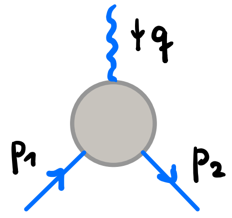
VII. The Sudakov Form Factor in SCET

We now return to the case of the off-shell Sudakov form factor in QCD,



$|q^2| \gg |p_i^2|$
$(m_i = 0)$

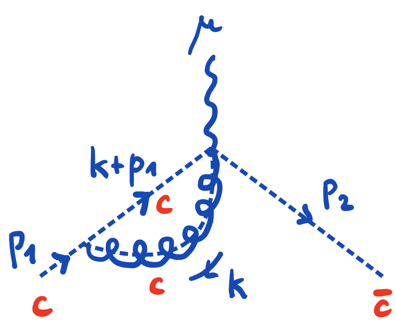
this time performing the calculation in SCET (and not ignoring the numerator terms). We thus evaluate the quark matrix element

$$C_V(Q^2) \; \langle q(p_2) | (\bar{\xi}_{\bar{n}} W_{\bar{c}})(0) \, \gamma_\perp^\mu \, (W_c^\dagger \xi_n)(0) | q(p_1) \rangle$$

at one-loop order, where $C_V(Q^2) = 1 + \mathcal{O}(\alpha_s)$ is the hard matching coefficient.

Collinear contribution:

The SCET Feynman rules allow for a single one-loop diagram:



(we work in Feynman gauge $\xi = 1$)

We find:

$$D_c = \int \frac{d^D k}{(2\pi)^D}\, \bar{u}_{\bar{n}}(p_2)\, \gamma^\mu\, \frac{\not{n}}{2}\, \frac{i\,\bar{n}\cdot(k+p_1)}{(k+p_1)^2+i0}$$

$$\times\, i g_s t_a \left(n^\alpha + \frac{\gamma_\perp^\alpha\, \not{p}_{1\perp}}{\bar{n}\cdot p_1} + \frac{(\not{k}_\perp+\not{p}_{1\perp})\,\gamma_\perp^\alpha}{\bar{n}\cdot(k+p_1)} - \bar{n}^\alpha\, \frac{(\not{k}_\perp+\not{p}_{1\perp})\,\not{p}_{1\perp}}{\bar{n}\cdot(k+p_1)\;\bar{n}\cdot p_1} \right) \frac{\not{\bar{n}}}{2}\, u_n(p_1)$$

$$\times\, \underbrace{(-g_s t_b)\, \frac{\bar{n}^\beta}{\bar{n}\cdot k}}_{\langle k|\, W_c^\dagger\, |0\rangle \quad (\text{cf. p. } 64)}\, \frac{(-i g_{\alpha\beta})}{k^2+i0}\, \delta_{ab}$$

$$= -i C_F g_s^2\, \bar{u}_{\bar{n}}(p_2)\, \overbrace{\gamma^\mu\, \frac{\not{n}\not{\bar{n}}}{4}}^{=\gamma_\perp^\mu}\, u_n(p_1)\, \overbrace{n\cdot\bar{n}}^{2}$$

$$\times \int \frac{d^D k}{(2\pi)^D}\, \frac{\bar{n}\cdot(k+p_1)}{(k^2+i0)\,[(k+p_1)^2+i0]\;\bar{n}\cdot k}$$

If we multiply numerator and denominator by $n\cdot p_2$ we get:

$$\frac{\bar{n}\cdot(k+p_1)}{\bar{n}\cdot k} \;\rightarrow\; \frac{\bar{n}\cdot(k+p_1)\; n\cdot p_2}{\bar{n}\cdot k\;\, n\cdot p_2} = \frac{2k\cdot p_{2+} + 2p_{1-}\cdot p_{2+}}{2k\cdot p_{2+}}$$

Apart from the factor $2k\cdot p_{2+}$ in the numerator (which we had previously ignored), the loop integral coincides with the integral I_c for the collinear region on page 33. Evaluating the integral leads to:

$$D_c = \bar{u}_{\bar{n}}(p_2)\,\gamma_\perp^\mu\, u_n(p_1)\,\frac{C_F\alpha_s}{4\pi}$$
$$\times\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\ln\frac{\mu^2}{p_1^2}+\frac{2}{\epsilon}+\ln^2\frac{\mu^2}{p_1^2}+2\ln\frac{\mu^2}{p_1^2}+4-\frac{\pi^2}{6}\right]$$

We work in the $\overline{MS}$ scheme, where $\mu^{2\epsilon}\longrightarrow\mu^{2\epsilon}(4\pi)^{-2\epsilon}e^{\epsilon\gamma_E}$.

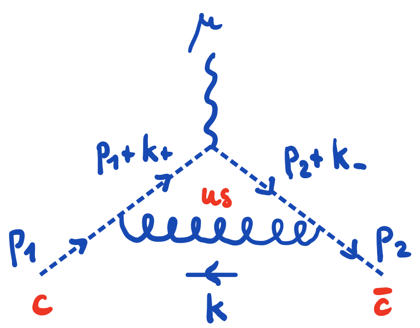
Anti-collinear contribution:

An analogous calculation leads to:

$$D_{\bar{c}} = \bar{u}_{\bar{n}}(p_2)\,\gamma_\perp^\mu\, u_n(p_1)\,\frac{C_F\alpha_s}{4\pi}$$
$$\times\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\ln\frac{\mu^2}{p_2^2}+\frac{2}{\epsilon}+\ln^2\frac{\mu^2}{p_2^2}+2\ln\frac{\mu^2}{p_2^2}+4-\frac{\pi^2}{6}\right]$$

Ultra-soft contribution:

The SCET Feynman rules allow for a single one-loop diagram:



(we work in Feynman gauge $\xi = 1$)

Recall that 4-momentum is not conserved at these vertices.

We find:

$$D_{us} = \int \frac{d^D k}{(2\pi)^D} \, \bar{u}_{\bar{n}}(p_2) \, i g_s t_a \, \bar{n}^\alpha \frac{\not{n}}{2} \frac{\not{\bar{n}}}{2} \frac{i \, \overbrace{n\cdot(k_- + p_2)}^{=0}}{(k_- + p_2)^2 + i0} \gamma^\mu$$

$$\times \frac{\not{n}}{2} \frac{i \, \overbrace{\bar{n}\cdot(k_+ + p_1)}^{=0}}{(k_+ + p_1)^2 + i0} \, i g_s t_b \, n^\beta \frac{\not{\bar{n}}}{2} u_n(p_1) \frac{(-i g_{\alpha\beta})}{k^2 + i0} \delta_{ab}$$

$$= -i C_F g_s^2 \, \bar{u}_{\bar{n}}(p_2) \overbrace{\frac{\not{n}\not{\bar{n}}}{4} \gamma^\mu \frac{\not{n}\not{\bar{n}}}{4}}^{\gamma_\perp^\mu} u_n(p_1) \, \overbrace{n\cdot\bar{n}}^{2}$$

$$\times \int \frac{d^D k}{(2\pi)^D} \frac{\bar{n}\cdot p_1 \; n\cdot p_2}{(k^2 + i0)(2k_+\cdot p_1 + p_1^2 + i0)(2k_-\cdot p_2 + p_2^2 + i0)}$$

$$\uparrow$$

$$\frac{2 p_{1-}\cdot p_{2+}}{(k^2 + i0)(2k\cdot p_{1-} + p_1^2 + i0)(2k\cdot p_{2+} + p_2^2 + i0)}$$

This expression coincides with the integral I_{us} for the ultra-soft region on page 36. Evaluating the integral one finds:

$$D_{us} = \bar{u}_{\bar{n}}(p_2) \gamma_\perp^\mu u_n(p_1) \frac{C_F \alpha_s}{4\pi}$$

$$\times \left[-\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \ln \frac{\mu^2 Q^2}{p_1^2 p_2^2} - \ln^2 \frac{\mu^2 Q^2}{p_1^2 p_2^2} - \frac{\pi^2}{2} \right]$$

Wave-function renormalization:

$$Z_q^{1/2}\, Z_{\bar q}^{1/2}\; \bar u_{\bar n}(p_2)\, \gamma_\perp^\mu\, u_n(p_1)$$

same as in QCD

In Feynman gauge, the WFR factor for an off-shell quark with momentum p^μ is:

some constant

$$Z_q = 1 + \frac{C_F \alpha_s}{4\pi}\left(-\frac{1}{\epsilon} - \ln\frac{\mu^2}{-p^2 - i0} - c\right)$$

We thus obtain:

$$\bar u_{\bar n}(p_2)\, \gamma_\perp^\mu\, u_n(p_1) \left\{ 1 + \frac{C_F \alpha_s}{4\pi}\left[-\frac{1}{\epsilon} - \frac{1}{2}\left(\ln\frac{\mu^2}{P_1^2} + \ln\frac{\mu^2}{P_2^2}\right) - c \right]\right\}$$

One-loop SCET matrix element:

Adding up all pieces, we find at one-loop order:

$$\langle q(p_2)|\, (\bar\xi_{\bar n} W_{\bar c})(0)\, \gamma_\perp^\mu\, (W_c^\dagger \xi_n)(0)\, | q(p_1)\rangle$$

$$= \bar u_{\bar n}(p_2)\, \gamma_\perp^\mu\, u_n(p_1)$$

$$\times \left\{ 1 + \frac{C_F \alpha_s}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\left(\ln\frac{\mu^2}{P_1^2} + \ln\frac{\mu^2}{P_2^2} - \ln\frac{\mu^2 Q^2}{P_1^2 P_2^2}\right) + \frac{3}{\epsilon} \right.\right.$$

$$+ \ln^2\frac{\mu^2}{P_1^2} + \ln^2\frac{\mu^2}{P_2^2} - \ln^2\frac{\mu^2 Q^2}{P_1^2 P_2^2}$$

$$\left.\left. + \frac{3}{2}\ln\frac{\mu^2}{P_1^2} + \frac{3}{2}\ln\frac{\mu^2}{P_2^2} + 8 - c - \frac{5\pi^2}{6} \right]\right\}$$

This is a matrix element of a "bare" SCET operator, which still needs to be renormalized. The fact that the coefficient of the $1/\epsilon$ pole depends on the collinear and ultra-soft (and hence low-energy) scales appears to be troublesome at first sight, since the counterterms removing the divergences must be of UV nature. On second look, however, we see that

$$\ln\frac{\mu^2}{P_1^2} + \ln\frac{\mu^2}{P_2^2} - \ln\frac{\mu^2 Q^2}{P_1^2 P_2^2} = \ln\frac{\mu^2}{Q^2}$$

only depends on the hard scale Q^2! It is thus consistent to interpret these poles as UV divergences.

We define the renormalized SCET current operator as:

$$V^{\mu,\,\mathrm{bare}}_{\mathrm{SCET}} = Z_V(\mu)\, V^{\mu}_{\mathrm{SCET}}(\mu)$$

The matrix elements of the renormalized current operator are finite, i.e., free of $1/\epsilon$ poles. In the $\overline{\mathrm{MS}}$ scheme, one obtains:

$$\boxed{Z_V(\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{3}{\epsilon}\right] + \mathcal{O}(\alpha_s^2)}$$

The matrix element of the renormalized current is then given by:

$$\langle q(p_2)|\, V^\mu_{SCET}(\mu)\,|q(p_1)\rangle$$

$$= \bar{u}_{\bar{n}}(p_2)\, \gamma^\mu_\perp\, u_n(p_1)$$

$$\times \left\{ 1 + \frac{C_F \alpha_s}{4\pi} \left[\ln^2 \frac{\mu^2}{p_1^2} + \ln^2 \frac{\mu^2}{p_2^2} - \ln^2 \frac{\mu^2 Q^2}{p_1^2 p_2^2} \right.\right.$$

$$\left.\left. + \frac{3}{2} \ln \frac{\mu^2}{p_1^2} + \frac{3}{2} \ln \frac{\mu^2}{p_2^2} + 8 - c - \frac{5\pi^2}{6} \right] \right\}$$

Derivation of the Wilson coefficient:

The matching relation for the renormalized vector current takes the form:

$$\bar{\psi} \gamma^\mu \psi \;\to\; C_V(Q^2, \mu)\, V^\mu_{SCET}(\mu)$$

We can derive the one-loop expression for the hard matching coefficient $C_V(Q^2, \mu)$ by comparing the above result for the renormalized SCET matrix element with the result for the Sudakov form factor in QCD, which reads:

$$\langle q(p_2)|\, \bar{\psi}\gamma^\mu\psi\,|q(p_1)\rangle = \bar{u}(p_2)\,\gamma^\mu u(p_1)$$

$$\times \left\{ 1 + \frac{C_F \alpha_s}{4\pi} \left[-2 \ln \frac{Q^2}{p_1^2} \ln \frac{Q^2}{p_2^2} + \frac{3}{2}\left(\ln \frac{Q^2}{p_1^2} + \ln \frac{Q^2}{p_2^2} \right) - \frac{2\pi^2}{3} - c \right] \right\}$$

$$+ \mathcal{O}\left(p_i^2 / Q^2 \right)$$

Matching the two expressions, we find:

$$\boxed{C_V(Q^2,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-\ln^2\frac{Q^2}{\mu^2} + 3\ln\frac{Q^2}{\mu^2} - 8 + \frac{\pi^2}{6}\right]}$$

$\rightarrow$ only depends on hard scale Q^2 ✓

VIII. RG Evolution Equations

The scale dependence of the renormalized SCET current operator is determined by the differential equation:

$$\mu\frac{d}{d\mu}V^\mu_{SCET}(\mu) = -\Gamma_V(Q^2,\mu)\, V^\mu_{SCET}(\mu)$$

really: $n\cdot\mathcal{P}_{\bar{c}}\ \bar{n}\cdot\mathcal{P}_c$

One can show that, to all orders of perturbation theory, the "anomalous dimension" Γ_V is given by:

coefficient of $1/\epsilon$ pole

$$\Gamma_V(Q^2,\mu) = -2\alpha_s\frac{\partial}{\partial\alpha_s}Z_V^{[1]}(\mu)$$

(see e.g. my Les Houches lectures in 1901.06573)

Using our result from above, we find at one-loop order:

$$\Gamma_V(Q^2,\mu) = -\frac{C_F\alpha_s}{\pi}\left(\ln\frac{\mu^2}{Q^2} + \frac{3}{2}\right) + \mathcal{O}(\alpha_s^2)$$

The appearance of a logarithm of μ^2 in an anomalous dimension is a new feature of SCET. It is characteristic of Sudakov problems, in which the perturbation series contains two powers of logarithms for each power of α_s. One can show that to all orders:

$$\boxed{\Gamma_V(Q^2;\mu) = -\Gamma_{cusp}(\alpha_s)\ln\frac{\mu^2}{Q^2} + \gamma_V(\alpha_s)}$$

single log !

From the fact that the vector current in full QCD is not renormalized (Noether's theorem), it then follows that:

$$\mu\frac{d}{d\mu}\left[C_V(Q^2;\mu)\,V^{\mu}_{SCET}(\mu)\right] = 0$$

$$\Downarrow$$

$$\boxed{\mu\frac{d}{d\mu}C_V(Q^2;\mu) = \left[\Gamma_{cusp}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]C_V(Q^2;\mu)}$$

("renormalization-group equation")

Γ_{cusp} is called the light-like cusp anomalous dimension. It is known to 4-loop order in QCD. The quantity γ_V is known at 3-loop order. At leading order we have:

$$\Gamma_{cusp}(\alpha_s) = \frac{C_F\alpha_s}{\pi}, \qquad \gamma_V(\alpha_s) = -\frac{3}{2}\frac{C_F\alpha_s}{\pi}$$

The general solution to the RGE is:

$$C_V(Q^2,\mu) = C_V(Q^2,\mu_h) \text{ "initial condition"}$$

$$\times \exp\left[\int_{\mu_h}^{\mu} \frac{d\mu'}{\mu'}\left(\Gamma_{cusp}(\alpha_s(\mu'))\ \ln\frac{Q^2}{\mu'^2} + \gamma_V(\alpha_s(\mu'))\right)\right]$$

At the "hard matching scale" $\mu_h^2 \approx Q^2$, the initial condition $C_V(Q^2,\mu_h)$ for the Wilson coefficient is free of large logarithms and can be calculated reliably using perturbation theory. For instance, with $\mu_h = Q$ we have:

$$C_V(Q^2,\mu_h) = 1 + \frac{C_F\alpha_s}{4\pi}\left(-8+\frac{\pi^2}{6}\right) + \mathcal{O}(\alpha_s^2)$$

The above solution can then be used to evolve the Wilson coefficients to scales $\mu \ll Q$ in such a way that the large logarithms

$$\alpha_s^n \ln^k\frac{Q^2}{\mu^2} \quad ; \quad k \leq 2n$$

are resummed to all orders in α_s.

(↳ see second tutorial for more details)

IX. Decoupling of Ultra-Soft Gluons

Ultra-soft gluons couple to collinear fields through the eikonal interaction:

$$\bar{\xi}_n(x)\,\frac{\not{\bar{n}}}{2}\left(in\cdot\partial + g_s\,n\cdot A_c(x) + g_s\,n\cdot A_{us}(x_-)\right)\xi_n(x)$$

In analogy with HQET, this coupling can be removed by a <u>field redefinition</u>, i.e.

new fields

$$\xi_n(x) \longrightarrow S_n(x_-)\,\xi_n^{(0)}(x)$$

$$A_c^\mu(x) \longrightarrow S_n(x_-)\,A_c^{\mu(0)}(x)\,S_n^\dagger(x_-)$$

where

note: n^μ, <u>not</u> $\bar{n}^\mu$!

$$\boxed{S_n(x) = \mathcal{P}\exp\left(ig_s\int_{-\infty}^{0} dt\; n\cdot A_{us}(x+nt)\right)} \quad \text{(unitary)}$$

is a soft Wilson line along the light-like direction n^μ. Note that this has the form of an ultra-soft gauge transformation with $U_s(x) = S_n(x)$. Using the property

$$\left(in\cdot\partial + g_s\,n\cdot A_{us}(x_-)\right)S_n(x_-) = S_n(x_-)\,in\cdot\partial$$

which follows from $[in\cdot D_{us}\,S_n(x)] = 0$, one finds that:

$$\bar{\xi}_n(x)\,\frac{\bar{\not{n}}}{2}\left(in\cdot\partial + g_s\,n\cdot A_c(x) + g_s\,n\cdot A_{us}(x_-)\right)\xi_n(x)$$

$$\rightarrow\ \bar{\xi}_n^{(0)}(x)\,\frac{\bar{\not{n}}}{2}\,S_n^\dagger(x_-)\,S_n(x_-)\left(in\cdot\partial + g_s\,n\cdot A_c^{(0)}(x)\right)\xi_n^{(0)}(x)$$

$$=\ \bar{\xi}_n^{(0)}(x)\,\frac{\bar{\not{n}}}{2}\,in\cdot D_c^{(0)}(x)\,\xi_n^{(0)}(x)$$

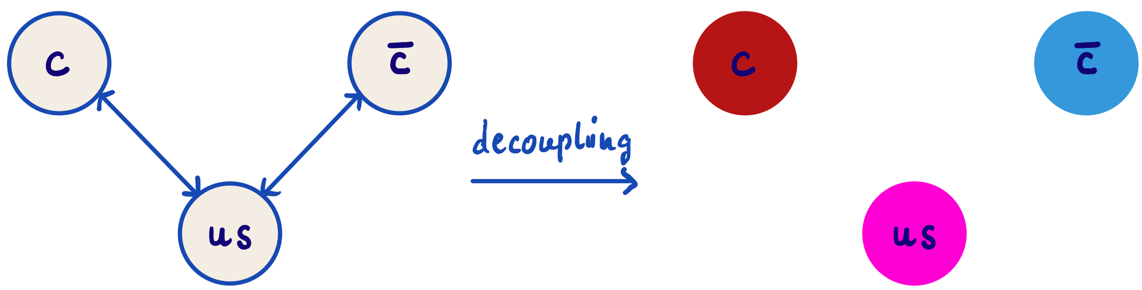
This field redefinition thus removes the ultra-soft gluon field from the leading-order SCET Lagrangian! The same trick also works for the pure gluon terms in the Lagrangian.

The "ultra-soft decoupling transformation" is the key to deriving factorization theorems in SCET! Like in HQET, it does not imply that ultra-soft interactions disappear entirely. Rather, it means that, as far as their couplings to ultra-soft gluon are concerned, collinear particles behave like light-like Wilson lines. The ultra-soft gluons will reappear when we consider external operators (such as currents) built out of two or more types of collinear fields.

For the example of the 2-jet vector current, the decoupling transformation implies:

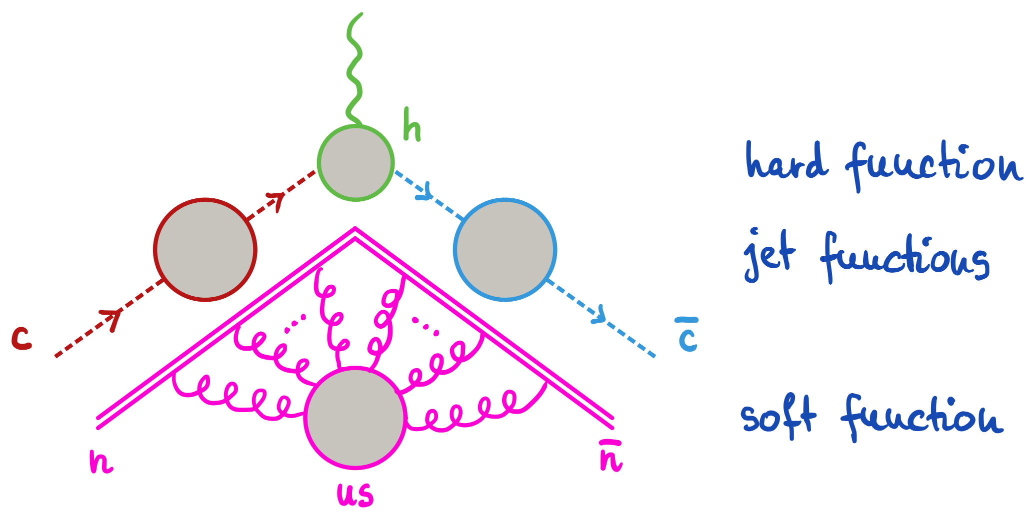
$$\bar\psi\gamma^\mu\psi \;\to\; C_V(Q^2)\,(\bar\xi_{\bar n}W_{\bar c})(0)\,\gamma_\perp^\mu\,(W_c^\dagger\xi_n)(0)$$

$$\to\; C_V(Q^2)\,(\bar\xi_{\bar n}^{(0)}W_{\bar c}^{(0)})(0)\,\gamma_\perp^\mu\,S_{\bar n}^\dagger(0)\,S_n(0)\,(W_c^{(0)\dagger}\xi_n^{(0)})(0)$$



"messenger" three decoupled sectors !

The Sudakov form factor then factorizes into four distinct objects each characterized by a single scale:

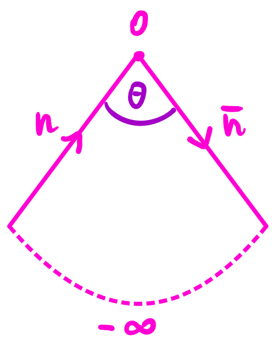


This is an example of a <u>factorization formula</u>.

In the following two lectures we will discuss specific examples of factorization theorems for some concrete physical processes.

To finish off this lecture, let me note that the appearance of the two soft Wilson lines is responsible for the cusp anomalous dimension in the anomalous dimension of the SCET current operator:

$$S_{\bar n}^{\dagger} S_n(0) =$$



↪ closed Wilson loop with a cusp at $x=0$ with angle θ

$$\boxed{\cosh\theta = \frac{n\cdot\bar n}{\sqrt{n^2\,\bar n^2}} = \infty}$$

The quantity $\Gamma_{cusp}(\alpha_s)$ is related to the time-like cusp anomalous dimension, which we have discussed in the context of HQET (see page 24), by:

$$\lim_{\theta\to\infty} \frac{1}{\theta}\, \underbrace{\Gamma_{cusp}(\alpha_s,\theta)}_{\theta = v\cdot v' \text{ for HQET}} = \Gamma_{cusp}(\alpha_s)$$

↑ light-like cusp anomalous dimension of SCET