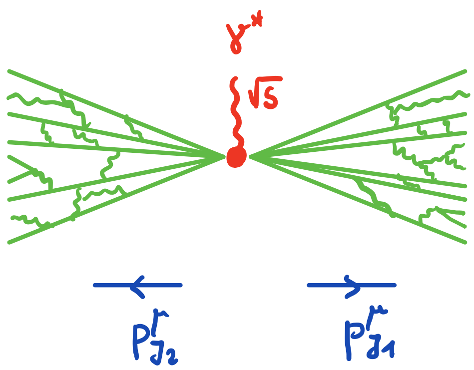
# IV. Construction of Soft-Collinear Effective Theory

A long-standing problem in QCD is how to systematically account for long-distance effects (including power corrections) in processes which do not admit a local OPE.

→ OPE provides rigorous framework for an expansion in powers and logarithms of the large scale for euclidean processes

→ processes involving energetic light particles pose new challenges: $p^\mu$ has some large components, but $p^2 \approx 0$

Consider $e^+e^- \to 2$ jets:




highly collimated jets of particles: large energy along jet axis, small invariant mass

$$p^\mu_{J_1} = \left(E_1,\, 0,\, 0,\, \sqrt{E_1^2 - m_{J_1}^2}\,\right)$$
$$p^\mu_{J_2} = \left(E_2,\, 0,\, 0,\, -\sqrt{E_2^2 - m_{J_2}^2}\,\right)$$

; $E_i \approx \frac{\sqrt{s}}{2}$ , $m_{J_i}^2 \ll s$

↳ intrinsically Minkowskian process

↳ what is there to integrate out?

Introduce small expansion parameter:

$$\boxed{\lambda \sim \frac{m_J}{Q} \ll 1} \qquad (Q = \sqrt{s})$$

Define two light-like reference vectors along jet directions:

$$n^\mu = (1,0,0,1), \quad \bar{n}^\mu = (1,0,0,-1)$$

$$n^2 = 0, \quad \bar{n}^2 = 0, \quad n\cdot\bar{n} = 2$$

Decompose 4-vectors in a <u>light-cone basis</u> spanned by $n^\mu, \bar{n}^\mu$ and two perpendicular directions:

$$\boxed{p^\mu = (n\cdot p)\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\frac{n^\mu}{2} + p_\perp^\mu}$$

$$\Rightarrow \begin{cases} n\cdot p_{J_1} = E_1 - \sqrt{E_1^2 - m_{J_1}^2} \approx \dfrac{m_{J_1}^2}{2E_1} \approx \dfrac{m_{J_1}^2}{\sqrt{s}} \sim \lambda^2 Q \\ \bar{n}\cdot p_{J_1} = E_1 + \sqrt{E_1^2 - m_{J_1}^2} \approx 2E_1 \approx \sqrt{s} = 1\cdot Q \\ p_{J_1}^\perp = 0 \end{cases}$$

Similarly: $\quad n\cdot p_{J_2} \sim 1\cdot Q, \quad \bar{n}\cdot p_{J_2} \sim \lambda^2 Q, \quad p_{J_2}^\perp = 0$

Individual partons inside the jets can carry momenta with the same scaling rules, but these can also have transverse components as long as $p_\perp^2 < m_{J_i}^2$. We thus find:

partons inside jet 1: $(n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (\lambda^2, 1, \lambda)\, Q$

↪ collinear ($n$-collinear) particles

partons inside jet 2: $(n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (1, \lambda^2, \lambda)\, Q$

↪ anti-collinear ($\bar{n}$-collinear) particles

↪ note: $p_i^2 = (n\cdot p_i)(\bar{n}\cdot p_i) + p_{\perp,i}^2 \sim \lambda^2 Q^2$ (both cases)

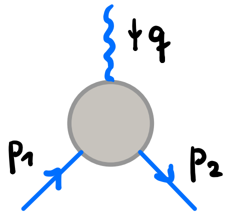
These collinear particles have virtualities much less than the hard scale $Q^2 = s$. But in virtual diagrams we can also exchange hard particles with:

$$p_i^\mu \sim Q\,, \quad \text{i.e.} \quad (n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (1,1,1)\, Q$$

↪ hard particles

We could try to integrate out these hard quantum fluctuations and construct an EFT built out of collinear and anti-collinear particles. However, one finds that this is not the whole story.

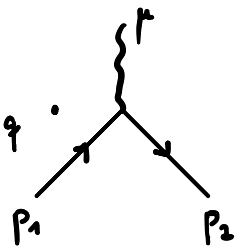
It is instructive at this point to consider a concrete example, the (off-shell) Sudakov form factor:




$|q^2| \gg |p_i^2|$

$(m_i = 0)$

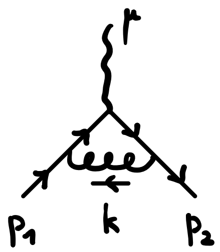
## One-loop calculation:

$Z_q \cdot$  $= Z_q \; \bar{u}(p_1)\, \gamma^\mu \, u(p_2)$ ; $\quad Z_q = 1 + \mathcal{O}(\alpha_s)$

WFR

(off-shell WFR factor)

from coupling renormalization



$$= -i\, C_F\, g_s^2\, \mu^{2\epsilon} \int \frac{d^Dk}{(2\pi)^D} \frac{\bar{u}(p_1)\, \gamma^\alpha (\not{k}+\not{p}_2)\, \gamma^\mu (\not{k}+\not{p}_1)\, \gamma_\alpha\, u(p_2)}{(k^2+i0)\,[(k+p_1)^2+i0]\,[(k+p_2)^2+i0]}$$

use Feynman gauge $\xi = 1$

$\boxed{D = 4-2\epsilon}$

For our purposes, it suffices to focus on the scalar loop integral:

$$I = i\, \pi^{-D/2} \mu^{2\epsilon} \int d^Dk \, \frac{2\, p_1 \cdot p_2}{(k^2+i0)\,[(k+p_1)^2+i0]\,[(k+p_2)^2+i0]}$$

$$= \ln\frac{Q^2}{P_1^2}\, \ln\frac{Q^2}{P_2^2} + \frac{\pi^2}{3} + \mathcal{O}\!\left(\frac{P_i^2}{Q^2}\right) \qquad \text{finite for } \epsilon \to 0$$

Sudakov double log

with:

$$Q^2 = -q^2 - i0 = -s - i0 < 0 \quad \to \text{ non-zero imag. part}$$

$$P_i^2 = -p_i^2 - i0 > 0 \quad (\text{off-shell} \to \text{IR regulators})$$

↪ note that $Q^2 = -(p_2 - p_1)^2 \simeq 2\, p_1 \cdot p_2$ up to $\mathcal{O}(\lambda^2)$ terms

In the CMS: $(n\cdot q,\ \bar{n}\cdot q,\ q_\perp) = (1, 1, 0)\sqrt{s}$ hard

$(n\cdot p_1,\ \bar{n}\cdot p_1,\ p_1^\perp) \sim (\lambda^2, 1, 0)\sqrt{s}$ collinear

$(n\cdot p_2,\ \bar{n}\cdot p_2,\ p_2^\perp) \sim (1, \lambda^2, 0)\sqrt{s}$ anti-collinear

We will now decompose this result into a sum of contributions each depending on only a single scale. The hard contribution (depending on $Q^2$) arises from hard quantum fluctuations. Contributions from lower scales will later be associated with low-frequency modes in the EFT.

## Method of regions: (Smirnov 1990s)

Systematic method for performing "Taylor expansions" of Feynman graphs $F_\Gamma$ by decomposing them into "regions":

$$F_\Gamma \sim \sum_\gamma M_\gamma \, F_\Gamma$$

sum over sets $\gamma$ of subgraphs

Taylor expansion in variables that are small in $\gamma$

Practical procedure (roughly):

1. determine large and small scales in the graph

2. introduce factorization scales $\mu_i$ and divide the loop integrals into regions in which each loop momentum is related to one of these scales

3. perform a Taylor expansion in parameters that are small in a given region

4. after the expansion, ignore the factorization scales and integrate over the entire loop integration domain in each region

For step 4 to be valid, it is essential that $F_\Gamma$ is defined in <u>dimensional regularization</u>!

↳ follows from vanishing of scaleless integrals:

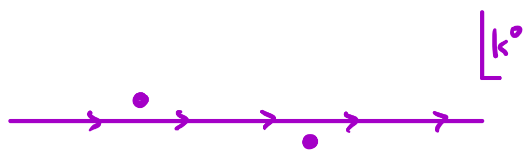
$$\boxed{\int d^D k \, \frac{1}{(k^2)^a} \equiv 0}$$

<u>Comment:</u>

A more rigorous treatment uses the notion of "singular surfaces":

- loop integrals (in dim.reg.) are contour integrals in the complex momentum plane

- non-zero contributions result from pinch singularities, where contours cannot be deformed so as to avoid the poles:




- this happens when some propagators go on shell (happens only when some loop momenta match external scales)

- other, off-shell propagators can be expanded about the singular surfaces (no need for hard cutoffs)
- graph $F_\Gamma$ is then equal to the sum of all singular subgraphs

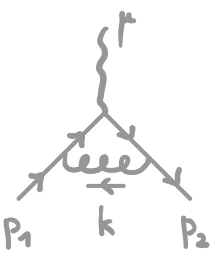
(see e.g. Beneke, Smirnov 1997)

We will see how this procedure works in concrete examples. Every region we identify (hard, collinear, anti-collinear, ...) will be associated with either a Wilson coefficient (hard region) or a field in the low-energy EFT. $\longrightarrow$ see exercises for more details

## Region analysis of the Sudakov form factor:

We now decompose the scalar one-loop integral

$$I = i\pi^{-D/2}\mu^{2\epsilon}\int d^Dk \, \frac{2p_1\cdot p_2}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[(k+p_2)^2+i0\right]}$$




into regions where the loop momentum $k^\mu$ is hard, collinear, and anti-collinear.

a) hard region: $k^\mu \sim (1,1,1)\,Q$

Expansions of propagators:

$$(k+p_1)^2 = k^2 + 2k\cdot p_1 + p_1^2$$

$$= \underset{1}{k^2} + \underset{1}{2k}\cdot\Big((n\cdot p_1)\underset{\lambda^2}{\frac{\bar n}{2}} + \underset{1}{\bar n\cdot p_1}\,\frac{n}{2} + \underset{\lambda}{p_1^\perp}\Big) + \underset{\lambda^2}{p_1^2}$$

$$= k^2 + (n\cdot k)(\bar n\cdot p_1) + \mathcal{O}(\lambda) \simeq (k+p_{1-})^2$$

drop

$$(k+p_2)^2 = k^2 + (\bar n\cdot k)(n\cdot p_2) + \mathcal{O}(\lambda) \simeq (k+p_{2+})^2$$

with: $\quad p_{1-}^\mu = (\bar n\cdot p_1)\,\frac{n^\mu}{2}\,, \quad p_{2+}^\mu = (n\cdot p_2)\,\frac{\bar n^\mu}{2}$ (null vectors)

This gives:

Taylor expansion of $2p_1\cdot p_2$

$$\boxed{\begin{aligned} I_h &= i\,\pi^{-D/2}\mu^{2\epsilon}\int d^Dk\,\frac{2p_{1-}\cdot p_{2+}}{(k^2+i0)\,[(k+p_{1-})^2+i0]\,[(k+p_{2+})^2+i0]} \\ &= \Gamma(1+\epsilon)\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right] \end{aligned}}$$

↳ appearance of double and single poles in $\epsilon$ (IR divergences, since integral is UV-finite)

↳ result depends on hard scale $Q^2$ only (and on the factorization scale $\mu$)

b) collinear region: $k^\mu \sim (\lambda^2, 1, \lambda)\, Q$

Expansions of propagators:

$$(k+p_1)^2 = k^2 + 2k\cdot p_1 + p_1^2$$

$$= \underset{\lambda^2}{k^2} + \underset{\lambda^2}{n\cdot k}\,\underset{1}{\bar{n}\cdot p_1} + \underset{1}{\bar{n}\cdot k}\,\underset{\lambda^2}{n\cdot p_1} + 2\underset{\lambda}{k_\perp}\cdot\underset{\lambda}{p_{1\perp}} + \underset{\lambda^2}{p_1^2} \sim \lambda^2$$

↪ nothing to expand

$$(k+p_2)^2 = \underset{\lambda^2}{k^2} + \underset{\lambda^2}{n\cdot k}\,\underset{\lambda^2}{\bar{n}\cdot p_2} + \underset{1}{\bar{n}\cdot k}\,\underset{1}{n\cdot p_2} + 2\underset{\lambda}{k_\perp}\cdot\underset{\lambda}{p_{2\perp}} + \underset{\lambda^2}{p_2^2} \sim \lambda^0$$

$$= \bar{n}\cdot k\; n\cdot p_2 + \underbrace{\mathcal{O}(\lambda^2)}_{\text{drop}} \simeq 2k\cdot p_{2+}$$

This gives:

$$I_c = i\pi^{-D/2}\mu^{2\epsilon}\int d^Dk\,\frac{2p_{1-}\cdot p_{2+}}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[2k\cdot p_{2+}+i0\right]}$$

$$= \Gamma(1+\epsilon)\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

$$(P_1^2 \equiv -p_1^2)$$

↪ appearance of double and single poles in $\epsilon$ (IR divergences, since integral is UV-finite)

↪ result depends on collinear scale $P_1^2$ only

c) anti-collinear region: $k^\mu \sim (1, \lambda^2, \lambda)\, Q$

We find an analogous contribution:

$$I_{\bar{c}} = \Gamma(1+\epsilon)\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

$$(P_2^2 \equiv -p_2^2)$$

⇒ sum of the three contributions:

$$I_h + I_c + I_{\bar{c}}$$

$$= \Gamma(1+\epsilon)\left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{\mu^2}{Q^2} - \ln\frac{\mu^2}{P_1^2} - \ln\frac{\mu^2}{P_2^2}\right)\right.$$

$$\left. + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

Surprisingly, this does not reproduce the exact result on p. 24, and also uncancelled IR divergences remain. It follows that we have failed to identify (at least) one relevant region. Combining the three logs in the coefficient of the $1/\epsilon$ pole, we get:

$$[\text{ above }] = -\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{P_1^2 P_2^2} + \ldots$$

↪ suggest that missing region corresponds to scale:

$$\frac{P_1^2 P_2^2}{Q^2} \sim \lambda^4 Q^2 \ll \text{collinear scale } P_i^2 \sim \lambda^2 Q^2$$

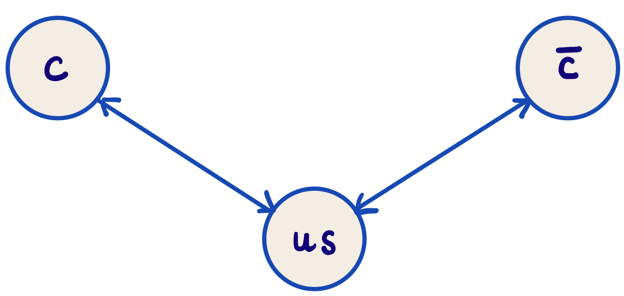
## d) ultra-soft contribution:

There is a strong physics reason suggesting that we need another mode (corresponding to a momentum region) in the low-energy effective theory. An EFT built out of colliner and anti-collinear particles would contain two disjunct sectors, because no vertices connecting both types of particles are allowed:

$$\underset{(\lambda^2,1,\lambda)}{p_c^\mu} + \underset{(1,\lambda^2,\lambda)}{p_{\bar{c}}^\mu} \sim (1,1,\lambda) \quad \text{hard!}$$

Physically, it would be strange if the two jets could not interact in the low energy theory, since they need to neutralize their color. The "largest" on-shell mode that can connect to both collinear and anti-collinear particles without taking them far off-shell is the <u>ultra-soft mode</u>:

$$p_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \Rightarrow p_{us}^2 \sim \lambda^4 Q^2$$




"messenger"

Let us evaluate the ultra-soft contribution to the Sudakov form factor: $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$

$$(k+p_1)^2 = \underset{\lambda^4}{k^2} + \underset{\lambda^2}{n\cdot k}\, \underset{1}{\bar{n}\cdot p_1} + \underset{\lambda^2}{\bar{n}\cdot k}\, \underset{\lambda^2}{n\cdot p_1} + 2\, \underset{\lambda^2}{k_\perp}\cdot \underset{\lambda}{p_{1\perp}} + \underset{\lambda^2}{p_1^2} \sim \lambda^2$$

$$= n\cdot k\; \bar{n}\cdot p_1 + p_1^2 + \mathcal{O}(\lambda^3) \simeq 2k\cdot p_{1-} + p_1^2$$

drop

$$(k+p_2)^2 = \bar{n}\cdot k\; n\cdot p_2 + p_2^2 + \mathcal{O}(\lambda^3) \simeq 2k\cdot p_{2+} + p_2^2$$

This gives:

$$I_{us} = i\pi^{-D/2}\mu^{2\epsilon}\int d^Dk\, \frac{2p_{1-}\cdot p_{2+}}{(k^2+i0)(2k\cdot p_{1-} + p_1^2 + i0)(2k\cdot p_{2+} + p_2^2 + i0)}$$

$$= \Gamma(1+\epsilon)\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{p_1^2 p_2^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{p_1^2 p_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

Adding this result to the expression on page 30, we find:

$$I_h + I_c + I_{\bar{c}} + I_{us}$$

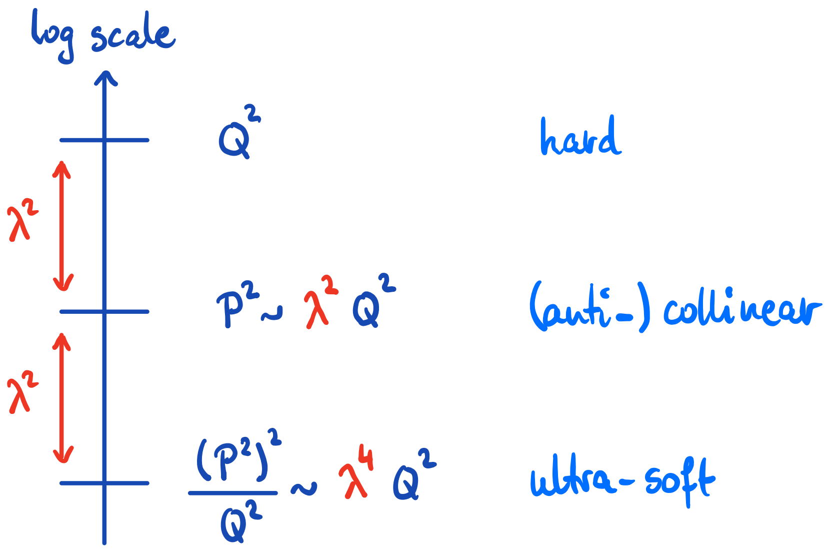
$$= \underbrace{\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}}_{\text{hard}} - \underbrace{\frac{1}{2}\ln^2\frac{\mu^2}{p_1^2}}_{\text{collinear}} - \underbrace{\frac{1}{2}\ln^2\frac{\mu^2}{p_2^2}}_{\text{anti-collinear}} + \underbrace{\frac{1}{2}\ln^2\frac{\mu^2 Q^2}{p_1^2 p_2^2}}_{\text{ultra-soft}} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)$$

$$= \frac{1}{2}\ln\frac{Q^2}{p_1^2}\ln\frac{Q^2}{p_2^2} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)$$

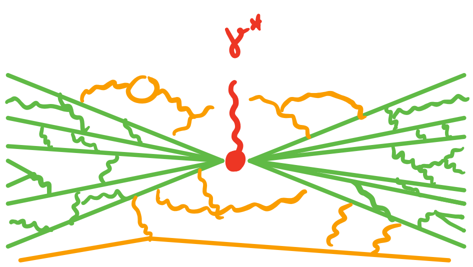
This agrees with the original expression on p.24!

## Comments:

The decomposition of Sudakov double logarithms into a sum of logarithms depending on a single physical scale requires the presence of <u>three correlated scales</u>:




The ultra-soft scale is physical and characterizes soft exchanges between the two jets:




ultra-soft exchanges are needed for color neutralization

The process can be calculated in perturbative QCD only if the ultra-soft scale is much larger that $\Lambda_{QCD}$.

↪ else, need nonperturbative soft functions

## Note on conventions:

Different authors use different names and choices of $\lambda$ to define the various modes. A common alternative convention is to choose $\lambda \sim \frac{m_J^2}{Q^2} = \lambda^2_{\text{above}}$ and define:

$$p_h^\mu \sim (1,1,1)\,Q \quad \text{hard}$$

$$p_{hc}^\mu \sim (\lambda, 1, \lambda^{1/2})\,Q \quad \text{hard-collinear}$$

$$p_{\overline{hc}}^\mu \sim (1, \lambda, \lambda^{1/2})\,Q \quad \text{anti-hard-collinear}$$

$$p_s^\mu \sim (\lambda, \lambda, \lambda)\,Q \quad \text{soft}$$

The names are different, but the physics is the same!