Effective Lagrangian of Soft-Collinear Effective Theory (SCET)

Our goal is to construct an effective Lagrangian built out of collinear, anti-collinear and ultra-soft quark and gluon fields (and ghost fields, but we will not write them out explicitly). Momentum conservation allows the following interactions involving different modes:

$$\underbrace{\phi_c \dots \phi_c}_{n_c \geq 2} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \qquad \checkmark$$

$$\underbrace{\phi_{\bar{c}} \dots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 2} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \qquad \checkmark$$

but not:

$$\underbrace{\phi_c \dots \phi_c}_{n_c \geq 1} \; \underbrace{\phi_{\bar{c}} \dots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 1} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 0}$$

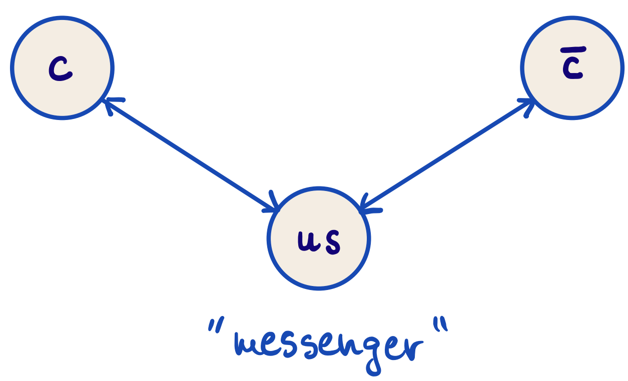
forbidden (hard interaction)

$$\phi_c \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \, , \qquad \phi_{\bar{c}} \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1}$$

momentum not conserved

It follows that:

$$\mathcal{L}_{SCET}^{(c,\bar{c},us)} = \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{\bar{c}} + \mathcal{L}_{c+us} + \mathcal{L}_{\bar{c}+us}$$



It suffices to study the Lagrangian of collinear and ultra-soft fields.

Collinear quark field:

The spinor of a highly energetic (along z-axis), light fermion satisfies:

$$\not{p}\, u_s(p) = m\, u_s(p) \quad ; \quad m \ll E = p^0 \simeq p^3$$

$$\Rightarrow \quad \not{n}\, u_s(p) \simeq 0 \qquad (\text{up to } m/E \text{ corrections})$$

In analogy with HQET, we identify the large and small components of such a spinor using projection operators:

$$P_n = \frac{\not{n}\not{\bar{n}}}{4} \quad , \quad P_{\bar{n}} = \frac{\not{\bar{n}}\not{n}}{4} \qquad (\bar{P}_n = \gamma^0 P_n^\dagger \gamma^0 = P_{\bar{n}})$$

with:

$$P_n + P_{\bar{n}} = \frac{\{\not{n}, \not{\bar{n}}\}}{4} = \frac{2\, n\cdot\bar{n}}{4} = 1$$

$$P_n^2 = P_n \; , \quad P_{\bar{n}}^2 = P_{\bar{n}} \; , \quad P_n P_{\bar{n}} = 0 = P_{\bar{n}} P_n$$

We define:

modes restricted to collinear region

$$\xi_n = P_n \psi_c \quad , \quad \eta_n = P_{\bar{n}} \psi_c$$

$$\Rightarrow \quad \not{n}\, \xi_n = 0 \qquad (\xi_n \text{ will describe a collinear quark in SCET})$$

$$\not{\bar{n}}\, \eta_n = 0$$

To derive the power counting in λ, we consider (massless fermion):

$$n\cdot p\,\frac{\bar{n}}{2} + \bar{n}\cdot p\,\frac{n}{2} + \not{p}_\perp$$

$$\langle 0|\, T\{\psi_c(x)\,\bar{\psi}_c(0)\}\,|0\rangle = \underbrace{\int\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\; \underbrace{e^{-ip\cdot x}}_{1}\; \underbrace{\frac{i\not{p}}{p^2+i\epsilon}}_{\frac{1}{\lambda^2}\,(\lambda^2,\,1,\,\lambda)}$$

Hence:

$$\langle 0|\, T\{\xi_n(x)\,\bar{\xi}_n(0)\}\,|0\rangle = \underbrace{\int\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\; \underbrace{e^{-ip\cdot x}}_{1}\; \underbrace{\frac{i\,\bar{n}\cdot p}{p^2+i\epsilon}}_{\lambda^{-2}}\;\frac{\not{n}}{2} \sim \lambda^2$$

$$\langle 0|\, T\{\eta_n(x)\,\bar{\eta}_n(0)\}\,|0\rangle = \underbrace{\int\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\; \underbrace{e^{-ip\cdot x}}_{1}\; \underbrace{\frac{i\,n\cdot p}{p^2+i\epsilon}}_{\lambda^{0}}\;\frac{\not{\bar{n}}}{2} \sim \lambda^4$$

$$\langle 0|\, T\{\xi_n(x)\,\bar{\eta}_n(0)\}\,|0\rangle = \underbrace{\int\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\; \underbrace{e^{-ip\cdot x}}_{1}\; \underbrace{\frac{i\,p_\perp^\mu}{p^2+i\epsilon}}_{\lambda^{-1}}\; P_n\,\gamma_\mu\,P_{\bar{n}} \sim \lambda^3$$

It follows that:

large components   small components

$$\boxed{\xi_n \sim \lambda\,, \qquad \eta_n \sim \lambda^2}$$

Note that these rules do <u>not</u> agree with naive dimensional analysis!

In analogy with HQET, we will integrate out the small components η_n and use the field ξ_n to describe a collinear quark in SCET.

Collinear gluon field:

The relevant two-point function to consider is:

general cov. gauge

$$\langle 0 | T\{ A_c^{\mu,a}(x)\, A_c^{\nu,b}(0) \} | 0 \rangle = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[\underbrace{-g^{\mu\nu}}_{1} + (1-\xi)\, \frac{p^\mu p^\nu}{p^2} \Big]$$

various scalings

It would be incorrect to work in Feynman gauge ($\xi = 1$) and conclude that $A_a^\mu \sim \lambda$ (as suggested by NDA).

Rather, we need to decompose:

$$A_c^\mu = n\cdot A_c\, \frac{\bar{n}^\mu}{2} + \bar{n}\cdot A_c\, \frac{n^\mu}{2} + A_{c,\perp}^\mu$$

It follows from above that:

$$\underbrace{\langle 0 | T\{ n\cdot A_c(x)\; n\cdot A_c(0) \} | 0 \rangle}_{\boxed{\sim \lambda^4}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[0 + (1-\xi)\, \underbrace{\frac{(n\cdot p)^2}{p^2}}_{\frac{\lambda^4}{\lambda^2}} \Big]$$

$$\underbrace{\langle 0 | T\{ \bar{n}\cdot A_c(x)\; \bar{n}\cdot A_c(0) \} | 0 \rangle}_{\boxed{\sim \lambda^0}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[0 + (1-\xi)\, \underbrace{\frac{(\bar{n}\cdot p)^2}{p^2}}_{\frac{\lambda^0}{\lambda^2}} \Big]$$

$$\underbrace{\langle 0 | T\{ A_{c,\perp}^{\mu,a}(x)\, A_{c,\perp}^{\nu,b}(0) \} | 0 \rangle}_{\boxed{\sim \lambda^2}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[\underbrace{-g_\perp^{\mu\nu}}_{1} + (1-\xi)\, \underbrace{\frac{p_\perp^\mu p_\perp^\nu}{p^2}}_{\frac{\lambda^2}{\lambda^2}} \Big]$$

same

$$\Rightarrow \quad \boxed{n\cdot A_c \sim \lambda^2, \quad \bar{n}\cdot A_c \sim 1, \quad A_{c,\perp}^\mu \sim \lambda}$$

This also works for the mixed terms, e.g.:

$$\underset{\sim \lambda^2}{\langle 0|T\{n\cdot A_c(x)\,\bar{n}\cdot A_c(0)\}|0\rangle} = \int \underset{\lambda^4}{\frac{d^4p}{(2\pi)^4}}\, e^{-ip\cdot x}\, \underset{\lambda^{-2}}{\frac{i\delta_{ab}}{p^2+i\epsilon}}$$

$$\times \left[\underset{1}{-n\cdot\bar{n}} + (1-\xi)\, \underset{\frac{\lambda^2\cdot 1}{\lambda^2}}{\frac{n\cdot p\ \bar{n}\cdot p}{p^2}} \right]$$

etc.

We observe that the different components of the collinear gluon field scale like the components of a collinear momentum, such that the covariant collinear derivative

$$\boxed{iD_c^\mu = i\partial^\mu + g_s A_c^{\mu,a} t^a \sim (\lambda^2, 1, \lambda)}$$

has homogeneous power counting in λ !

Important note:

The presence of a field with unsuppressed power counting, $\bar{n}\cdot A_c \sim 1$, is worrisome, since this will lead to an infinite number of operators with the same power counting in the EFT. We will see the implications of this observation later.

Ultra-soft quark field:

We have:

$$\langle 0 | T\{ q_{us}(x)\, \bar{q}_{us}(0) \} | 0 \rangle = \underbrace{\int \frac{d^4k}{(2\pi)^4}}_{\lambda^8} \underbrace{e^{-ik\cdot x}}_{1} \underbrace{\frac{i\not{k}}{k^2+i\epsilon}}_{\frac{\lambda^2}{\lambda^4}} \sim \lambda^6$$

This gives:

$$\boxed{q_{us} \sim \lambda^3}$$

There are no large or small components in this case.

Ultra-soft gluon fields:

We find:

$$\langle 0 | T\{ A^{\mu,a}_{us}(x), A^{\nu,b}_{us}(0) \} | 0 \rangle = \underbrace{\int \frac{d^4k}{(2\pi)^4}}_{\lambda^8} e^{-ik\cdot x} \underbrace{\frac{i\delta_{ab}}{k^2+i\epsilon}}_{\lambda^{-4}} \Big[\underbrace{-g^{\mu\nu}}_{1} + (1-\xi) \underbrace{\frac{k^\mu k^\nu}{k^2}}_{\lambda^0} \Big]$$

$$\sim \lambda^4$$

This gives:

$$\boxed{A^{\mu,a}_{us} \sim \lambda^2}$$

In the following we will derive the effective Lagrangian of collinear and ultra-soft fields, focussing on the Dirac Lagrangian. The Yang-Mills Lagrangian can be discussed in an analogous way.

Introducing the decompositions

$$\psi \longrightarrow \psi_c + q_{us} = \xi_n + \eta_n + q_{us}$$

$$A^\mu \longrightarrow A^\mu_c + A^\mu_{us}$$

in the Dirac Lagrangian, we obtain:

$$\bar{\psi}\, i\slashed{D}\, \psi \longrightarrow (\overset{\lambda}{\bar{\xi}_n} + \overset{\lambda^2}{\bar{\eta}_n} + \overset{\lambda^3}{\bar{q}_{us}})\, i\slashed{D}_{c+us}\, (\overset{\lambda}{\xi_n} + \overset{\lambda^2}{\eta_n} + \overset{\lambda^3}{q_{us}})$$

where:

$$i\slashed{D}_{c+us} = (\overset{\lambda^2}{i n\cdot D_c} + g_s\, \overset{\lambda^2}{n\cdot A_{us}})\frac{\bar{\slashed{n}}}{2} + (\overset{1}{i\bar{n}\cdot D_c} + g_s\, \overset{\lambda^2}{\bar{n}\cdot A_{us}})\frac{\slashed{n}}{2}$$
$$+\, \overset{\lambda}{i\slashed{D}^\perp_c} + g_s\, \overset{\lambda^2}{\slashed{A}^\perp_{us}}$$

$\mathcal{O}(\lambda^4)$:

$$\bar{\xi}_n \frac{\bar{\slashed{n}}}{2}(i n\cdot D_c + g_s n\cdot A_{us})\, \xi_n + \bar{\eta}_n \frac{\slashed{n}}{2}\, i\bar{n}\cdot D_c\, \eta_n$$
$$+\, \bar{\xi}_n\, i\slashed{D}^\perp_c\, \eta_n + \bar{\eta}_n\, i\slashed{D}^\perp_c\, \xi_n$$

$\mathcal{O}(\lambda^5)$:

$$\bar{\xi}_n\, g_s \slashed{A}^\perp_{us}\, \eta_n + \bar{q}_{us}\, g_s \slashed{A}^\perp_c\, \xi_n + \bar{q}_{us} \frac{\slashed{n}}{2} g_s\, \bar{n}\cdot A_c\, \eta_n + \text{h.c.}$$

In the $\mathcal{O}(\lambda^5)$ Lagrangian we have used that terms involving only a single collinear field are not allowed by momentum conservation. Note that <u>all</u> terms of $\mathcal{O}(\lambda^5)$ and higher contain at least one ultra-soft field.

From now on we focus on the leading-order SCET Lagrangian:

$$\mathcal{L}_{SCET} = \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{c+us}$$

with:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\not{\bar{n}}}{2} i n\cdot D_c \xi_n + \bar{\eta}_n \frac{\not{n}}{2} i \bar{n}\cdot D_c \eta_n$$
$$+ \bar{\xi}_n i \not{D}_c^{\perp} \eta_n + \bar{\eta}_n i \not{D}_c^{\perp} \xi_n + (\text{pure glue terms}) \sim \lambda^4$$

$$\mathcal{L}_{c+us} = \bar{\xi}_n \frac{\not{\bar{n}}}{2} g_s n\cdot A_{us} \xi_n + (\text{pure glue terms}) \sim \lambda^4$$

$$\mathcal{L}_{us} = \bar{q}_{us} i \not{D}_{us} q_{us} + (\text{pure glue terms}) \sim \lambda^8$$

The leading-order action is:

$$S_{SCET} = \underset{\lambda^{-4}}{\int d^4x} \underbrace{(\mathcal{L}_c + \mathcal{L}_{c+us})}_{\lambda^4} + \underset{\lambda^{-8}}{\int d^4x} \underset{\lambda^8}{\mathcal{L}_{us}} \sim \lambda^0$$

Since the field ξ_n contains the large components of the collinear spinor field, we can use it to describe collinear quarks and integrate out the power-suppressed field η_n in the generating functional. Like in HQET, the functional determinant is just an irrelevant (divergent) constant. The resulting Lagrangian is obtained by using the solution of the classical equation of motion:

$$\frac{\delta \mathcal{L}_c}{\delta \bar{\eta}_n} = 0 \;\Rightarrow\; \underset{1}{\frac{\not{n}}{2}\, i\bar{n}\cdot D_c}\, \underset{\lambda^3}{\eta_n} + \underset{\lambda}{i\not{D}_c^{\perp}}\, \underset{\lambda^2}{\xi_n} = 0$$

To solve this equation for η_n we introduce an auxilliary regulator $i\delta$ to obtain:

$$\underbrace{\frac{\not{\bar{n}}\not{n}}{4}}_{\text{replace by: } P_{\bar{n}} + P_n = 1} (i\bar{n}\cdot D_c + i\delta)\, \eta_n = -\frac{\not{n}}{2}\, i\not{D}_c^{\perp}\, \xi_n$$

P_n: vanishes when acting on η_n

$$\Rightarrow\; \boxed{\eta_n = -\frac{1}{\underbrace{i\bar{n}\cdot D_c}_{\mathcal{O}(\lambda^0)} + i\delta}\, \frac{\not{n}}{2}\, i\not{D}_c^{\perp}\, \xi_n}$$

$i\delta$: arbitrary sign, since pole is unphysical

It is instructive to compare this to the corresponding expression in HQET:

$$H_v = \frac{1}{\underset{1}{2m_Q} + \underset{\lambda}{iv\cdot D_s}}\, i\not{D}_s^{\perp}\, h_v \qquad (\to \text{p. } 16)$$

In that case, the inverse differential operator could be expanded in powers of $iv\cdot D_s/m_Q = \mathcal{O}(\lambda)$. In the case of SCET, such an expansion parameter is lacking.

Inserting the above solution into our expression for $\mathcal{L}_c$, we find:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\not{\bar{n}}}{2} \, i n\cdot D_c \, \xi_n - \bar{\xi}_n \, i \not{D}_c^{\perp} \, \frac{\not{\bar{n}}}{2} \, \frac{1}{i\bar{n}\cdot D_c + i\delta} \, i\not{D}_c^{\perp} \, \xi_n$$

$$+ \text{(pure glue terms)}$$

The inverse of a derivative is an integral, but what is the inverse of a covariant derivative?

To define the above expression properly, we introduce the <u>collinear Wilson line</u>:

large component $\bar{n}\cdot A_c \sim \lambda^0$ light-like direction $\bar{n}^\mu$

$$\boxed{W_c(x) = \mathbf{P} \exp\left(i g_s \int_{-\infty}^{0} dt \; \bar{n}\cdot A_c(x + t\bar{n}) \right)}$$

This is analogous to the definition of the soft Wilson line in HQET (see p. 21). It follows that:

$$\left[i\bar{n}\cdot D_c \, W_c(x) \right] = 0$$

$$\Rightarrow \quad i\bar{n}\cdot D_c \, W_c(x) \, \phi_c(x) = W_c(x) \, i\bar{n}\cdot\partial \, \phi_c(x)$$

arb. function of collinear fields

As a differential operator, it follows that:

$$\boxed{W_c^\dagger(x)\; i\bar{n}\cdot D_c\; W_c(x) = i\bar{n}\cdot\partial}$$

This in turn implies:

$$\frac{1}{i\bar{n}\cdot D_c + i\delta} = W_c \frac{1}{i\bar{n}\cdot\partial + i\delta} W_c^\dagger$$

↳ proof: apply $W_c^\dagger\, i\bar{n}\cdot D_c \ldots W_c$ on both sides

The second term in the Lagrangian can now be written in the form:

$$\bar{\xi}_n\, i\not{D}_c^\perp \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c + i\delta}\, i\not{D}_c^\perp\, \xi_n(x)$$

$$= \bar{\xi}_n\, i\not{D}_c^\perp W_c \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot\partial + i\delta} W_c^\dagger\, i\not{D}_c^\perp\, \xi_n(x)$$

$$= \left(\bar{\xi}_n\, i\not{D}_c^\perp W_c\right)(x)\, \frac{\not{\bar{n}}}{2}\, (-i)\int_{-\infty}^{0} dt\, \left(W_c^\dagger\, i\not{D}_c^\perp\, \xi_n\right)(x + t\bar{n})$$

↳ check:

$$i\bar{n}\cdot\partial_x\, (-i)\int_{-\infty}^{0} dt\; \phi(x + t\bar{n}) = \bar{n}\cdot\partial_x \int_{-\infty}^{\frac{n\cdot x}{2}} dt'\; \phi\left(\frac{\bar{n}\cdot x}{2} n + x_\perp + t'\bar{n}\right)$$

$$= \frac{\bar{n}\cdot n}{2}\,\phi(x) = \phi(x)$$

Note that the lower integration limit "$-\infty$" is appropriate for our choice of the "$i\delta$" regulator. If the collinear fields $(W_c^\dagger \, i\not{D}_c^\perp \, \xi_n)(x+t\bar{n})$ carry total momentum p_c, then the t-integral gives:

$$(-i)\int_{-\infty}^{0} dt \; e^{-i p_c \cdot (x + t\bar{n})}$$

regulator to ensure convergence at $t \to -\infty$

$$= (-i)\, e^{-i p_c \cdot x} \int_{-\infty}^{0} dt \; e^{-i(\bar{n}\cdot p_c + i\delta)\, t} = \frac{e^{-i p_c \cdot x}}{\bar{n}\cdot p_c + i\delta}$$

This indeed corresponds to the action of the inverse differential operator:

$$\frac{1}{i\bar{n}\cdot\partial + i\delta} \; e^{-i p_c \cdot x}$$

This leads to the final form of the leading-order SCET Lagrangian:

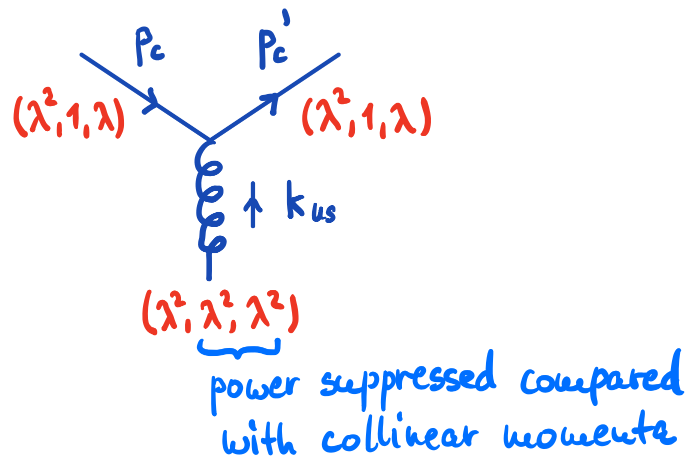
$$\boxed{\begin{aligned} \mathcal{L}_c(x) = {} & \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c \, \xi_n(x) \\ & + \left(\bar{\xi}_n \, i\not{D}_c^\perp W_c\right)(x) \frac{\not{\bar{n}}}{2}\, i \int_{-\infty}^{0} dt \left(W_c^\dagger \, i\not{D}_c^\perp \, \xi_n\right)(x + t\bar{n}) \\ & + (\text{pure glue terms}) \end{aligned}}$$

Coupling to ultra-soft gluons:

Besides the collinear Lagrangian, the leading-order SCET Lagrangian contains interactions of collinear fields with the component $n\cdot A_{us}$ of the ultra-soft gluon field:

$$\mathcal{L}_{c+us} = \bar{\xi}_n \frac{\bar{n}}{2} g_s\, n\cdot A_{us}\, \xi_n \quad + \text{(pure glue terms)}$$

Let us look at the structure of these interactions in more detail:



In the method of regions it is important that we expand the Feynman integrands consistently to leading order in λ. This means that we should expand:

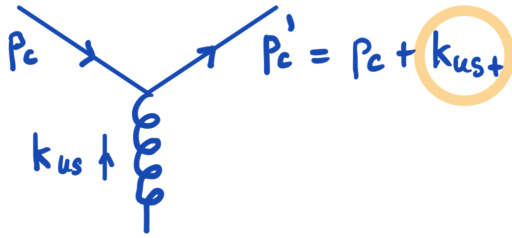
$$p_c'^{\,\mu} = p_c^\mu + k_{us}^\mu = (\overset{\lambda^2}{n\cdot p_c} + \overset{\lambda^2}{n\cdot k_{us}})\frac{\bar{n}^\mu}{2} + (\overset{1}{\bar{n}\cdot p_c} + \overset{\lambda^2}{\bar{n}\cdot k_{us}})\frac{n^\mu}{2} + \overset{\lambda}{p_{c\perp}^\mu} + \overset{\lambda^2}{k_{us\perp}^\mu}$$

$$= (n\cdot p_c + n\cdot k_{us})\frac{\bar{n}^\mu}{2} + \bar{n}\cdot p_c\,\frac{n^\mu}{2} + p_{c\perp}^\mu + \text{higher orders}$$

must expand away!

$$\longrightarrow\; p_c^\mu + k_{us+}^\mu \;; \qquad k_+^\mu \equiv n\cdot k\,\frac{\bar{n}^\mu}{2}$$

This implies that 4-momentum is not conserved at the vertex:



To implement this rule at the Lagrangian level, we must perform a <u>multipole expansion</u> of the ultra-soft fields whenever they interact with collinear fields:

$$x^\mu = \underbrace{n\cdot x \frac{\bar n^\mu}{2}}_{} + \underbrace{\bar n \cdot x \frac{n^\mu}{2}}_{} + x_\perp^\mu$$

$$\equiv \underset{1}{x_+^\mu} + \underset{\lambda^{-2}}{x_-^\mu} + \underset{\lambda^{-1}}{x_\perp^\mu} \quad \leftarrow \text{in interactions with collinear fields}$$

(since $x \cdot p_c \sim 1$)

$$\Rightarrow \quad \phi_{us}(x) = \phi_{us}(x_-) + \overset{\lambda^{-1}}{x_\perp} \cdot \overset{\lambda^2}{\partial_\perp} \phi_{us}(x_-)$$

$$+ \Big(\underset{1}{x_+} \cdot \underset{\lambda^2}{\partial_-} + \underset{\lambda^{-2}}{\frac{x_\perp^\mu x_\perp^\nu}{2}} \underset{\lambda^2}{\partial_\mu^\perp} \underset{\lambda^2}{\partial_\nu^\perp} \Big) \phi_{us}(x_-) + \ldots$$

↳ generates series of higher-order terms in λ

At leading order in λ, the correct form of the effective Lagrangian thus contains:

$$\boxed{\mathcal{L}_{c+us}(x) = \bar\xi_n(x) \frac{\bar n\!\!\!/}{2}\, g_s\, n\cdot A_{us}(\overset{!}{x_-})\, \xi_n(x) + (\text{pure glue terms})}$$

For the vertex shown above, the action $\int d^4x\, \mathcal{L}_{c+us}(x)$ generates:

$$\frac{1}{2}\bar{n}\cdot x\; n\cdot k_{us} = x\cdot\left(\frac{\bar{n}}{2}\, n\cdot k_{us}\right) = x\cdot k_{us+}$$

$$\int d^4x\; e^{i(p_c'\cdot x - p_c\cdot x - \underbrace{k_{us}\cdot x_-})}$$

$$= \int d^4x\; e^{i(p_c'\cdot x - p_c\cdot x - k_{us+}\cdot x)} = (2\pi)^4\, \delta^{(4)}(p_c' - p_c - k_{us+}) \quad ✓$$

Leading-order SCET Lagrangian:

Collecting our results, and adding back the anti-collinear sector, we obtain:

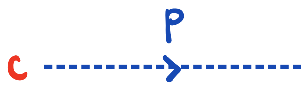
$$\begin{aligned}
\mathcal{L}_{SCET} &= \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n(x) \\
&\quad + (\bar{\xi}_n\, i\not{D}_c^\perp W_c)(x)\, \frac{\not{\bar{n}}}{2}\, i\int_{-\infty}^{0} dt\, (W_c^\dagger\, i\not{D}_c^\perp\, \xi_n)(x+t\bar{n}) && \Big\}\ \mathcal{L}_c \\
&\quad + \left[\text{same with } n\leftrightarrow\bar{n},\ c\leftrightarrow\bar{c}\right] && \mathcal{L}_{\bar{c}} \\
&\quad + \bar{q}_{us}\, i\not{D}_{us}\, q_{us}(x) && \mathcal{L}_{us} \\
&\quad + \bar{\xi}_n(x)\, \frac{\not{\bar{n}}}{2}\, g_s\, n\cdot A_{us}(x_-)\, \xi_n(x) && \mathcal{L}_{c+us} \\
&\quad + \bar{\xi}_{\bar{n}}(x)\, \frac{\not{n}}{2}\, g_s\, \bar{n}\cdot A_{us}(x_+)\, \xi_{\bar{n}}(x) && \mathcal{L}_{\bar{c}+us} \\
&\quad + (\text{pure glue terms}) && \text{same structure as above}
\end{aligned}$$

Note the important fact that ultra-soft quarks do not interact with collinear fields at leading order!

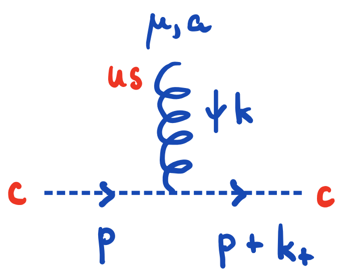
In principle, it is possible to go to higher orders in the expansion in λ ($\rightarrow$ a topic of intensive current research), but we will focus on the leading terms in this course.

Feynman rules:

$$\frac{\not{n}\not{\bar{n}}}{4}\,\not{p}\,\frac{\not{\bar{n}}\not{n}}{4} = \frac{\not{n}}{2}\,\bar{n}\cdot p$$

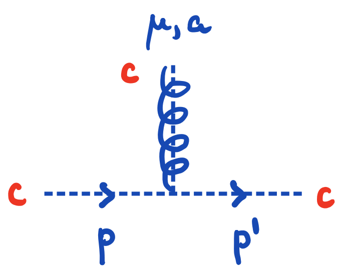


$$= \frac{\not{n}}{2}\,\frac{i\,\bar{n}\cdot p}{n\cdot p\;\bar{n}\cdot p + p_\perp^2 + i0} = P_n\,\frac{i\not{p}}{p^2 + i0}\,P_n^\dagger$$

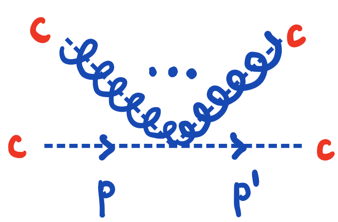


$$= i g_s t^a\, n^\mu\, \frac{\not{\bar{n}}}{2}$$

(momentum not conserved)



$$= i g_s t^a \left(n^\mu + \frac{\gamma_\perp^\mu \not{p}_\perp}{\bar{n}\cdot p} + \frac{\not{p}_\perp'\gamma_\perp^\mu}{\bar{n}\cdot p'} - \bar{n}^\mu\,\frac{\not{p}_\perp'\not{p}_\perp}{\bar{n}\cdot p'\;\bar{n}\cdot p}\right)\frac{\not{\bar{n}}}{2}$$



= complicated

plus pure glue and ghost vertices

Due to the presence of the Wilson lines, there are vertices involving any number of collinear gluons!