V. Important Facts about the SCET Lagrangian

Residual gauge invariance:

The SCET Lagrangian is no longer invariant under arbitrary local gauge transformations:

$$\psi(x) \quad \longrightarrow \quad U(x)\,\psi(x)$$

$$A^\mu(x) \quad \longrightarrow \quad U(x)\,A^\mu(x)\,U^\dagger(x) + \frac{i}{g_s}\,U(x)\left(\partial^\mu U^\dagger(x)\right)$$

$$\uparrow$$

$$A^{\mu,a}\,t^a \qquad\qquad = \frac{i}{g_s}\,U(x)\left(D^\mu(x)\,U^\dagger(x)\right)$$

The Fourier transform $\tilde{U}(p)$ could contain modes with arbitrary momenta (including hard ones), which would mix up the various modes in the EFT. However, the SCET Lagrangian is invariant (order by order in λ) under a set of collinear, anti-collinear and ultra-soft gauge transformations, which preserve the scaling properties of the various fields. (It is also invariant under global gauge transformations, of course.)

The appropriate form of the residual gauge transformations can be obtained by performing suitable expansions in the QCD relations shown above.

Collinear gauge transformations:

$$\xi_n(x) \xrightarrow{U_c} U_c(x)\,\xi_n(x)$$

$$\bar{n}\cdot A_c(x) \xrightarrow{U_c} U_c(x)\,\bar{n}\cdot A_c(x)\,U_c^\dagger(x) + \frac{i}{g_s}\,U_c(x)\left(\bar{n}\cdot\partial\,U_c^\dagger(x)\right)$$

$$A^\mu_{c\perp}(x) \xrightarrow{U_c} U_c(x)\,A^\mu_{c\perp}(x)\,U_c^\dagger(x) + \frac{i}{g_s}\,U_c(x)\left(\partial^\mu_\perp\,U_c^\dagger(x)\right)$$

$$n\cdot A_c(x) + n\cdot A_{us}(x_-)$$

$$\xrightarrow{U_c} U_c(x)\left(n\cdot A_c(x) + n\cdot A_{us}(x_-)\right)U_c^\dagger(x) + \frac{i}{g_s}\,U_c(x)\left(n\cdot\partial\,U_c^\dagger(x)\right)$$

The last relation can be rewritten in the form:

$$n\cdot A_c(x) \xrightarrow{U_c} U_c(x)\,n\cdot A_c(x)\,U_c^\dagger(x) + \frac{i}{g_s}\,U_c(x)\left[n\cdot D_{us}(x_-),\,U_c^\dagger(x)\right]$$

The ultra-soft fields do not transform at all:

$$\left.\begin{aligned} q_{us}(x) &\xrightarrow{U_c} q_{us}(x) \\ A^\mu_{us}(x) &\xrightarrow{U_c} A^\mu_{us}(x) \end{aligned}\right\}$$

The same is true for anti-collinear fields.

The collinear Wilson line transforms as:

$$W_c(x) \xrightarrow{U_c} U_c(x)\,W_c(x)\,U_c^\dagger(-\infty\bar{n}) = U_c(x)\,W_c(x)$$

consider gauge transformations that vanish at infinity

Ultra-soft gauge transformations:

$$\xi_n(x) \xrightarrow{U_{us}} U_{us}(x_-)\, \xi_n(x)$$

$$A_c^\mu(x) \xrightarrow{U_{us}} U_{us}(x_-)\, A_c^\mu(x)\, U_{us}^\dagger(x_-)$$

The collinear gluon field transforms as a background field. The ultra-soft fields transform in the usual way:

$$q_{us}(x) \xrightarrow{U_{us}} U_{us}(x)\, q_{us}(x)$$

$$A_{us}^\mu(x) \xrightarrow{U_{us}} U_{us}(x)\, A_{us}^\mu(x)\, U_{us}^\dagger(x) + \frac{i}{g_s} U_{us}(x) \left(\partial^\mu U_{us}^\dagger(x)\right)$$

Note that the collinear Wilson line transforms as:

$$W_c(x) \xrightarrow{U_{us}} U_{us}(x_-)\, W_c(x)\, U_{us}^\dagger(x_-)$$

This follows from the factor that the gluon fields in the path-ordered exponential all live at the same value of x_-:

$$A_c^\mu(\underbrace{x+t\bar{n}}) \xrightarrow{U_{us}} U_{us}(x_-)\, A_c^\mu(x+t\bar{n})\, U_{us}^\dagger(x_-)$$

$$(x+t\bar{n})_- = \frac{n}{2}\, \bar{n}\cdot(x+t\bar{n}) = \frac{n}{2}\, \bar{n}\cdot x = x_-$$

It is straightforward to show that $\mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{c+us}$ is invariant under these residual gauge transformations. Note, in particular, that (as differential operators):

$$
\begin{aligned}
i D_{c\perp}^\mu &\xrightarrow{U_c} U_c(x)\, i D_{c\perp}^\mu\, U_c^\dagger(x) \\
&\xrightarrow{U_{us}} U_{us}(x_-)\, i D_{c\perp}^\mu\, U_{us}^\dagger(x_-) \\
i n\cdot D_c + g_s n\cdot A_{us}(x_-) &\xrightarrow{U_c} U_c(x) \left(i n\cdot D_c + g_s n\cdot A_{us}(x_-) \right) U_c^\dagger(x) \\
&\xrightarrow{U_{us}} U_{us}(x_-) \left(i n\cdot D_c + g_s n\cdot A_{us}(x_-) \right) U_{us}^\dagger(x_-)
\end{aligned}
$$

Nonrenormalization theorem:

There is an important difference between SCET and HQET. In the absence of ultra-soft interactions, the effective Lagrangian

$$
\begin{aligned}
\mathcal{L}_c(x) = {} & \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n(x) \\
& + \left(\bar{\xi}_n\, i \not{D}_c^\perp W_c \right)(x) \frac{\not{\bar{n}}}{2}\, i \int_{-\infty}^{0} dt \left(W_c^\dagger\, i \not{D}_c^\perp\, \xi_n \right)(x + t\bar{n}) \\
& + (\text{pure glue terms})
\end{aligned}
$$

is <u>exact to all orders in λ</u>, i.e. it does not receive any power corrections! In fact, this Lagrangian does not contain any small ratio of scales, unlike in HQET, where $iv\cdot D_s/m_Q \ll 1$. By a Lorentz boost, a set of highly boosted collinear particles with momenta $p_c^\mu \sim (\lambda^2, 1, \lambda)$ can be transformed into a set of particles with momenta $\Lambda^\mu{}_\nu\, p_c^\nu \sim (\lambda, \lambda, \lambda)$. Thus, the Lagrangian $\mathcal{L}_c$ is, in fact, completely equivalent to the QCD Lagrangian, and as a result its vertices do not receive any hard matching corrections.

Remarkably, this statement remains true when ultra-soft interactions are included.

(see: Beneke, Chapovsky, Diehl, Feldmann 2002)

Reparameterization invariance:

The choice of the light-like reference vectors $n^\mu, \bar{n}^\mu$ ($n^2 = 0 = \bar{n}^2$, $n\cdot\bar{n} = 2$) in the construction of SCET is not unique. For instance, one can rescale:

$$\boxed{n^\mu \to \xi\, n^\mu, \quad \bar{n}^\mu \to \xi^{-1}\, \bar{n}^\mu} \qquad (\mathrm{III})$$

(with $\xi \sim \lambda^0$)

One can also rotate n^μ or $\bar{n}^\mu$ by small amounts away from the z-axis. In infinitesimal form:

$$\boxed{\begin{aligned} n^\mu &\to n^\mu + \Delta_\perp^\mu \\ \bar{n}^\mu &\to \bar{n}^\mu \end{aligned}} \quad (\mathrm{I}) \qquad \boxed{\begin{aligned} n^\mu &\to n^\mu \\ \bar{n}^\mu &\to \bar{n}^\mu + \bar{\Delta}_\perp^\mu \end{aligned}} \quad (\mathrm{II})$$

$(\Delta_\perp \sim \lambda)$ $\qquad$ $(\bar{\Delta}_\perp \sim \lambda)$

The most general transformation is a combination of these three.

The SCET Lagrangian is invariant under these reparameterization transformations as a consequence of Lorentz invariance. The transformations (I) and (II) relate the coefficients of operators arising at different orders in λ. Invariance under type-III transformations, on the other hand, must be satisfied order by order in λ.

(see: Manohar, Mehen, Pirjol, Stewart 2002)

VI. Matching of the 2-Jet Current

As a very important application of our formalism, we now reconsider the process $e^+e^- \to \gamma^* \to 2$ jets. What is the proper representation of the vector current $\bar{\psi}\gamma^\mu\psi$ in SCET? The naive guess

$$\bar{\psi}\gamma^\mu\psi \;\to\; \bar{\xi}_{\bar{n}}\,\gamma_\perp^\mu\,\xi_n$$

anti-collinear quark   collinear quark

is not gauge invariant in SCET, because ξ_n and $\xi_{\bar{n}}$ transform under <u>different</u> sets of residual gauge transformations. A gauge-invariant current operator is:

$$(\bar{\xi}_{\bar{n}} W_{\bar{c}})(0)\,\gamma_\perp^\mu\,(W_c^\dagger \xi_n)(0)$$

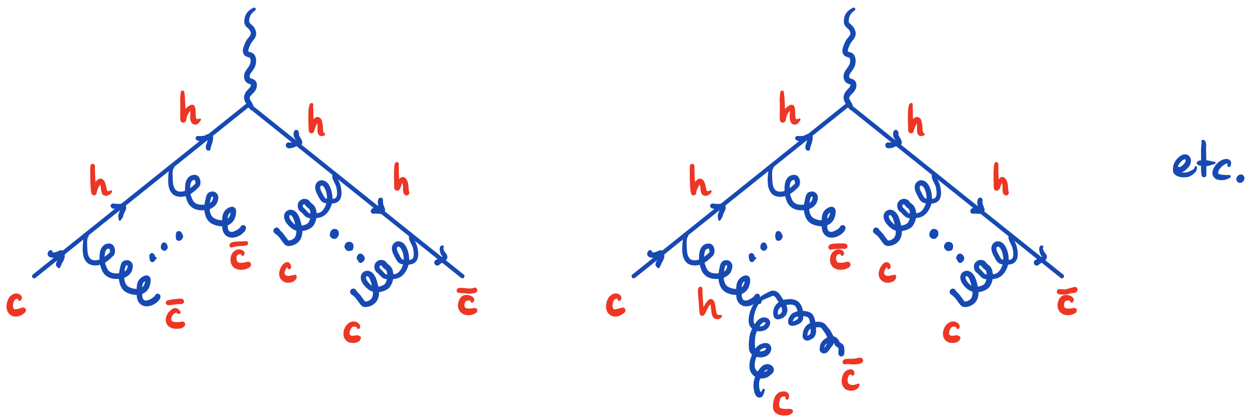
To see this, note that:

$$W_c^\dagger(x)\,\xi_n(x) \xrightarrow{U_c} W_c^\dagger(x)\,U_c^\dagger(x)\,U_c(x)\,\xi_n(x)$$
$$= W_c^\dagger(x)\,\xi_n(x)$$
$$\xrightarrow{U_{us}} U_{us}(x_-)\,W_c^\dagger(x)\,U_{us}^\dagger(x_-)\,U_{us}(x_-)\,\xi_n(x)$$
$$= U_{us}(x_-)\,W_c^\dagger(x)\,\xi_n(x)$$

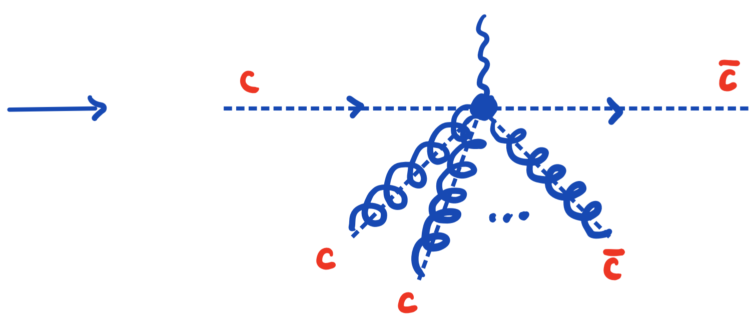
The anti-collinear field $\bar{\xi}_{\bar{n}} W_{\bar{c}}(x)$ transforms in a similar way.

For the special choice $x=0$ the operator is invariant under all three types of gauge transformations ($c, \bar{c}$, us). The case $x \neq 0$ is a bit more tricky, since it requires a multipole expansion of the collinear and anti-collinear fields themselves ($\hookrightarrow$ see the tutorial on THU).

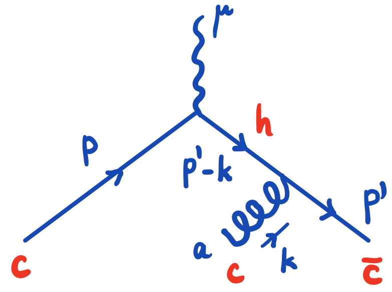
The (anti-) collinear Wilson lines are required by gauge invariance, but what do they represent physically? In fact, they account for an infinite set of QCD graphs of the type:



Despite the many hard propagators, these diagrams contain leading-power pieces in λ, which in the EFT are accounted for by the Wilson lines.



To see how this works, consider the attachment of a single collinear gluon to an anti-collinear quark:



$$p' \sim (1, \lambda^2, \lambda)$$
$$k \sim (\lambda^2, 1, \lambda)$$
$$\epsilon \sim (\lambda^2, 1, \lambda)$$

$$= \bar u(p')\, i g_s\, \not\epsilon(k)\, t_a \frac{i(\not p' - \not k)}{(p'-k)^2}\, \gamma^\mu u(p)$$

$$\underset{\lambda^2}{p'^2} + \underset{\lambda^2}{k^2} - \big(\underset{\lambda^0}{n\cdot p'}\;\underset{\lambda^0}{\bar n\cdot k} + \underset{\lambda^2}{\bar n\cdot p'}\;\underset{\lambda^2}{n\cdot k} + 2\underset{\lambda}{p'_\perp}\cdot\underset{\lambda}{k_\perp}\big)$$

$$= \bar u(p')\, i g_s \Big(\underset{\lambda^0}{\bar n\cdot\epsilon}\,\frac{\not n}{2} + \underset{\lambda^2}{n\cdot\epsilon}\,\frac{\not{\bar n}}{2} + \underset{\lambda}{\not\epsilon_\perp}\Big)\, t_a \frac{i}{(p'-k)^2}$$

$$\times \Big(\bar n\cdot(\underset{\lambda^2}{p'} - \underset{\lambda^0}{k})\frac{\not n}{2} + n\cdot(\underset{\lambda^0}{p'} - \underset{\lambda^2}{k})\frac{\not{\bar n}}{2} + \underset{\lambda}{(\not p' - \not k)_\perp}\Big)\gamma^\mu u(p)$$

$$= \bar u(p')\, \underbrace{\frac{\not n \not{\bar n}}{4}}_{P_n = P_{\bar n}^+}\, (-g_s t_a)\, \frac{\bar n\cdot\epsilon\; \cancel{n\cdot p'}}{-\cancel{n\cdot p'}\;\bar n\cdot k}\, \gamma^\mu u(p) + \text{power-suppressed terms}$$

drop

$$\not n \frac{\bar n^\mu}{2} + \cancel{\not{\bar n}\frac{n^\mu}{2}} + \gamma_\perp^\mu$$

0

drop, since $\not n\, u(p) \approx 0$ is suppressed

$$= \underbrace{\bar u(p')\, P_{\bar n}^+}_{\text{from } \bar\xi_{\bar n} = \bar\psi_{\bar c} P_{\bar n}^+} \boxed{\left(\frac{g_s t_a\; \bar n\cdot\epsilon(k)}{\bar n\cdot k}\right)} \gamma_\perp^\mu\, \underbrace{P_n\, u_n(p)}_{\text{from } \xi_n = P_n \psi_c} + \ldots$$

The highlighted factor is nothing but the one-gluon matrix element of the collinear Wilson line $W_c^\dagger$:

$$\langle 0 | W_c^\dagger(0) | \epsilon, k \rangle = \langle 0 | \bar{\mathbf{P}} \exp\left(-i g_s \int_{-\infty}^{0} dt \, \bar{n}\cdot A_c(t\bar{n})\right) | \epsilon, k \rangle$$

opposite ordering than for W_c

$$= -i g_s t_a \, \bar{n}\cdot\epsilon(k) \int_{-\infty}^{0} dt \, e^{-it\bar{n}\cdot k} = \frac{g_s t_a \, \bar{n}\cdot\epsilon(k)}{\bar{n}\cdot k} \quad ✓$$

Hard matching corrections:

The 2-jet current carries a hard momentum transfer $q^2 = -Q^2$. Unlike for the SCET Lagrangian, we thus expect that there should be a non-trivial Wilson coefficient $C_V(Q^2)$ accounting for the effects of hard gluons, which have been integrated out, i.e.:

$$\boxed{\bar{\psi}\gamma^\mu\psi(0) \;\to\; C_V(Q^2)\, (\bar{\xi}_{\bar{n}} W_{\bar{c}})(0)\, \gamma_\perp^\mu\, (W_c^\dagger \xi_n)(0)}$$

hard quantum fluctuations

This is indeed the correct relation, but the question arises how such a dependence of the Wilson coefficient on $q^2 = (p_c - p_{\bar{c}})^2$ can arise, since q^2 depends on the momenta of light particles in the low-energy EFT.

This question is connected with another worry we had when constructing SCET: the presence of fields $\bar{n}\cdot A_c \sim \lambda^0$, $n\cdot A_{\bar{c}} \sim \lambda^0$ with unsuppressed power counting. In fact, the infinite set

$$\left[\bar{\xi}_{\bar{n}}\,(i n\cdot D_{\bar{c}})^{m_1}\, W_{\bar{c}}\right](0)\ \gamma_\perp^\mu\ \left[W_c^\dagger\,(i\bar{n}\cdot D_c)^{m_2}\,\xi_n\right](0)$$

of gauge-invariant operators, with $m_1, m_2 \in \mathbb{N}_0$, can equally well arise in the matching condition for the vector current at leading power in λ. Using the relations

$$W_c^\dagger\, i\bar{n}\cdot D_c\, W_c = i\bar{n}\cdot\partial$$

$$W_{\bar{c}}^\dagger\, i n\cdot D_{\bar{c}}\, W_{\bar{c}} = i n\cdot\partial$$

(see p. 45)

these operators can be rewritten as:

$$\left[\bar{\xi}_{\bar{n}}\, W_{\bar{c}}\right](0)\,(-i n\cdot\overleftarrow{\partial})^{m_1}\,\gamma_\perp^\mu\,(i\bar{n}\cdot\partial)^{m_2}\left[W_c^\dagger\,\xi_n\right](0)$$

$$\equiv \mathcal{O}_{m_1 m_2}(0)$$

The most general leading-order matching relation is thus of the form:

$$\bar{\psi}\gamma^\mu\psi(0) \to \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} c_{m_1 m_2}\, O_{m_1 m_2}(0)$$

An equivalent way of writing this result uses the <u>non-local</u> expression:

$$\bar{\psi}\gamma^\mu\psi(0)$$

$$\to \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\; \tilde{C}_V(t_1,t_2)\, \left[\bar{\xi}_{\bar{n}} W_{\bar{c}}\right](t_1 n)\, \gamma_\perp^\mu \left[W_c\, \xi_n\right](t_2 \bar{n})$$

Using the Taylor series

$$f(t\bar{n}) = e^{t\bar{n}\cdot\partial_x} f(x)\Big|_{x=0} = \sum_{m=0}^{\infty} \frac{t^m}{m!}(\bar{n}\cdot\partial)^m f(0)$$

we find:

$$c_{m_1 m_2} = \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\; \tilde{C}_V(t_1,t_2)\, \frac{(-it_1)^{m_1}}{m_1!}\,\frac{(it_2)^{m_2}}{m_2!}$$

We can now use the fact that the large component of the total (anti-) collinear momentum of each jet (or in each sector) is fixed by kinematics. Let us call these momenta $\bar{n}\cdot\mathcal{P}_c$ and $n\cdot\mathcal{P}_{\bar{c}}$. We can then use translational invariance to write:

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; \tilde{C}_V(t_1,t_2) \; [\bar{\xi}_{\bar{n}} W_{\bar{c}}](t_1 n) \; \gamma_\perp^\mu \; [W_c \xi_n](t_2 \bar{n})$$

$$= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; \tilde{C}_V(t_1,t_2) \; e^{i t_1 n\cdot \mathcal{P}_{\bar{c}}} \; e^{-i t_2 \bar{n}\cdot \mathcal{P}_c}$$

$$\times \; (\bar{\xi}_{\bar{n}} W_{\bar{c}})(0) \; \gamma_\perp^\mu \; (W_c^\dagger \xi_n)(0)$$

$$\equiv \; C_V(n\cdot \mathcal{P}_{\bar{c}}, \; \bar{n}\cdot \mathcal{P}_c) \; (\bar{\xi}_{\bar{n}} W_{\bar{c}})(0) \; \gamma_\perp^\mu \; (W_c^\dagger \xi_n)(0)$$

Type-III reparameterization invariance requires that the coefficient C_V can only depend on the product of its two arguments:

as eigenvalue

$$n\cdot \mathcal{P}_{\bar{c}} \; \bar{n}\cdot \mathcal{P}_c \simeq -q^2 = Q^2$$

Hence, we have derived the matching condition shown on p. 64.

Note:

In the literature the objects $\bar{n}\cdot \mathcal{P}_c$ and $n\cdot \mathcal{P}_{\bar{c}}$ are often called "label operators". These operators project out the large components of the sum of all (anti-) collinear particles in a given process.