C(z,y)\, \Omega^{-1}(y)\,. \tag{9}$$

Especially interesting are closed curves C, for which $z = y$. The phases about such closed loops are, like their abelian counterparts, sensitive to the magnetic flux that they enclose, even when the field strengths vanish on the curve.

5 QCD at the Shortest and Longest Distances

Much of the fascination of QCD is its extraordinary variation of behavior at differing distance scales. Its discovery is linked to asymptotic freedom, which characterizes the theory at the shortest scales. Asymptotic freedom also suggests (and in part provides) a bridge to longer distances.

Most analyses in QCD begin with a path integral formulation in terms of the elementary fields $\Phi_a = q_f \ldots$,

$$G_n\left(x_i, (z_j, y_j)\right) = \int \left[\prod_{a=q,\bar{q},G,c,\bar{c}} \mathcal{D}\Phi_a \right] \prod_i \Phi_i(x_i) \prod_j U_{C_j}(z_j, y_j)\, \mathrm{e}^{iS_{\mathrm{QCD}}}\,, \quad (10)$$

with S_{QCD} the action. Perturbation theory keeps only the kinetic Lagrangian, quadratic in fields, in the exponent, and expands the potential terms in the coupling. This procedure produces Feynman diagrams, with vertices corresponding to the cubic and quartic terms in the QCD Lagrangian (1).

Most nonperturbative analyses of QCD require studying the theory on a Eucliean, rather than Minkowski space, related by an analytic continuation in the times x^0, y^0, z^0 in G_n from real to imaginary values. In Euclidean space, we find, for example, classical solutions to the equations of motion, known as instantons, that provide nonperturbative contributions to the path integral. Perhaps the most flexible nonperturbative approach approximates the action and the measure at a lattice of points in four dimensional space. For this purpose, integrals over the gauge fields are replaced by averages over "gauge links", of the form of Eq. (8) between neighboring points.

Perturbation theory is most useful for processes that occur over short time scales and at high relative energies. Lattice QCD, on the other hand, can simulate processes that take much longer times, but is less useful when large momentum transfers are involved. The gap between the two methods remains quite wide, but between the two they have covered enormous ground, enough to more than confirm QCD as the theory of strong interactions.

5.1 Asymptotic freedom

Quantum chromodynamics is a renormalizable field theory, which implies that the coupling constant g must be defined by its value at a *renormalization scale*, and is denoted $g(\mu)$. Usually, the magnitude of $\alpha_s(\mu) \equiv g^2/4\pi$, is quoted at $\mu = m_Z$, where it is approximately 0.12. In effect, $g(\mu)$ controls the amplitude that connects any state to another state with one more or one fewer gluon, including quantum corrections that occur over time scales from zero up to $\hbar/\mu$ (if we measure μ in units of energy). The QCD Ward identities mentioned above ensure that the coupling is the same for both quarks and gluons, and indeed remains the same in all terms in the Lagrangian, ensuring that the symmetries of QCD are not destroyed by renormalization.

Quantum corrections to gluon emission are not generally computable directly in renormalizable theories, but their dependence on μ is computable, and is a power series in $\alpha_s(\mu)$ itself,

$$\mu^2 \frac{d\alpha_s(\mu)}{d\mu^2} = -b_0 \frac{\alpha_s^2(\mu)}{4\pi} - b_1 \frac{\alpha_s^3(\mu)}{(4\pi)^2} + \ldots \equiv \beta(\alpha_s)\,, \tag{11}$$

where $b_0 = 11 - 2n_f/3$ and $b_1 = 2(31 - 19n_f/3)$. The celebrated minus signs on the right-hand side are associated with both the spin and self-interactions of the gluons.

The solution to this equation provides an expression for α_s at any scale μ_1 in terms of its value at any other scale μ_0. Keeping only the lowest-order, b_0, term, we have

$$\alpha_s(\mu_1) = \frac{\alpha_s(\mu_0)}{1 + (b_0/4\pi)\,\ln(\mu_1^2/\mu_0^2)} = \frac{4\pi}{b_0 \ln(\mu_1^2/\Lambda_{\rm QCD}^2)}\,, \tag{12}$$

where in the second form, we have introduced $\Lambda_{\rm QCD}$, the scale parameter of the theory, which embodies the condition that we get the same coupling at scale μ_1 no matter which scale μ_0 we start from. Asymptotic freedom consists of the observation that at larger renormalization masses μ, or correspondingly shorter time scales, the coupling weakens, and indeed vanishes in the limit $\mu \to \infty$. The other side of the coin is that over longer times or lower momenta the coupling grows. Eventually, near the pole at $\mu_1 = \Lambda_{\rm QCD}$, the lowest-order approximation to the running fails, and the theory becomes essentially nonperturbative. Thus the discovery of asymptotic freedom suggested, although it certainly doesn't prove, that QCD is capable of producing very strong forces, and confinement at long distances. Current estimates of $\Lambda_{\rm QCD}$ are around 200 MeV.

5.2 Spontaneous breaking of chiral symmetry

The number of quarks and their masses is an external input to QCD. In the Standard Model masses are provided by the Higgs mechanism, but in QCD they are simply parameters. Because the Standard Model has chosen several of the quarks to be especially light, QCD incorporates the chiral symmetries implied by Eq. (4) (with $P = 1$). In the limit of zero quark masses, these symmetries becomes exact, respected to all orders of perturbation theory, that is, for any finite number of gluons emitted or absorbed.

At distances on the order to $1/\Lambda_{\rm QCD}$, however, QCD cannot respect chiral symmetry, which would require each state to have a degenerate partner with the opposite parity, something not seen in nature. Rather, QCD produces, nonperturbatively, nonzero values for matrix elements that mix right- and left-handed fields, such as $\langle 0|\bar{u}_L u_R|0\rangle$, with u the up-quark field. Pions are the Goldstone bosons of this symmetry, and may be thought of as ripples in the chiral condensate, rotating it locally as they pass along. The observation that these Goldstone bosons are not exactly massless is due to the *current* masses of the quarks, their values in $\mathcal{L}_{\rm QCD}$. The (chiral perturbation theory) expansion in these light quark masses also enables us to estimate them quantitatively: $1.5 \leq m_u \leq 4$ MeV, $4 \leq m_d \leq 8$ MeV, $80 \leq m_s \leq 155$ MeV. These are the light quarks, with masses smaller than $\Lambda_{\rm QCD}$. (Like α_s, the masses are renormalized; these are quoted from Eidelman (2004) with $\mu = 2$ GeV.) For comparison, the heavy quarks have masses $m_c \sim 1 - 1.5$ GeV, $m_b \sim 4 - 4.5$ GeV, and $m_t \sim 180$ GeV (the giant among the known elementary particles).

Although the mechanism of the chiral condensate (and in general other nonperturbative aspects of QCD) has not yet been demonstrated from first principles, a very satisfactory description of the origin of the condensate, and indeed of much hadronic structure has been given in terms of the attractive forces between quarks provided by instantons. The actions of instanton solutions provide a dependence $\exp[-8\pi^2/g_s^2]$ in Euclidean path integrals, and so are characteristically nonperturbative.

5.3 Mechanisms of confinement

As described above, confinement is the absence of asymptotic states that transform nontrivially under color. The full spectrum of QCD, however, is a complex thing to study, and so the problem has been approached somewhat indirectly. A difficulty is the same light quark masses associated with approximate chiral symmetry. Because the masses of the light quarks are far below the scale $\Lambda_{\rm QCD}$ at which the perturbative coupling blows up, light quarks are created freely from the vacuum and the process of "hadronization", by which quarks and gluons form mesons and baryons, is both nonperturbative and relativistic. It is therefore difficult to approach in both perturbation theory *and* lattice simulations.

Tests and studies of confinement are thus normally formulated in truncations of QCD, typically with no light quarks. The question is then reformulated in a way that is somewhat more tractable, without relativistic light quarks popping in and out of the vacuum all the time. In the limit that its mass becomes infinite compared to the natural scale of fluctuations in the QCD vacuum, the propagator of a quark becomes identical to a phase operator, (8), with a path C corresponding to a constant velocity. This observation suggests a number of tests for confinement that can be implemented in the lattice theory. The most intuitive is the vacuum expectation value of a "Wilson loop", consisting of a rectangular path, with sides along the time direction, corresponding to a heavy quark and antiquark at rest a distance R apart, and closed at some starting and ending times with straight lines. The vacuum expectation value of the loop then turns out to be the exponential of the potential energy between the quark pair, multiplied by the elapsed time,

$$\langle 0|\, P \exp\left[-ig_s \oint_C A_\mu(x) dx^\mu \right] |0\rangle = \exp(-V(R)T/\hbar)\,. \tag{13}$$

When $V(R) \propto R$ ("area law" behavior), there is a linearly-rising, confining potential. This behavior, not yet proven analytically yet well-confirmed on the lattice, has an appealing interpretation as the energy of a "string", connecting the quark and antiquark, whose energy is proportional to its length.

Motivation for such a string picture was also found from the hadron spectrum itself, before of any of the heavy quarks were known, and even before the discovery of QCD, from the observation that many mesonic ($\bar{q}q'$) states lie along "Regge trajectories", which consist of sets of states of spin J and mass m_J^2 that obey a relation

$$J = \alpha' m_J^2\,, \tag{14}$$

for some constant α'. Such a relation can be modeled by two light particles ("quarks") revolving around each other at some constant (for simplicity fixed nonrelativistic) velocity v_0 and distance $2R$, connected by a "string" whose energy per unit length is a constant ρ.

Suppose the center of the string is stationary, so the overall system is at rest. Then neglecting the masses, the total energy of the system is $M = 2R\rho$. Meanwhile, the momentum density per unit length at distance r from the center is $v(r) = (r/R)v_0$, and the total angular momentum of the system is

$$J = 2\rho v_0 \int_0^R dr\, r^2 = \frac{2\rho v_0}{3}\, R^2 = \frac{v_0}{6\rho}\, M^2\,, \tag{15}$$

and for such a system, (14) is indeed satisfied. Quantized values of angular momentum J give quantized masses m_J, and we might take this as a sort of "Bohr model" for a meson. Indeed, string theory has its origin in related consideration in the strong interactions.

Lattice data are unequivocal on the linearly rising potential, but it requires further analysis to take a lattice result and determine what field configurations, string-like or not, gave that result. Probably the most widely accepted explanation is in terms of an analogy to the Meissner effect in superconductivity, in which type II superconductors isolate magnetic flux in quantized tubes, the result of the formation of a condensate of Cooper pairs of electrons. If the strings of QCD are to be made of the gauge field, they must be electric ($F^{\mu 0}$) in nature to couple to quarks, so the analogy postulates a "dual" Meissner effect, in which electric flux is isolated as the result of a condensate of objects with magnetic charge (producing nonzero F^{ij}). Although no proof of this mechanism has been provided yet, the role of magnetic fluctuations in confinement has been widely investigated in lattice simulations, with encouraging results. Of special interest are magnetic field configurations, monopoles or vortices, in the Z_3 center of $SU(3)$, $\exp[i\pi k/3]\ I_{3\times 3}$, $k = 0, 1, 2$. Such configurations, even when localized, influence closed gauge loops (13) through the nonabelian Aharonov-Bohm effect. Eventually, of course, the role of light quarks must be crucial for any complete description of confinement in the real world, as emphasized by Gribov.

Another related choice of closed loop is the "Polyakov loop", implemented at finite temperature, for which the path integral is taken over periodic field configurations with period $1/T$, where T is the temperature. In this case, the curve C extends from times $t = 0$ to $t = 1/T$ at a fixed point in space. In this formulation it is possible to observe a phase transition from a confined phase, where the expectation is zero, to a deconfined phase, where it is nonzero. This phase transition is currently under intense experimental study in nuclear collisions.

6 Using Asymptotic Freedom: Perturbative QCD

It is not entirely obvious how to use asymptotic freedom in a theory that should (must) have confinement. Such applications of asymptotic freedom go by the term perturbative QCD, which has many applications, not the least as a window to extensions of the Standard Model.

6.1 Lepton annihilation and infrared safety

The electromagnetic current, $J_\mu = \sum_f e_f \bar{q}_f \gamma_\mu q_f$ is a gauge invariant operator, and its correlation functions are not limited by confinement. Perhaps the simplest application of asymptotic freedom, yet of great physical relevance, is the scalar two-point function,

$$\pi(Q) = \frac{-i}{3} \int d^4x \, \mathrm{e}^{-iQ\cdot x} \, \langle 0 \,|\, \mathrm{T}\left(J^\mu(0)\, J_\mu(x)\right) |\, 0\rangle \, . \tag{16}$$

The imaginary part of this function is related to the total cross section for the annihilation process $\mathrm{e}^+\mathrm{e}^- \to$ hadrons in the approximation that only one photon takes part in the reaction. The specific relation is $\sigma_{\mathrm{QCD}} = (e^4/Q^2)\, \mathrm{Im}\pi(Q^2)$, which follows from the optical theorem, illustrated in Fig. 1. The perturbative expansion of the function $\pi(Q)$ depends, in general, on the mass scales Q and the quark masses m_f as well as on the strong coupling $\alpha_s(\mu)$ and on the renormalization scale μ. We may also worry about the influence of other, truly nonperturbative scales, proportional to powers of Λ_{QCD}. At large values of Q^2, however, the situation simplifies greatly, and dependence on all scales below Q is suppressed by powers of Q. This may be expressed in terms of the operator product expansion,

$$\langle 0 \,|\, T\, \left(J^\mu(0)\, J_\mu(x)\right) |\, 0\rangle = \sum_{O_I} (x^2)^{-3+d_I/2}\, C_I(x^2\mu^2, \alpha_s(\mu))\, \langle 0 \,|\, O_I(0) \,|\, 0\rangle \, , \tag{17}$$

where d_I is the mass dimension of operator O_I, and where the dimensionless coefficient functions C_I incorporate quantum corrections. The sum over operators begins with the identity ($d_I = 0$), whose coefficient function is identified with the sum of quantum corrections in the approximation of zero masses. The sum continues with quark mass corrections, which are suppressed by powers of at least m_f^2/Q^2, for those flavors with masses below Q. Any QCD quantity that has this property, remaining finite in perturbation theory when all particle masses are set to zero, is said to be *infrared safe*.

The effects of quarks whose masses are above Q are included indirectly, through the couplings and masses observed at the lower scales. In summary, the leading power behavior of $\pi(Q)$, and hence of the cross section, is a function of Q, μ and $\alpha_s(\mu)$ only. Higher order operators whose vacuum matrix elements receive nonperturbative corrections include the *gluon condensate*, identified as the product $\alpha_s(\mu) G_{\alpha\beta} G^{\alpha\beta} \propto \Lambda_{\rm QCD}^4$.

Once we have concluded that Q is the only physical scale in π, we may expect that the right choice of the renormalization scale is $\mu = Q$. Any observable quantity is independent of the choice of renormalization scale, μ, and neglecting quark masses, the chain rule gives

$$\mu \frac{d\sigma(Q/\mu, \alpha_s(\mu))}{d\mu} = \mu \frac{\partial \sigma}{\partial \mu} + 2\beta(\alpha_s) \frac{\partial \sigma}{\partial \alpha_s} = 0 \,, \tag{18}$$

which shows that we can determine the beta function directly from the perturbative expansion of the cross section. Defining $a \equiv \alpha_s(\mu)/\pi$, such a perturbative calculation gives

$$\mathrm{Im}\, \pi(Q^2) = \frac{3}{4\pi} \sum_f e_f^2 \left(1 + a + a^2 \left(1.986 - 0.115 n_f - (b_0/4\pi) \ln \frac{Q^2}{\mu^2}\right)\right) , \tag{19}$$

with b_0 as above. Now, choosing $\mu = Q$, we see that asymptotic freedom implies that when Q is large, the total cross section is given by the lowest order, plus small and calculable QCD corrections, a result that is borne out in experiment. Comparing experiment to an expression like (19), one can measure the value of $\alpha_s(Q)$, and hence, with Eq. (12), $\alpha_s(\mu)$ for any $\mu \gg \Lambda_{\rm QCD}$. Fig. 2 shows a recent compilation of values of α_s from this kind of analysis in different experiments at different scales, clearly demonstrating asymptotic freedom.

6.2 Factorization, scaling and parton distributions

One step beyond vacuum matrix elements of currents are their expectation values in single particle states, and here we make contact with the discovery of QCD, through scaling. Such expectations are relevant to the class of experiments known as deep-inelastic scattering, in which a high energy electron exchanges a photon with a nucleon target. All QCD information is contained in the tensor matrix element

$$W_N^{\mu\nu}(p,q) \equiv \frac{1}{8\pi} \sum_\sigma \int d^4x \, \mathrm{e}^{-iq\cdot x} \, \langle p, \sigma \,|\, J^\mu(0)\, J^\nu(x) \,|\, p, \sigma \rangle \;, \tag{20}$$

with q the momentum transfer carried by the photon, and p, σ the momentum and spin of the target nucleon, N. This matrix element is not infrared safe, since in depends in principle on entire history of the nucleon state. Thus, it is not accessible to direct perturbative calculation.

Nevertheless, when the scattering involves a large momentum transfer compared to $\Lambda_{\rm QCD}$ we may expect a quantum-mechanical incoherence between the scattering reaction, which occurs (by the uncertainty principle) at short distances, and the forces that stabilize the nucleon. After all, we have seen that the latter, strong forces, should be associated with long distances. Such a separation of dynamics, called factorization, can be implemented in perturbation theory, and is assumed to be a property of full QCD. Factorization is illustrated schematically in Fig. 3. Of course, short- and long distances are relative concepts, and the separation requires the introduction of a so-called factorization scale, μ_F, not dissimilar to the renormalization scale described above. For many purposes, it is convenient to choose the two equal, although this is not required.

The expression of factorization for deep-inelastic scattering is

$$W_N^{\mu\nu}(p,q) = \sum_{i=q_f,\bar{q}_f,G} \int_\xi^1 d\xi\ C_i^{\mu\nu}(\xi p, q, \mu_F, \alpha_s(\mu_F))\ f_{i/N}(\xi,\mu_F)\,, \tag{21}$$

where the functions $C_i^{\mu\nu}$ (the coefficient functions) can be computed as an expansion in $\alpha_s(\mu_F)$, and describe the scattering of the *partons*, quarks and gluons, of which the target is made. The variable ξ ranges from unity down to $x \equiv -q^2/2p\cdot q > 0$, and has the interpretation of the fractional momentum of the proton carried by parton i. (Here $-q^2 = Q^2$ is positive.) The parton distributions $f_{i/N}$ can be defined in terms of matrix elements in the nucleon, in which the currents are replaced by quark (or antiquark or gluon) fields, as

$$f_{q/N}(x,\mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\lambda\, {\rm e}^{-i\lambda x p^+}\ \langle p,\sigma\,|\,\bar{q}(\lambda n) U_n(n\lambda,0)\, n\cdot\gamma\, q(0)\,|\,p,\sigma\rangle\ . \tag{22}$$

n^μ is a light-like vector, and U_n a phase operator whose path C is in the n-direction. The dependence of the parton distribution on the factorization scale is through the renormalization of the composite operator consisting of the quark fields, separated along the light-cone, and the nonabelian phase operator $U_n(n\lambda,0)$, which renders the matrix element gauge invariant by Eq. (9). By combining the calculations of the C's and data for $W_N^{\mu\nu}$, we can infer the parton distributions, $f_{i/N}$. Important factorizations of a similar sort also apply to some exclusive processes, including amplitudes for elastic pion or nucelon scattering at large momentum transfer.

Eq. (21) has a number of extraordinary consequences. First, because the coefficient function is an expansion in α_s, it is natural to choose $\mu_F^2 \sim Q^2 \sim p\cdot q$ (when x is of order unity). When Q is large, we may approximate $C_i^{\mu\nu}$ by its lowest order, which is first order in the electromagnetic coupling of quarks to photons, and zeroth order in α_s. In this approximation, dependence on Q is entirely in the parton distributions. But such dependence is of necessity weak (again for x not so small as to produce another scale), because the μ_F dependence of $f_{i/N}(\xi,\mu_F)$ must be compensated by the μ_F dependence of $C_i^{\mu\nu}$, which is order α_s. This means that the overall Q dependence of the tensor $W_N^{\mu\nu}$ is weak for Q large when x is moderate. This is the scaling phenomenon that played such an important role in the discovery of QCD.

6.3 Evolution: beyond scaling

Another consequence of the factorization (21), or equivalently of the operator definition, (22) is that the μ_F-dependence of the coefficient functions and the parton distributions are linked. As in the lepton annihilation cross section, this may be thought of as due to the independence of the physically-observable tensor $W_N^{\mu\nu}$ from the choice of factorization and renormalization scales. This implies that the μ_F-dependence of $f_{i/N}$ may be calculated perturbatively since it must cancel the corresponding dependence in C_i. The resulting relation is coventionally expressed in terms of the *evolution equations*,

$$\mu\frac{df_{a/N}(x,\mu)}{d\mu} = \sum_c \int_x^1 d\xi\ P_{ac}(x/\xi,\alpha_s(\mu))\ f_{c/N}(\xi,\mu)\,, \tag{23}$$

where $P_{ab}(\xi)$ are calculable as power series, now known up to α_s^3. This relation expands the applicability of QCD from scales where parton distributions can be inferred directly from experiment, to arbitrarily high scales, reachable in accelerators under construction or in the imagination, or even on the cosmic level.

At very high energy, however, the effective values of the variable x can become very small and introduce new scales, so that eventually the evolution of Eq. (23) fails. The study of nuclear collisions may provide a new high-density regime for quantum chromodynamics, which blurs the distinction between perturbative and nonperturbative dynamics.

6.4 Inclusive production

Once we have evolution at our disposal, we can take yet another step, and replace electroweak currents with any operator from any extension of QCD, in the standard model or beyond, that couples quarks and gluons to the particles of as-yet unseen fields. Factorization can be extended to these situations as well, providing predictions for the production of new particles, F of mass M, in the form of factorized inclusive cross sections,

$$\begin{aligned}\sigma_{AB\to F(M)}(M,p_A,p_B) &= \sum_{i,j=q_f\bar{q}_f,G} \int d\xi_a d\xi_b\, f_{i/A}(\xi_a,\mu)\, f_{j/B}(\xi_b,\mu) \\ &\quad \times H_{ij\to F(M)}(x_a p_A, x_b p_B, M, \mu, \alpha_s(\mu))\,, \quad (24)\end{aligned}$$

where the functions $H_{ij\to F}$ may be calculated perturbatively, while the $f_{i/A}$ and $f_{j/B}$ parton distributions are known from a combination of lower-energy observation and evolution. In this context, they are said to be *universal*, in that they are the same functions in hadron-hadron collisions as in the electron-hadron collisions of deep-inelastic scattering. In general the calculation of hard-scattering functions H_{ij} is quite nontrivial beyond lowest order in α_s. The exploration of methods to compute higher orders, currently as far as α_s^2, has required extraordinary insight into the properties of multidimensional integrals.

The factorization method helped predict the observation of the W and Z bosons of electroweak theory, and the discovery of the top quark. The extension of factorization from deep-inelastic scattering to hadron production is nontrivial; and indeed it only holds in the limit that the velocities, β_i, of the colliding particles approach the speed of light in the center-of-momentum frame of the produced particle. Corrections to the relation (24) are then at the level of powers of $\beta_i - 1$, which translates into inverse powers of the invariant mass(es) of the produced particle(s) M. Factorizations of this sort do not apply to low velocity collisions. Arguments for this result rely on relativistic causality and the uncertainty principle. The creation of the new state happens over time scales of order $1/M$. Before that well-defined event, the colliding particles are approaching at nearly the speed of light, and hence cannot affect the distributions of each others' partons. After the new particle is created, the fragments of the hadrons recede from each other, and the subsequent time development, when summed over all possible final states that include the heavy particle, is finite in perturbation theory as a direct result of the unitarity of QCD.

6.5 Structure of hadronic final states

A wide range of semi-inclusive cross sections are defined by measuring properties of final states that depend only on the flow of energy, and which bring QCD perturbation theory to the threshold of nonperturbative dynamics. Schematically, for a state $N = |k_1 \ldots k_N\rangle$, we define $\mathcal{S}(N) = \sum_i s(\Omega_i) k_i^0$, where $s(\Omega)$ is some smooth function of directions. We generalize the e^+e^- annihilation case above, and define a cross section in terms of a related, but highly nonlocal, matrix element,

$$\frac{d\sigma(Q)}{d\mathcal{S}} \equiv \sigma_0 \int d^4x \, e^{-iQ\cdot x} \left\langle 0 \left| J^\mu(0)\, \delta \left(\int d^2\Omega \, s(\Omega)\, \mathcal{E}(\Omega) - \mathcal{S} \right) J_\mu(x) \right| 0 \right\rangle , \tag{25}$$

where σ_0 is a zeroth order cross sections, and where $\mathcal{E}$ is an operator at spatial infinity, which measures the energy flow of any state in direction Ω: $\mathcal{E}(\Omega)|k_1 \ldots k_N\rangle = (1/Q) \sum_i k_i^0 \delta^2(\Omega - \Omega_i)$. This may seem a little complicated, but like the total annihilation cross section, the only dimensional scale on which it depends is Q. The operator $\mathcal{E}$ can be defined in a gauge invariant manner, through the energy-momentum tensor for example, and has a meaning independent of partonic final states. At the same time, this sort of cross section may be implemented easily in perturbation theory, and like the total annihilation cross section, it is infrared safe. To see why, notice that when a massless ($k^2 = 0$) particle decays into two particles of momenta xk and $(1-x)k$ ($0 \le x \le 1$), the quantity $\mathcal{S}$ is unchanged, since the sum of the new energies is the same as the old. This makes the observable $\mathcal{S}(N)$ insensitive to processes at low momentum transfer.

For the case of leptonic annihilation, the lowest-order perturbative contribution to energy flow requires no powers of α_s, and consists of an oppositely-moving quark and antiquark pair. Any measure of energy flow that includes these configurations will dominate over correlations that require α_s corrections. As a result, QCD predicts that in most leptonic annihilation events energy will flow in two back-to-back collimated sets of particles, known as *jets*. In this way, quarks and gluons are observed clearly, albeit indirectly.

With varying choices of $\mathcal{S}$, many properties of jets, such as their distributions in invariant mass, and the probabilities and angular distributions of multijet events, and even the energy dependence of their particle multiplicities, can be computed in QCD. This is in part because hadronization is dominated by the production of light quarks, whose production from the vacuum requires very little momentum transfer. Paradoxically, the very lightness of the quarks is a boon to the use of perturbative methods. All these considerations can be extended to hadronic scattering, and jet and other semi-inclusive properties of final states also computed and compared to experiment.

7 Conclusions

Quantum chromodynamics is an extremely broad field, and this article has hardly scratched the surface. The relation of QCD-like theories to supersymmetric and string theories, and implications of the latter for confinement and the computation of higher order perturbative amplitudes, have been some of the most exciting developments of recent years. As another example, we note that the reduction of the heavy quark propagator to a nonabelian phase, noted in our discussion of confinement, is related to an additional symmetries of heavy quarks in QCD, with many consequences for the analysis of their bound states. Of the bibliography given below, one may mention the four volumes of Shifman (2001,2002), which communicate in one place a sense of the sweep of work in QCD.

Our confidence in QCD as the correct description of the strong interactions is based on a wide variety of experimental and observational results. At each stage in the discovery, confirmation and exploration of QCD, the mathematical analysis of relativistic quantum field theory entered new territory. As is the case for gravity or electromagnetism, this period of exploration is far from complete, and perhaps never will be.

Further Reading

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Figure Captions

Fig. 1 First line: schematic relation of lowest order e^+e^- annihilation to sum over quarks q, each with electric charge e_q. Second line perturbative unitarity for the current correlation function $\pi(Q)$.

Fig. 2 Experimental variation of the strong coupling with scales; from Bethke (2004).

Fig. 3 Schematic depiction of factorization in deep-inelastic scattering.

See also

AdS/CFT Correspondence. Aharanov-Bohm effect. BRST Quantization. Chiral symmetry. Current algebras. Dirac field and Dirac operator. Euclidean field theory. Gauge theory. Instantons in gauge theory. Lattice gauge theory. Operator product expansion. Perturbation theory and techniques. Perturbative renormalization and BRST. Renormalization, general theory. Scattering: fundamental concepts and tools. Scattering: asymptotic completeness and bound states. Quantum field theory overview. Seiberg-Witten theory. Standard model of particle physics.

Key Words

Quantum chromodynamics

Gauge theory

Asymptotic freedom

Confinement

Infrared safety

Factorization

Evolution

Jets

Renormalization

Chiral symmetry

Strong coupling

High energy scattering

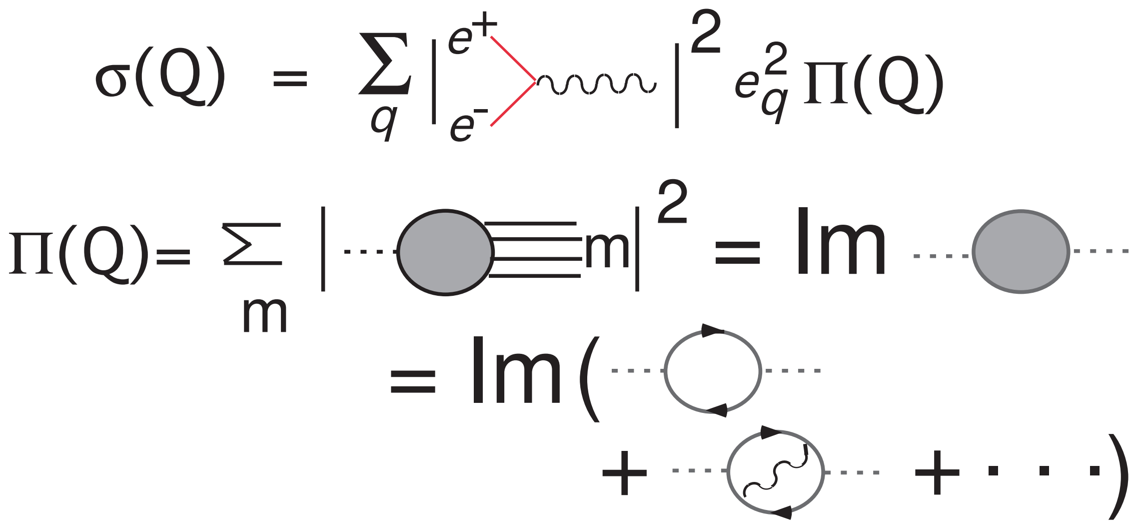
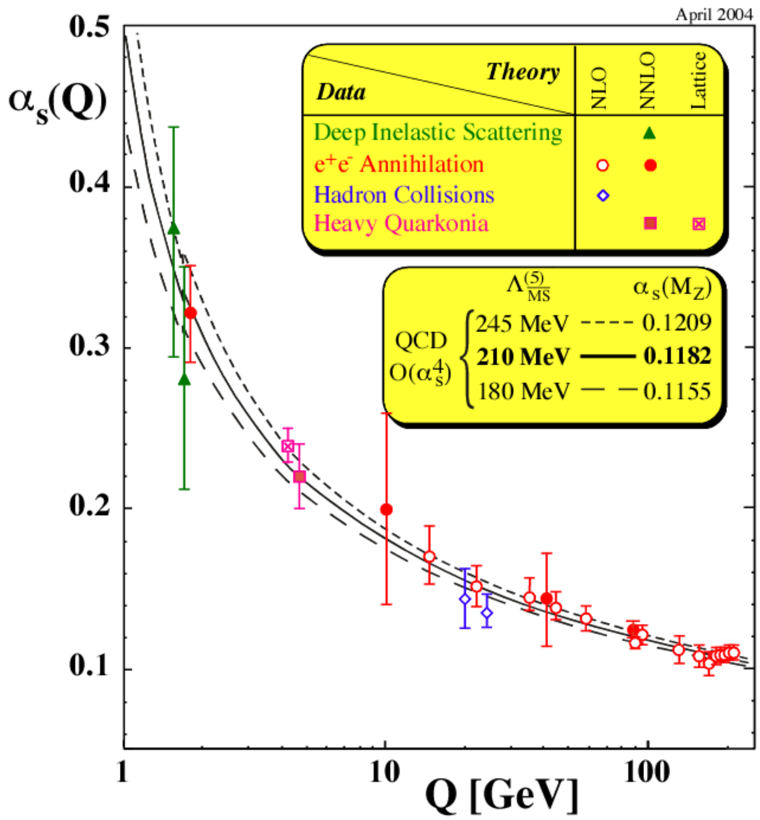


Figure 1:



April 2004
$\alpha_s(Q)$
Theory
Data
NLO
NNLO
Lattice
Deep Inelastic Scattering
e+e- Annihilation
Hadron Collisions
Heavy Quarkonia
$\Lambda^{(5)}_{\overline{MS}}$
$\alpha_s(M_Z)$
QCD
$O(\alpha_s^4)$
245 MeV 0.1209
210 MeV 0.1182
180 MeV 0.1155
0.5
0.4
0.3
0.2
0.1
1
10
100
Q [GeV]

Figure 2:

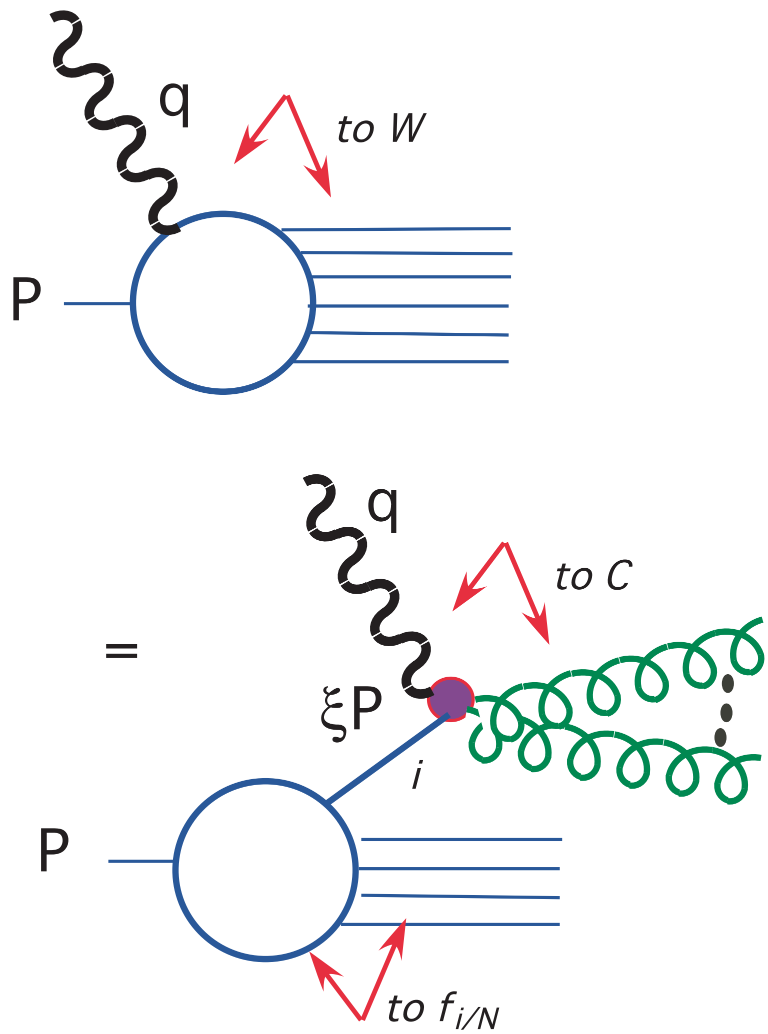


Figure 3: