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RENORMALIZATION OF THE WILSON LOOPS BEYOND THE LEADING ORDER

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We study the renormalization properties of the Wilson loops containing cusp singularities. In particular, we calculate the two-loop contribution to the cusp anomalous dimension $\Gamma_{\text{cusp}}(\gamma, g)$ and investigate its behaviour in the limit of large and small cusp angles γ. The general form of $\Gamma_{\text{cusp}}(\gamma, g)$ is established in the limit of large minkowskian cusp angles γ, and the analyticity properties of Γ_{cusp} with respect to γ are investigated. The relation of these results to the nonleading infrared behaviour of the quark form factor is demonstrated.

1. Introduction

One of the most promising approaches for studying the infrared behaviour of quantum chromodynamics is the attempt to formulate the non-abelian gauge theory in the loop-space. Instead of the gauge-dependent entities and Yang-Mills equations, one studies in such a formulation the properties of the gauge-invariant functionals

$$W(\mathrm{C}) = \frac{1}{N}\mathrm{Tr}\langle 0|\mathrm{TP}\exp\left(ig\oint_{\mathrm{C}} \mathrm{d}x^{\mu}\hat{A}_{\mu}(x)\right)|0\rangle \tag{1}$$

(the Wilson loops) and functional equations for $W(\mathrm{C})$ [1,2]. This approach, however, faces many problems. In particular, $W(\mathrm{C})$ is a nonlocal divergent functional of the gauge potential: it cannot be renormalized by the ordinary R-operation [3] restricted to the local operators. The renormalization properties of $W(\mathrm{C})$ for an arbitrary contour C were studied earlier e.g., in refs. [4–8] and the main conclusion contained therein is the following: $W(\mathrm{C})$ is multiplicatively renormalizable to all orders of perturbation theory (PT). More specifically, if the loop is smooth and simple (i.e. without self-intersections) the divergent quantity $W(\mathrm{C})$ can be made finite by expressing it in terms of the renormalized QCD coupling constant (in some renormalization schemes [4] it is necessary to multiply the result also by the $\exp(-KL(\mathrm{C}))$ factor to remove linear divergence K related to a "testing particle" mass renormalization, $L(\mathrm{C})$ being the length of the contour C). Furthermore, it was proved [4–7] that a Wilson loop is multiplicatively renormalizable in the case of it having a finite number of self-intersection points and cusps corresponding to angles $\{\gamma_i\}$ (the relevant infinities are referred to as cusp singularities).

In the present paper we restrict our analysis to a simple loop (without self-intersections) and study the structure of the cusp singularities in higher orders of PT. In sect. 2 we define the regularization procedure for singularities that appear in a perturbative expansion of eq. (1); we also construct there the subtraction procedure and study some properties of both. In sect. 3 we describe the calculation of the cusp anomalous dimension to order α_s and formulate a general scheme for explicit higher-order calculations. We present further our results for the two-loop cusp anomalous dimension. In sect. 4 we study the general form of the cusp anomalous dimension in the limit $\gamma \to \infty$ (where γ is the minkowskian cusp angle) for an arbitrary order of PT. In sect. 5 we analyze some properties of our results for the "time-like" cusp angles related to the Glauber singularities. In the conclusion we formulate our main results.

2. Regularization and subtraction procedure

2.1. PRELIMINARIES

If one expands $W(\mathrm{C})$ in the PT series

$$W(\mathrm{C}) = 1 + \frac{1}{N} \sum_{n=2}^{\infty} (ig)^n \oint_{\mathrm{C}} \mathrm{d}x_1^{\mu_1} \dots \oint_{\mathrm{C}} \mathrm{d}x_n^{\mu_n}\, \theta_{\mathrm{C}}(x_1 > \cdots > x_n) \operatorname{Tr} D_{\mu_1 \dots \mu_n}(x_1, \dots, x_n), \tag{2}$$

there appear the ultraviolet (UV) singularities both from the ultraviolet integration regions for the Green function $D_{\mu_1 \dots \mu_n}(x_1, \dots, x_n)$ and from "contraction into a point" of some set of contour integrations. In what follows it is always implied that all integrals are dimensionally regularized. To analyze the UV divergences of eq. (2), we incorporate the approach [5] in which the one-dimensional fermions living on the contour C are introduced. In this approach eq. (2) can be rewritten as

$$W(\mathrm{C}) = \langle 0 | \mathrm{T}\, \bar{z}(L) z(0) | 0 \rangle$$

$$= \int \mathscr{D}\bar{z}(\sigma)\, \mathscr{D}z(\sigma)\, \mathscr{D}\hat{A}_\mu\, \mathscr{D}C\, \mathscr{D}\bar{C} \exp\left[iS_{\mathrm{YM}}(A, C, \bar{C}) + iS_{\mathrm{eff}}(A, z, \bar{z}) \right],$$

where the modified action S_{eff} is

$$S_{\mathrm{eff}} = \int_0^L \mathrm{d}\sigma \left[i\bar{z}(\sigma) \frac{\partial z(\sigma)}{\partial \sigma} - g\bar{z}(\sigma) \dot{x}_\mu(\sigma) \hat{A}_\mu(x(\sigma)) z(\sigma) \right] \tag{3}$$

and furthermore the boundary conditions $x_\mu(L) = x_\mu(0)$, $z(L) = -z(0)$ are imposed.

To study the renormalization properties of the local lagrangian (3), one can apply the ordinary R-operation since the counterterms resulting from its application have (for a smooth simple loop C) structure of the original lagrangian [5]. In other words, after the renormalization one has

$$A^\mu \to A^\mu_R = Z_3^{-1/2} A^\mu, \qquad C \to C_R = \tilde{Z}_3^{-1/2} C, \qquad g \to g_R = Z_1^{-1} Z_3^{3/2} \mu^{-\varepsilon/2} g,$$

$$z(\sigma) \to z_R(\sigma) = \left(Z_3^F\right)^{-1/2} z(\sigma), \qquad \varepsilon = 4 - n, \tag{4}$$

and incorporating in addition the Slavnov-Taylor identities

$$Z_1/Z_3 = \tilde{Z}_1/\tilde{Z}_3 = Z_1^F/Z_3^F$$

(where Z_1, $\tilde{Z}_1$, Z_1^F are the renormalization constants for the three-gluon, four-gluon and fermion-gluon vertices, respectively) one obtains the expression for the Wilson loop (defined on a smooth contour) which is finite in the limit $\varepsilon \to 0$, ε being the dimensional regularization parameter*. Thus, for a simple smooth contour the renormalized contour average $W_R(C; g_R, \mu)$ is given by

$$W_R(C; g_R, \mu) = \lim_{\varepsilon \to 0} \tilde{W}(C; g_R, \mu, \varepsilon), \qquad \tilde{W}(C; g_R, \mu, \varepsilon) = RW(C; g, \varepsilon), \tag{5}$$

where $W(C; g, \varepsilon)$ is a regularized, but not renormalized r.h.s. of eq. (2) and μ is a subtraction point. In what follows we use the $\overline{\text{MS}}$ subtraction scheme [9] for which the renormalization constants Z_1^F, Z_3^F are known in Feynman gauge at the two-loop level [8].

2.2. SUBTRACTION OF CUSP SINGULARITIES

However, if the loop C has a cusp characterized by angle γ then W_R, even after applying to it the R-operation defined by eqs. (4), (5) possesses the cusp singularities resulting from integration in vicinity of the cusp. The relevant divergent subgraphs are those containing the singular point (the cusp) and which are furthermore the two-particle (rainbow) irreducible ones with respect to the one-dimensional fermions lines. The general structure of these subgraphs is illustrated in fig. 1. To construct the renormalized Wilson loop, we incorporate in this case the subtraction procedure K_γ proposed in refs. [2,7]. The action of K_γ on the functional $\tilde{W}(C; g_R, \mu, \varepsilon)$ defined in eq. (5) produces the renormalized contour average with

* The regularization used in ref. [4] in contradistinction to the dimensional regularization violates the chiral invariance of the $z(\sigma)$-field lagrangian and requires an additional renormalization of the mass in eq. (4).

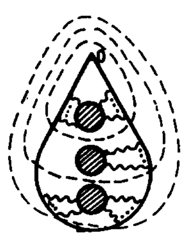


Fig. 1. General structure of the rainbow-irreducible subgraphs. The dashed line denotes the contour integration in a vicinity of the cusp 0. The blob denotes an arbitrary gluon subprocess.

the cusp singularities subtracted for each divergent subgraph of fig. 1:

$$W_{\mathrm{R}}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \overline{\mathrm{C}}_\gamma) = \lim_{\varepsilon \to 0} K_\gamma \tilde{W}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) = \lim_{\varepsilon \to 0} K_\gamma R W(\mathrm{C}_\gamma; g, \varepsilon), \tag{6}$$

where $\overline{\mathrm{C}}_\gamma$ denotes a generalized subtraction point of the K_γ procedure. The cusp divergences are multiplicatively renormalizable, and the action of K_γ on a loop functional containing a single cusp singularity is defined by

$$K_\gamma \tilde{W}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) = Z_{\mathrm{cusp}}(g_{\mathrm{R}}, \gamma; \mu, \overline{\mathrm{C}}_\gamma, \varepsilon) \tilde{W}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon). \tag{7}$$

The r.h.s. of eq. (6) would be finite if the nth term of the PT expansion $Z_{\mathrm{cusp}}(g_{\mathrm{R}}, \gamma; \mu, \overline{\mathrm{C}}_\gamma, \varepsilon) = 1 + \sum_{n=1}^{\infty} g^{2n} Z_n$ equals (up to finite terms and taken with an opposite sign) the cusp divergence of the whole nth-order graph contributing to $\tilde{W}$ with all subdivergences subtracted before. Fixing the finite part of Z_n one fixes a particular K_γ subtraction scheme. We shall use the two schemes described below.

K_γ^{MS} *scheme.* The cusp singularities of an arbitrary subgraph are given in the dimensional regularization by a sum of pole terms. If one defines Z_n to be given just by the sum of the poles

$$Z_n(\gamma, \varepsilon) = \sum_{k=1}^{n} \varepsilon^{-k} a_{kn}(\gamma), \tag{8}$$

one arrives at an MS-like scheme to be referred to from now on as K_γ^{MS} with the generalized subtraction point $\overline{\mathrm{C}}_\gamma$ coinciding with the R-operation parameter μ of eq. (5). Note that the coefficients of expansion (8) as well as Z_n themselves depend in the K_γ^{MS} scheme only on the cusp angle γ, since the UV singularities of eq. (2) for an arbitrary loop C depend only on the first derivative $\dot{x}_\mu(\sigma)$ [6], i.e. on the cusp angle in our case.

K_γ^{MOM} *scheme.* Owing to this important property of the contour averages we can define $(-Z_n)$ corresponding to some arbitrary graph $\tilde{W}_n$ ordered along C_γ to be equal to the contribution of the same graph (with the subdivergences subtracted beforehand) but ordered along another fixed contour $\bar{C}_\gamma$ also possessing a single cusp point with angle γ and having the length $1/\mu$. It is easy to realize that in the subtraction scheme $K_\gamma^{\rm MOM}$ defined in this way (and being an analog of the standard MOM-scheme) the following boundary condition

$$W_{\rm R}(C_\gamma; g_{\rm R}, \mu, \bar{C}_\gamma) = 1 \tag{9}$$

is fulfilled. Furthermore

$$Z_{\rm cusp}(g_{\rm R}, \gamma; \mu, \bar{C}_\gamma, \varepsilon) = \left(\tilde{W}(\bar{C}_\gamma; g_{\rm R}, \mu, \varepsilon)\right)^{-1} = \left(\tilde{W}(1, \gamma, \{\bar{\eta}\}, g_{\rm R}, \varepsilon)\right)^{-1}, \tag{10}$$

where an arbitrary loop is characterized by its length, cusp angle γ, and by a set of some dimensionless parameters $\{\eta\}$.

The subtraction procedures described above possess all the necessary properties of the R-operation. As a result, the renormalized loop average (6) satisfies the renormalization group equation [2]:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g_{\rm R})\frac{\partial}{\partial g_{\rm R}} + \Gamma_{\rm cusp}(\gamma, g_{\rm R})\right) W_{\rm R}(L\mu, \gamma, \{\eta\}, g_{\rm R}) = 0, \tag{11}$$

where the anomalous dimension is given by

$$\Gamma_{\rm cusp}(\gamma, g_{\rm R}) = -\lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln\tilde{W}(C_\gamma; g_{\rm R}, \mu, \varepsilon). \tag{12}$$

As emphasized above, $\Gamma_{\rm cusp}$ depends only on a single contour parameter: the cusp angle γ.

2.3. RENORMALIZED EXPONENTIATION THEOREM

Some general properties of the anomalous dimension $\Gamma_{\rm cusp}$ in higher orders of PT can be established on the basis of the exponentiation theorem [6,10] for the non-abelian path-ordered exponentials (2). The theorem amounts to the statement that the dimensionally regularized (but nonrenormalized) contour average $W(C)$ can be represented in the form

$$W(C; g, \varepsilon) = \exp\left(\sum_{n=1}^{\infty}\alpha_s^n \sum_{W\in W(n)} C_n(W)F_n(W)\right), \tag{13}$$

where summation in the exponential is over all diagrams W of the set W(n) of the two-particle (rainbow) irreducible (2PI) contour averages of nth PT order. (It is straightforward to observe that the criterion of the two-particle irreducibility coincides with the definition of "webs" given in ref. [10].) Furthermore, F_n(W) denotes the contour integral present in the expression for W and $C_n(\mathrm{W})$ the "maximally non-abelian" [10] or the "colour-connected" [11] parts of the colour factor corresponding to the contribution yielded by a diagram W to the total expression (13) for the contour average. For lowest orders in α_s there exists an estimate

$$C_n(\mathrm{W}) \sim C_F N^{n-1} \tag{14}$$

(the exact definition of C_n is given in ref. [11]). The diagrams whose colour factor does not possess a term of eq. (14) type do not contribute to the sum over W in eq. (13).

Of course, eq. (13) is only a formal relation unless the renormalization prescription and the renormalized analogue of eq. (13) are defined. We are interested in loops possessing the cusp singularities. To this end we apply to both sides of eq. (12) the operation $K_\gamma R$ introduced above. Note now that the exponential factor in eq. (13) is given by a sum of contour integrals. Hence, the transformation given by eq. (4) is sufficient for a consistent renormalization, i.e.

$$RW(\mathrm{C}; g, \varepsilon) = \exp\left(\sum_{n=1}^{\infty} \alpha_s^n \sum_{\mathrm{W} \in \mathrm{W}(n)} C_n(\mathrm{W}) RF_n(\mathrm{W}) \right).$$

Denoting

$$\alpha_s^n F_n(\mathrm{W}) = W_n^{2\mathrm{PI}}(\mathrm{W}, \mathrm{C}_\gamma; g, \varepsilon), \qquad RW_n^{2\mathrm{PI}} = \tilde{W}^{2\mathrm{PI}}(\mathrm{W}, \mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon),$$

we find that

$$\tilde{W}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) = \exp\left(\sum_{n=1}^{\infty} \sum_{\mathrm{W} \in \mathrm{W}(n)} C_n(\mathrm{W}) \tilde{W}_n^{2\mathrm{PI}}(\mathrm{W}, \mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) \right). \tag{15}$$

Just as in the above discussion, the r.h.s. of eq. (15) possesses the uncompensated UV poles related to the cusp singularities removed by the K_α operation. Consider first the action of the K_γ^{MOM} procedure on eq. (15). By virtue of eq. (10) we have

$$\begin{aligned} &K_\gamma^{\mathrm{MOM}} \tilde{W}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) \\ &\quad = \left(\tilde{\mathrm{W}}(\bar{\mathrm{C}}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) \right)^{-1} \tilde{W}(\mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) \\ &\quad = \exp\left\{ \sum_{n=1}^{\infty} \sum_{\mathrm{W} \in \mathrm{W}(n)} C_n(\mathrm{W}) \right. \\ &\qquad\qquad \left. \times \left[\tilde{W}_n^{2\mathrm{PI}}(\mathrm{W}, \mathrm{C}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) - \tilde{W}_n^{2\mathrm{PI}}(\mathrm{W}, \bar{\mathrm{C}}_\gamma; g_{\mathrm{R}}, \mu, \varepsilon) \right] \right\}, \end{aligned} \tag{16}$$

where it is taken into account that all the topologically equivalent loops possessing the cusp singularity have the same colour factor $C_n(\mathrm{W})$. Note now that the 2PI contour averages present in the exponential factor of eq. (16) have no divergent subgraphs, and the action of the subtraction procedure K_γ^{MOM} in this case amounts to the subtraction of the contribution of the same graph containing a single pole $1/\varepsilon$ but ordered along $\overline{\mathrm{C}}_\gamma$, i.e.

$$\tilde{W}_n^{2\mathrm{PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\varepsilon) - \tilde{W}_n^{2\mathrm{PI}}(\mathrm{W},\overline{\mathrm{C}}_\gamma;\, g_\mathrm{R},\mu,\varepsilon) = K_\gamma^{\mathrm{MOM}}\tilde{W}_n^{2\mathrm{PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\varepsilon)$$

and, hence

$$\begin{aligned} W_\mathrm{R}(\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\overline{\mathrm{C}}_\gamma) &= \lim_{\varepsilon\to 0}\exp\left[\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in\mathrm{W}(n)} C_n(\mathrm{W})K_\gamma^{\mathrm{MOM}}RW_n^{2\mathrm{PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g,\varepsilon)\right] \\ &= \exp\left[\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in\mathrm{W}(n)} C_n(\mathrm{W})W_{\mathrm{R},n}^{2\mathrm{PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\overline{\mathrm{C}}_\gamma)\right]. \qquad (17) \end{aligned}$$

The validity of this important relation in the $\mathrm{K}_\gamma^{\mathrm{MS}}$ scheme is not obvious a priori because of the absence of the analogue of eq. (10) for this scheme. However, it can be demonstrated that there exists in this case a relation between the renormalization constants for the cusp singularities and the pole part of the 2PI contour averages:

$$Z_{\mathrm{cusp}}^{\mathrm{MS}}(g_\mathrm{R},\gamma,\mu,\varepsilon) = \exp\left(-\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in\mathrm{W}(n)} C_n(\mathrm{W})\tilde{W}_n^{2\mathrm{PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\varepsilon)\Big|_{\mathrm{poles}}\right). \qquad (18)$$

To prove it we note that the factor Z_{cusp} in the K_γ^{MS} scheme (given by a sum of poles of eq. (8)) can always be represented in the form

$$Z_{\mathrm{cusp}}(\gamma,\varepsilon) = \exp\left(-\sum_{\substack{l,n=1\\ l\leqslant n}}^{\infty}\frac{g_\mathrm{R}^{2n}}{\varepsilon^l}f_{nl}(\gamma)\right),$$

where $f_{nl}(\gamma)$ are some yet unknown functions of the cusp angle γ. Substituting this expression and the one for the regularized contour average determined by the contribution of the 2PI graphs

$$\tilde{W}(\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\varepsilon) = \exp\left(\tilde{W}^{2\mathrm{PI}}(\mathrm{C}_\gamma;\, g_\mathrm{R},\mu,\varepsilon)\right) = \exp\left(\sum_{n=1}^{\infty}\frac{g_\mathrm{R}^{2n}}{\varepsilon}\varphi_n(\gamma,\{\eta\},\mu L,\varepsilon)\right)$$

(where $\varphi_n(\gamma,\{\eta\},\mu L,\varepsilon)$ is some regular function of ε) into eq. (7) we find that

$$W_\mathrm{R}(\mathrm{C}_\gamma;\, g_\mathrm{R},\mu) = \lim_{\varepsilon\to 0}\exp\left(\sum_{n=1}^{\infty}\frac{g_\mathrm{R}^{2n}}{\varepsilon}\varphi_n(\gamma,\{\eta\},\mu L,\varepsilon) - \sum_{\substack{l,n=1\\ l\leqslant n}}^{\infty}\frac{g_\mathrm{R}^{2n}}{\varepsilon^l}f_{nl}(\gamma)\right).$$

The requirement that ε-poles be absent in both sides of this equation unambiguously fixes the f_{nl} coefficients;

$$f_{nl}(\gamma)=0, \qquad l>1; \qquad f_{n1}(\gamma)=\varphi_n(\gamma,\{\eta\},\mu L,0).$$

Hence, the final expression for the cusp singularity renormalization constant is given by an exponential of the pole part of the 2PI contour averages depending only on the cusp angle γ [6]. As a result,

$$K_\gamma^{\mathrm{MS}} R W(\mathrm{C}_\gamma;\, g,\, \varepsilon)=\exp\left[\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in \mathrm{W}(n)} C_n(\mathrm{W}) K_\gamma^{\mathrm{MS}} R W_n^{\mathrm{2PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g,\, \varepsilon)\right].$$

Thus, the exponentiation theorem (13) is valid for the renormalized contour averages, at least within the framework of the two subtraction procedures used in the present paper:

$$W_{\mathrm{R}}(\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu,\, \overline{\mathrm{C}}_\gamma)=\exp\left(W_{\mathrm{R}}^{\mathrm{2PI}}(\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu,\, \overline{\mathrm{C}}_\gamma)\right), \tag{19}$$

where

$$W_{\mathrm{R}}^{\mathrm{2PI}}(\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu,\, \overline{\mathrm{C}}_\gamma)=\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in \mathrm{W}(n)} C_n(\mathrm{W}) K_\gamma R W_n^{\mathrm{2PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g,\, \varepsilon).$$

2.4. CUSP ANOMALOUS DIMENSION

Incorporating now the RG equation for the nonrenormalized contour averages one obtains from eq. (19) the equation for $W_{\mathrm{R}}^{\mathrm{2PI}}$:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g_{\mathrm{R}})\frac{\partial}{\partial g_{\mathrm{R}}}\right) W_{\mathrm{R}}^{\mathrm{2PI}}(\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu,\, \overline{\mathrm{C}}_\gamma)=-\Gamma_{\mathrm{cusp}}(\gamma,\, g_{\mathrm{R}}), \tag{20}$$

which has the following important consequences:

(i) Using the explicit form of $W_{\mathrm{R}}^{\mathrm{2PI}}$ we obtain the relation between the cusp anomalous dimension and the contribution of the 2PI contour integrals

$$\begin{aligned}\Gamma_{\mathrm{cusp}}(\gamma,\, g_{\mathrm{R}}) &= -\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in \mathrm{W}(n)} C_n(\mathrm{W})\frac{\mathrm{d}}{\mathrm{d}\ln\mu} W_{\mathrm{R},n}^{\mathrm{2PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu,\, \overline{\mathrm{C}}_\gamma)\\ &= -\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\sum_{\mathrm{W}\in \mathrm{W}(n)} C_n(\mathrm{W})\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \tilde{W}_n^{\mathrm{2PI}}(\mathrm{W},\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu,\, \varepsilon).\end{aligned} \tag{21}$$

This means that $\Gamma_{\mathrm{cusp}}(\gamma,\, g_{\mathrm{R}})$, just as expected, does not depend on the generalized subtraction point $\bar{C}_\gamma$. The second observation is that in an nth order of the PT series expansion $\Gamma_{\text{cusp}}(\gamma, g_R)$ contains only the "maximally non-abelian" [10] or "colour-connected" [11] colour factors. In particular, in the QED case eq. (21) contains only the first term of the series.

(ii) The general form of the solution of eq. (20) is

$$W_R^{2PI}(C_\gamma; g_R, \mu, \bar{C}_\gamma) = W_R^{2PI}(C_\gamma; g_R, \bar{\mu}, \bar{C}_\gamma) - \int_{g_R(\bar{\mu})}^{g_R(\mu)} dg \frac{\Gamma_{\text{cusp}}(\gamma, g)}{\beta(g)}. \tag{22}$$

The expression for W_R^{2PI} contains the μ-dependence in g_R and in a single logarithm of cusp singularity. Hence, there exists some point $\bar{\mu}$ where

$$W_R^{2PI}(C_\gamma; g_R, \bar{\mu}, \bar{C}_\gamma) = 0. \tag{23}$$

Consequently the general solution of eq. (11) can be written as

$$W_R(C_\gamma; g_R, \mu, \bar{C}_\gamma) = \exp\left(- \int_{g_R(\bar{\mu})}^{g_R(\mu)} dg \frac{\Gamma_{\text{cusp}}(\gamma, g)}{\beta(g)} \right), \tag{24}$$

where $\bar{\mu}$ is the solution of eq. (23) depending on the contribution to W_R only from 2PI contour averages of eq. (19).

3. Calculation of the cusp anomalous dimension

As we established in the preceding section the cusp anomalous dimension depends only on a single characteristic of the loop, the cusp angle. Hence, to calculate it, one can use the simplest loop shown in fig. 2 formed by two lines and closed at infinity. Furthermore, we restrict our analysis to the 2PI graphs yielding a non-zero contribution to the expansion (21).

To begin with, we recall the Feynman rules for the modified action (3) in the case of the contour of fig. 2 both in momentum and configuration representations. Note, that the latter has the dimension $n = 4 - \varepsilon$ for gluonic lines and is one-dimensional for the z-fermions, the transformation from the configuration space into the

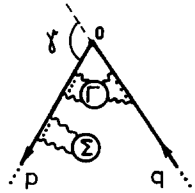
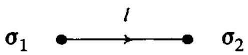


Fig. 2. Self-energy and vertex corrections to the contour average.

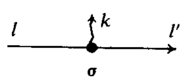
momentum one for the one-dimensional fermions being

$$z(l)=\int_0^{\infty}\mathrm{d}\sigma\,\mathrm{e}^{i\sigma(l+i0)}z(\sigma).$$

In addition to ordinary QCD gluon vertices the action (3) brings in two other elements, viz., the propagator of the one-dimensional fermions and the vertex describing the interaction between gluons and these fermions:



$$\theta(\sigma_2-\sigma_1) \qquad \frac{i}{l+i0},$$



$$ign^{\mu}\hat{A}_{\mu}(n\sigma) \qquad ign^{\mu}\hat{A}_{\mu}(k)\delta(l'+(kn)-l), \tag{25}$$

where $n_\mu=(p_\mu$ or $q_\mu)$ is one of the vectors characterizing the directions of the two lines shown in fig. 2.

It is worth noting here that fig. 2 may be treated as an amplitude of the elastic scattering of an on-mass-shell ($l=0$) one-dimensional fermion on a colour singlet potential. The contributions to this amplitude are due to both the self-energy corrections $\Sigma(l)=\Sigma(0)+l\partial\Sigma(0)/\partial l+\cdots$ to the fermion lines and vertex corrections $\Gamma(l,l';\gamma)$. The latter satisfy the equality

$$\Gamma(0,0;0)=-\left.\frac{\partial\Sigma(l)}{\partial l}\right|_{l=0}$$

following from the gauge properties of eq. (3). Hence, the total result for the cusp singularity of the diagram shown in fig. 2 is given by

$$\Gamma(0,0;\gamma)+\left.\frac{\Sigma(l)}{l}\right|_{l=0}=\Gamma(0,0;\gamma)-\Gamma(0,0;0) \tag{26}$$

and to calculate it one can consider only the vertex corrections which in the lowest nontrivial order in the strong coupling constant α_s are determined by the diagrams shown in fig. 3. In the following the $K_{\gamma}^{\mathrm{MS}}R_{\mathrm{MS}}$ subtraction procedure is used.

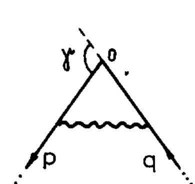


Fig. 3. Diagram contributing to the one-loop cusp anomalous dimension.

3.1. ONE-LOOP CALCULATION

The regularized expression for the contribution of the diagram in fig. 3 in Feynman gauge is

$$\mathcal{M}=(ig)^2(pq)C_{\mathrm{F}}\frac{\Gamma\left(\frac{1}{2}n-1\right)}{4\pi^{n/2}}\mu^{4-n}\int_0^{\infty}\mathrm{d}s\int_0^{\infty}\mathrm{d}t\left[(ps+qt)^2-i0\right]^{1-n/2}.$$

Using the scaling transformation $s+t=\lambda$, $s=\lambda x$ one can rewrite it as

$$\mathcal{M}=(ig)^2(pq)C_{\mathrm{F}}\frac{\Gamma\left(\frac{1}{2}n-1\right)}{4\pi^{n/2}}\mu^{4-n}\int_0^{\infty}\frac{\mathrm{d}\lambda}{\lambda^{n-3}}\int_0^{1}\mathrm{d}x\left[(px+q\bar{x})^2-i0\right]^{1-n/2},\quad(27)$$

where a convenient notation $\bar{x}=1-x$ is introduced. Generally speaking the integral over λ appearing in eq. (27) does not exist because it converges on the lower (ultraviolet) limit only if $n<4$ whereas on the upper (infrared) limit it converges only for $n>4$. The appearance of the IR divergences is a penalty for the (relative) simplicity of the contour chosen since the infinite length of this contour just determines the essential scale for wavelengths of the gluons exchanged by the one-dimensional fermions. To define the λ-integral of eq. (27) in the IR region, one can use another regularization scheme different from the dimensional one, e.g., ascribe a fictitious mass Λ to the gluons which corresponds to the following modification of the gluon propagator

$$\frac{1}{k^2+i0}\to\frac{1}{k^2-\Lambda^2+i0}.\qquad(28)$$

Calculation in this scheme suggests defining the λ-integral of eq. (27) as

$$\mu^{4-n}\int_0^{\infty}\frac{\mathrm{d}\lambda}{\lambda^{n-3}}=\frac{1}{4-n}\left(\frac{\mu}{\Lambda}\right)^{4-n},\qquad(29)$$

where Λ has just the meaning of the IR cut-off parameter (i.e. the scale inverse to the contour length L: $\Lambda\sim 1/L$). The x-integral remaining in eq. (27) can be easily calculated by using the following angular variables

$$\frac{x\sqrt{p^2}+\bar{x}\sqrt{q^2}\,\mathrm{e}^{\gamma}}{x\sqrt{p^2}+\bar{x}\sqrt{q^2}\,\mathrm{e}^{-\gamma}}=\mathrm{e}^{2\psi},\qquad\int_0^1\frac{\mathrm{d}x(pq)}{(px+q\bar{x})^2}=\coth\gamma\int_0^{\gamma}\mathrm{d}\psi.\qquad(30)$$

The angle γ between p_μ and q_μ (fig. 2) in the Minkowski space is defined by

$$\cosh\gamma=\frac{(pq)}{\sqrt{p^2q^2}}.\qquad(31)$$

The corresponding euclidean results can be obtained by a mere redefinition of the angles

$$\gamma_{\mathrm{M}} = i\gamma_{\mathrm{E}}. \tag{32}$$

The final result for the renormalized contribution of fig. 3 (up to the irrelevant finite part) is

$$\mathscr{M}_{\mathrm{R}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda) = K_{\gamma}^{\mathrm{MS}} R_{\mathrm{MS}} \mathscr{M} = -\frac{\alpha_{\mathrm{s}}}{2\pi} C_{\mathrm{F}} \gamma \coth\gamma \ln\frac{\mu^2}{\Lambda^2}, \tag{33}$$

where $\alpha_{\mathrm{s}} = g_{\mathrm{R}}^2/4\pi$. Taking into account also eq. (26) we find the exponential factor in eq. (19)

$$W_{\mathrm{R,\,one\text{-}loop}}^{\mathrm{2PI}}(C_{\gamma};\, g_{\mathrm{R}}, \mu) = -\frac{\alpha_{\mathrm{s}}}{2\pi} C_{\mathrm{F}} (\gamma\coth\gamma - 1) \ln\frac{\mu^2}{\Lambda^2} \tag{34}$$

and the one-loop cusp anomalous dimension

$$\Gamma_{\mathrm{cusp}}^{\mathrm{one\text{-}loop}}(\gamma, g_{\mathrm{R}}) = \frac{\alpha_{\mathrm{s}}}{\pi} C_{\mathrm{F}} (\gamma\coth\gamma - 1). \tag{35}$$

3.2. TWO-LOOP CALCULATION

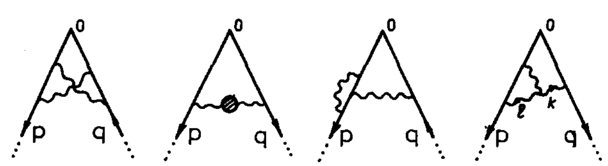
In the α_{s}^2 order the colour factor entering into eq. (19) is proportional to

$$C_2(\mathrm{W}) \sim C_{\mathrm{F}} N \tag{36}$$

and the set of the 2PI vertex diagrams containing the term displayed by eq. (36) in their colour factors is shown in fig. 4a–d. Below we present the results of their calculation in Feynman gauge.

(a) Crossed ladder graph. For the graph 4a we have the expression

$$\mathscr{M}_{\mathrm{a}} = (ig)^4 (pq)^2 C_{\mathrm{F}} \left(C_{\mathrm{F}} - \tfrac{1}{2}N\right) \frac{\Gamma^2\left(\tfrac{1}{2}n - 1\right)}{16\pi^n} \mu^{2(4-n)} \int_0^{\infty} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{\infty} \mathrm{d}s_4 \int_0^{s_4} \mathrm{d}s_3$$
$$\times \left[\left((ps_1 + qs_3)^2 - i0\right)\left((ps_2 + qs_4)^2 - i0\right)\right]^{1-n/2}.$$



(a) (b) (c) (d)

Fig. 4. Diagrams contributing to the two-loop cusp anomalous dimension.

After the scaling transformations $s_2 = xs_1$, $s_3 = ys_4$, $s_1 + s_4 = \lambda$, $s_1 = \lambda z$ it contains an integral over λ that can be defined in a way similar to eq. (29)

$$\mu^{2(4-n)} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{2n-7}} = \frac{1}{2(4-n)} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)}.$$

Calculating now the integrals over x and y gives

$$\mathcal{M}_{\mathrm{a}} = \frac{g^4}{64\pi^n} C_{\mathrm{F}}\left(C_{\mathrm{F}} - \tfrac{1}{2}N\right) \frac{\Gamma^2\left(\frac{1}{2}n - 1\right)}{2(4-n)} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)}$$
$$\times \coth^2\gamma \int_0^1 \frac{\mathrm{d}z}{z\bar{z}} \ln \frac{z\,\mathrm{e}^{\gamma} + \bar{z}}{z\,\mathrm{e}^{-\gamma} + \bar{z}} \ln \frac{z + \bar{z}\,\mathrm{e}^{\gamma}}{z + \bar{z}\,\mathrm{e}^{-\gamma}}. \tag{37}$$

Changing further the angular variables according to eq. (30) and applying the subtraction procedure we get the regularized version of eq. (37)

$$\mathcal{M}_{\mathrm{a,R}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda) = \left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 C_{\mathrm{F}}\left(C_{\mathrm{F}} - \tfrac{1}{2}N\right) \coth^2\gamma \int_0^{\gamma} \mathrm{d}\psi\, \psi(\gamma - \psi) \coth\psi \ln\frac{\mu^2}{\Lambda^2}. \tag{38}$$

(b) Self-energy inserted graph. The calculation of the diagram 4b can be most conveniently performed in the momentum representation

$$\mathcal{M}_{\mathrm{b}} = -(ig)^2 C_{\mathrm{F}} p^{\mu} q^{\nu} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\Pi_{\mu\nu}(k)}{(kp + i0)(kq + i0)} \left(-\frac{i}{k^2}\right)^2, \tag{39}$$

where

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} k^2 - k_{\mu} k_{\nu}\right) \frac{g^2 N}{2(2\pi)^n} \frac{i\pi^{n/2}}{(-k^2)^{2-n/2}} \frac{\Gamma\left(2 - \frac{1}{2}n\right)\Gamma^2\left(\frac{1}{2}n - 1\right)}{\Gamma(n-2)} \frac{3n-2}{n-1}$$

is the regularized gluon polarization operator possessing a UV pole removed by the renormalization procedure eq. (4) in the MS scheme with

$$Z_3 = 1 + \tfrac{5}{3}N \frac{g^2}{8\pi^2} \frac{1}{4-n}.$$

After the application of the R-operation to eq. (39) it is necessary to redefine the resulting IR divergent expression according to eq. (28). This gives

$$\mathcal{M}_{\mathrm{b,R}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda) = K_{\gamma}^{\mathrm{MS}} R_{\mathrm{MS}} \mathcal{M}_{\mathrm{b}} = -\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 \tfrac{1}{8} C_{\mathrm{F}} N \left[\tfrac{5}{6}\ln^2\frac{\mu^2}{\Lambda^2} + \tfrac{31}{9}\ln\frac{\mu^2}{\Lambda^2}\right] \gamma \coth\gamma. \tag{40}$$

(c) QED-type quark-gluon vertex correction graph. The regularized contribution of fig. 4c

$$\mathscr{M}_{\mathrm{c}}=(ig)^4(pq)C_{\mathrm{F}}\left(C_{\mathrm{F}}-\tfrac{1}{2}N\right)\int_0^{\infty}\mathrm{d}s_3\int_{s_3}^{\infty}\mathrm{d}s_2\int_{s_2}^{\infty}\mathrm{d}s_1\int_0^{\infty}\mathrm{d}s_4\,\frac{\Gamma^2\left(\frac{1}{2}n-1\right)}{16\pi^n}$$

$$\times\left[\left((ps_2+qs_4)^2-i0\right)\left(p^2(s_1-s_3)^2-i0\right)\right]^{1-n/2}$$

after integration over s_1, s_3 contains a UV pole corresponding to the fermion-gluon vertex correction

$$\mathscr{M}_{\mathrm{c}}=\frac{g^2}{16\pi^n}(pq)(p^2)^{(4-n)/2}C_{\mathrm{F}}\left(C_{\mathrm{F}}-\tfrac{1}{2}N\right)\Gamma^2\left(\tfrac{1}{2}n-1\right)$$

$$\times\mu^{2(4-n)}\int_0^{\infty}\mathrm{d}s_2\int_0^{\infty}\mathrm{d}s_4\,\frac{\left[(ps_2+qs_4)^2-i0\right]^{1-n/2}}{(4-n)(n-3)}s_2^{4-n} \tag{41}$$

and removed by the renormalization of eq. (4) with

$$Z_1^{\mathrm{F}}=1+\left(C_{\mathrm{F}}-\tfrac{1}{2}N\right)\frac{g^2}{4\pi^2}\frac{1}{4-n}\,.$$

Redefining the integral (41) we obtain the result of the action of the R-operation on M_{c}:

$$R\mathscr{M}_{\mathrm{c}}=\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2C_{\mathrm{F}}\left(C_{\mathrm{F}}-\tfrac{1}{2}N\right)\left[-\frac{\gamma\coth\gamma}{(4-n)^2}\left(\frac{\mu}{\Lambda}\right)^{4-n}+\frac{1}{2(4-n)^2}\left(\frac{\mu}{\Lambda}\right)^{2(4-n)}\right.$$

$$\left.\times\frac{1}{n-3}\int_0^1\frac{\mathrm{d}x(pq)\,(p^2x^2)^{2-n/2}}{\left[(px+q\bar{x})^2-i0\right]^{3-n/2}}\right].$$

Now, subtracting the cusp singularities and performing the change of angular variables (30) we find the renormalized expression for the graph 4c:

$$\mathscr{M}_{\mathrm{c,R}}(\gamma,g_{\mathrm{R}},\mu/\Lambda)=\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2C_{\mathrm{F}}\left(C_{\mathrm{F}}-\tfrac{1}{2}N\right)\left[\tfrac{1}{8}\gamma\coth\gamma\ln^2\frac{\mu^2}{\Lambda^2}+\tfrac{1}{2}\gamma\coth\gamma\ln\frac{\mu^2}{\Lambda^2}\right.$$

$$\left.-\tfrac{1}{2}\coth\gamma\int_0^{\gamma}\mathrm{d}\psi\,\psi\coth\psi\ln\frac{\mu^2}{\Lambda^2}\right]. \tag{42}$$

(d) Three-gluon vertex correction graph. The most complicated is the calculation of the diagram 4d containing the three-gluon vertex. We represent the corresponding factor $V_{\mu\nu\rho}(k, l, -k-l)$ in the form

$$V_{\mu\nu\rho}(k, l, -k-l) = \bar{V}_{\mu\nu\rho}(k, l) + D_{\mu\nu\rho}(k, l)$$

(cf. refs. [12, 13]), where

$$\bar{V}_{\mu\nu\rho}(k, l) = (2l+k)_\mu g_{\nu\rho} + 2k_\rho g_{\mu\nu} - 2k_\nu g_{\mu\rho}\,, \quad D_{\mu\nu\rho}(k, l) = -l_\nu g_{\mu\rho} - (l+k)_\rho g_{\mu\nu}$$

satisfies the simplest Ward identity

$$k^\mu \bar{V}_{\mu\nu\rho}(k, l) = \left[(k+l)^2 - l^2\right] g_{\nu\rho}\,. \tag{43}$$

The "D"-vertex produces the following contribution

$$\mathcal{M}_{\mathrm{d},D} - \tfrac{1}{2} g^4 C_\mathrm{F} N \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{d}^n l}{(2\pi)^n} \frac{1}{k^2 l^2 (k+l)^2} \left[\frac{2(pq)}{(qk)(pk)} + \frac{(pq)}{(qk)(pl)}\right]. \tag{44}$$

The first integral has the structure of that corresponding to fig. 4b and is easily calculated. The result of applying the R-operation to eq. (44) (after necessary redefinition) can be written in the form

$$\begin{aligned} R\mathcal{M}_{\mathrm{d},D} = \left(\frac{\alpha_\mathrm{s}}{\pi}\right)^2 C_\mathrm{F} N \Bigg[&- \frac{\gamma \coth\gamma}{4(4-n)^2(n-3)} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)} + \frac{\gamma\coth\gamma}{2(4-n)^2}\left(\frac{\mu}{\Lambda}\right)^{4-n} \\ &+ \frac{1}{16(4-n)} \left(\frac{\mu}{\Lambda}\right)^{2(4-n)} \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}x \\ &\times \frac{1}{(px + q\bar{x}y)^2} \ln \frac{(px+q\bar{x}y)^2 + q^2\bar{x}^2 y\bar{y}}{q^2 \bar{x}^2 y \bar{y}} \Bigg]. \end{aligned}$$

After the change of angular variables, integration by parts and subtraction of cusp divergences we get

$$\begin{aligned} K_\gamma R\mathcal{M}_{\mathrm{d},D} &= \mathcal{M}_{\mathrm{d},D,\mathrm{R}}(\gamma, g_\mathrm{R}, \mu/\Lambda) \\ &= \left(\frac{\alpha_\mathrm{s}}{\pi}\right)^2 C_\mathrm{F} N \Bigg[-\tfrac{1}{16}\gamma\coth\gamma \ln^2\frac{\mu^2}{\Lambda^2} - \tfrac{1}{4}\gamma\coth\gamma\ln\frac{\mu^2}{\Lambda^2} \\ &\qquad + \tfrac{1}{96}\pi^2\gamma\coth\gamma\ln\frac{\mu^2}{\Lambda^2} + \tfrac{1}{8}\sinh 2\gamma \\ &\qquad \times \int_0^\gamma \mathrm{d}\psi \frac{\psi\coth\psi}{\sinh^2\gamma - \sinh^2\psi} \ln\frac{\sinh\gamma}{\sinh\psi} \ln\frac{\mu^2}{\Lambda^2}\Bigg]. \end{aligned} \tag{45}$$

For the contribution of the $\bar{V}$-term we first rewrite the vertex factor in the form

$$\bar{V}_{\mu\nu\rho}(k,l)p^\nu p^\rho = p^2(2l+k)^\nu\left[\left(g_{\mu\nu} - \frac{p_\mu k_\nu}{(pk)}\right) + \frac{p_\mu k_\nu}{(pk)}\right].$$

Now, using eq. (43), it is easy to see that the second term in this equation (longitudinal with respect to k_ν) is cancelled by the corresponding contribution from the vertex function of diagram 4c. As a result, the sum of the $\bar{V}$-contribution from the diagram 4d and the total contribution of the diagram 4c after calculating the $\mathrm{d}^n l$ integral does not contain the UV-poles related to the fermion-gluon vertex

$$\begin{aligned}\mathcal{M}_{\mathrm{c}} + \mathcal{M}_{\mathrm{d},\bar{V}} = {} & iC_{\mathrm{F}}N\frac{g^4}{16\pi^2}\mu^{4-n}\Gamma(3-\tfrac{1}{2}n)\\ & \times\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{p^2}{k^2(kp+i0)(kq+i0)}q^\mu k^\nu\left(g_{\mu\nu} - \frac{p_\mu k_\nu}{pk}\right)\\ & \times\int_0^1\mathrm{d}\lambda\,\lambda^{4-n}\int_0^1\mathrm{d}x\,(1-2x)\left[(p\bar{\lambda} - k\lambda x)^2 - k^2\lambda^2 x - i0\right]^{n/2-3}, \end{aligned} \tag{46}$$

where $\bar{\lambda} = 1-\lambda$. Performing the IR redefinition and subtracting the cusp singularity one can rewrite eq. (46) in the form

$$\mathcal{M}_{\mathrm{c,R}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda) + M_{\mathrm{d},\bar{V},\mathrm{R}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda) = -\tfrac{1}{4}\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 C_{\mathrm{F}}N\left(\tfrac{1}{24}\pi^2 + I\right)\ln\frac{\mu^2}{\Lambda^2}, \tag{47}$$

where the π^2 term corresponds to the $g_{\mu\nu}$ part of the projector present in eq. (46), while its second term yields

$$\begin{aligned} I &= \coth\gamma\int_0^1\mathrm{d}x\,x(1-2x)\int_0^1\mathrm{d}y\,\bar{y}\int_0^\gamma\mathrm{d}\psi\,\frac{\sinh^2\psi}{\sinh^2\gamma}\left(x+\bar{x}y^2\frac{\sinh^2\psi}{\sinh^2\gamma}\right)^{-2}\\ &= \coth\gamma\int_0^\gamma\mathrm{d}\psi\,\frac{\sinh^2\psi}{\sinh^2\gamma - \sinh^2\psi}\ln\frac{\sinh\psi}{\sinh\gamma}. \end{aligned}$$

Thus, subtracting eqs. (47) and (42) we find the final result for the renormalized amplitude of the $\bar{V}$-part of diagram 4d

$$\begin{aligned}&\mathcal{M}_{\mathrm{d},\bar{V},\mathrm{R}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda)\\ &\quad = -\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 C_{\mathrm{F}}N\left(-\tfrac{1}{16}\gamma\coth\gamma\ln^2\frac{\mu^2}{\Lambda^2} - \tfrac{1}{4}\gamma\coth\gamma\ln\frac{\mu^2}{\Lambda^2} + \tfrac{1}{96}\pi^2\ln\frac{\mu^2}{\Lambda^2}\right.\\ &\qquad\left. + \tfrac{1}{8}\sinh 2\gamma\int_0^\gamma\mathrm{d}\psi\,\frac{1}{\sinh^2\gamma - \sinh^2\psi}\ln\frac{\sinh\gamma}{\sinh\psi}\ln\frac{\mu^2}{\Lambda^2}\right). \end{aligned} \tag{48}$$

It is worth noting here that the cancellation of double logarithms of μ^2 in the sum of eqs. (48), (45) is a consequence of the fact that the vertex correction of diagram 4d gives a zero contribution to the renormalization constant Z_1^{F} of eq. (9).

Substituting the total contribution of the 2PI vertex diagrams 4a–d calculated above (with a proper account of combinatorial factors) we obtain the following result for the two-loop contribution to the exponential factor of eq. (19):

$$W^{2\mathrm{PI}}_{\mathrm{R,two\text{-}loop}}(\gamma, g_{\mathrm{R}}, \mu/\Lambda) = -\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 \tfrac{11}{48} C_{\mathrm{F}} N(\gamma\coth\gamma - 1)\ln^2\frac{\mu^2}{\Lambda^2}$$
$$-\tfrac{1}{2}\Gamma^{\mathrm{two\text{-}loop}}_{\mathrm{cusp}}(\gamma, g_{\mathrm{R}})\ln\frac{\mu^2}{\Lambda^2}. \qquad (49)$$

For the gauge-invariant cusp anomalous dimension this gives

$$\Gamma^{\mathrm{two\text{-}loop}}_{\mathrm{cusp}}(\gamma, g_{\mathrm{R}}) = \left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 C_{\mathrm{F}} N$$
$$\times\left[\tfrac{1}{2} + \left(\tfrac{67}{36} - \tfrac{1}{24}\pi^2\right)(\gamma\coth\gamma - 1) - \coth\gamma\int_0^\gamma \mathrm{d}\psi\, \psi\coth\psi\right.$$
$$+\coth^2\gamma\int_0^\gamma \mathrm{d}\psi\, \psi(\gamma-\psi)\coth\psi - \tfrac{1}{2}\sinh 2\gamma$$
$$\left.\times\int_0^\gamma \mathrm{d}\psi \frac{\psi\coth\psi - 1}{\sinh^2\gamma - \sinh^2\psi}\ln\frac{\sinh\gamma}{\sinh\psi}\right]. \qquad (50)$$

Continuation of eqs. (49), (50) into the euclidean space can be performed by changing the angles as prescribed by eq. (32).

4. Asymptotic behaviour of the cusp anomalous dimension

4.1. ONE-LOOP AND TWO-LOOP TERMS

In this section we consider the behaviour of the cusp anomalous dimension in two limiting cases for the minkowskian angle γ:

(a) $\gamma \to \infty$: in this case

$$\gamma = \ln\frac{Q^2}{m^2}, \qquad Q^2 \gg m^2, \qquad (51)$$

where $Q^2 = -(p-q)^2$, $p^2 = q^2 = m^2$ and

(b) $\gamma \to 0$: in this case

$$\gamma = \left(\frac{Q^2}{m^2}\right)^{1/2}, \qquad Q^2 \ll m^2. \qquad (52)$$

In the limit (52) one can represent $\Gamma_{\mathrm{cusp}}(\gamma, g_{\mathrm{R}})$ as

$$\Gamma_{\mathrm{cusp}}(\gamma, g_{\mathrm{R}}) \underset{\gamma\to 0}{=} \frac{\alpha_s}{\pi} C_{\mathrm{F}} \tfrac{1}{3}\gamma^2 + 2\left(\frac{\alpha_s}{\pi}\right)^2 C_{\mathrm{F}} N \left[\tfrac{1}{16}\gamma^2\left(2 - \tfrac{4}{9}\pi^2\right) + \tfrac{67}{72}\tfrac{1}{3}\gamma^2\right] \tag{53}$$

and hence Γ_{cusp} vanishes as Q^2/m^2 when $\gamma \to 0$.

In the opposite limit (51) the two-loop term of the cusp anomalous dimension

$$\Gamma_{\mathrm{cusp}}(\gamma, g_{\mathrm{R}}) \underset{\gamma\to \infty}{=} \frac{\alpha_s}{\pi} C_{\mathrm{F}} \ln\frac{Q^2}{m^2} + 2\left(\frac{\alpha_s}{\pi}\right)^2 C_{\mathrm{F}} N \left[-\tfrac{1}{24}\pi^2 \ln\frac{Q^2}{m^2} + \tfrac{67}{72}\ln\frac{Q^2}{m^2}\right] \tag{54}$$

does not contain $\ln(Q^2/m^2)$ in a power higher than 1, because in Feynman gauge the $\ln^3(Q^2/m^2)$ terms due to the D-parts of the diagrams of fig. 4d (see eq. (45)) are cancelled by those due to the diagram 4a (eq. (38)), and the double logarithms due to the $\bar{V}$-parts of the diagram 4d (eq. (48)) are cancelled by those due to the diagram 4c (eq. (42)). The factor in the contribution proportional to the one-loop cusp anomalous dimension and $\frac{67}{72}$ presented in eqs. (50), (53), (54) is an artifact of the scheme employed because it appears after one applies the R_{MS} operation to remove the subdivergences from diagrams 4b, c.

It should be noted that the path-ordered exponential corresponding to the path shown in fig. 2 absorbs all the IR singularities of the amplitude of quark scattering by an external colour-singlet potential, the initial quark momentum being p_μ and the momentum transfer Q^2 (see ref. [14]). In the limits $\gamma \to 0, \infty$ this amplitude was calculated in ref. [13]. Our results (eqs. (53), (54)) are in complete agreement with those obtained there.

4.2. STRUCTURE OF Γ_{cusp} FOR $\gamma \to \infty$ IN HIGHER ORDERS OF PT

Let us prove now that the cusp anomalous dimension (eq. (21)) in the limit $\gamma \gg 1$ is linear in $\ln(Q^2/m^2)$ for arbitrary order of PT. To this end we incorporate the Feynman rules (eq. (25)) in momentum representation and note that the UV pole related to the cusp singularity of the 2PI contour averages (eq. (19)–(21)) is due to the integration over the UV region of the fermion and gluon momenta while its dependence on γ is determined by integration over small angles between the tangent vectors to the curve on which the fermions are "living" and momenta of the emitted gluons. A general structure of these angular integrals singular in the $Q^2 \gg m^2$ limit can be studied by using standard methods of the factorization technique [15, 16]. Note that the lagrangian of the one-dimensional fermions (eq. (3)) has the following properties: the one-dimensional fermions interacting with gluons cannot change their "helicities", and, hence, the emission of the collinear gluons with physical polarizations by the fermions is suppressed. There exist the whole class of the so-called contour gauges [14] defined by the gauge condition

$\mathrm{P}\exp(ig\int_C \mathrm{d}x^\mu \hat{A}_\mu(x)) = \mathbf{1}$ for which the gauge potential $\hat{A}_\mu$ is a linear functional of the field strength $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ig[\hat{A}_\mu, \hat{A}_\nu]$

$$\hat{A}_\mu(x) = \int_C \mathrm{d}z^\nu \frac{\partial z^\rho}{\partial x^\mu} \hat{F}_{\nu\rho}(z) \tag{55}$$

and, hence, the gauge field has only physical degrees of freedom. Some well-known physical gauges, namely, the axial gauge, Hamilton gauge and Fock-Schwinger [17] (or fixed point [18]) gauge are important particular examples of the contour gauge. Using the dimensional analysis of ref. [15] it is easy to find that in physical gauges the logarithms $(\ln(Q^2/m^2))^N$ arise from integrations inside the self-energy diagrams Σ of fig. 2*. The relevant power N is equal to the number of the one-particle irreducible (with respect to one-dimensional fermions) self-energy subgraphs $\Sigma^{1\mathrm{PI}}$ in Σ. For the 2PI subgraphs contributing to $W_{\mathrm{R}}^{2\mathrm{PI}}$ this number is 1.

Thus, the UV cusp singularity coefficient for the 2PI diagrams of eq. (21) in the limit $Q^2 \gg m^2$ is a single logarithm $\ln(Q^2/m^2)$ and, hence,

$$\Gamma_{\mathrm{cusp}}(\gamma, g_{\mathrm{R}}) = \sum_{n=1}^{\infty} \sum_{\mathrm{W}\in\mathrm{W}(n)} \alpha_s^n C_n(\mathrm{W}) a_n(\mathrm{W}) \ln\frac{Q^2}{m^2} + \mathrm{O}\left(\ln^0 \frac{Q^2}{m^2}\right). \tag{56}$$

The fact that in Feynman gauge the propagation of the longitudinally polarized gluons is also allowed leads to a more singular structure of angular integrals. However, higher powers appearing in separate diagrams cancel with each other in the gauge-invariant sum (eq. (38)).

It is worth emphasizing here that the cusp anomalous dimension (eqs. (35), (50)) is regular in γ everywhere except the point $\gamma = i\pi$ (or $\gamma = \pi$ in the euclidean space-time) corresponding to the "collapse" of the contour shown in fig. 2. The contour average (renormalized as well as nonrenormalized) equals 1 in this case. This means that in the regularization scheme used in ref. [4] the cusp singularity for $\gamma = i\pi$ possesses a linear divergence: $\exp(kL_{\mathrm{app}})$ (see the introduction), where L_{app} is the length of the "collapsed" part of the contour, i.e. in this limiting case the very definition of the cusp anomalous dimension is meaningless.

5. Glauber regime of the anomalous dimension

In the process of our calculations in sect. 4 it has been implied that all the intervals between any two points on the contour shown in fig. 2 have the same sign (this is equivalent to the statement that the three kinematic invariants $p^2, q^2, (pq)$ have the same sign). This allows one to continue analytically the results into the

* This situation is quite analogous to that in perturbative QCD when one calculates the asymptotic behaviour of the hard scattering amplitudes (see, e.g., ref. [15, 16]).

euclidean space using eq. (32). Note also that if the two points of the contour shown in fig. 2 are separated by a time-like interval, then the path-ordering along the contour coincides with T- (or anti-T-) ordering of the gauge potentials in eq. (1). Consider now the class of contours for which these properties are not valid, e.g., fig. 2 but with the change

$$q_\mu \to -q_\mu . \tag{57}$$

In euclidean space such a transformation leads only to the evident redefinition of the cusp angle

$$\gamma_{\mathrm{E}} \to \pi - \gamma_{\mathrm{E}} \tag{58}$$

in the final results. In the Minkowski space-time eq. (30) is not fulfilled and moreover, the residues of gluon propagator poles produce nonzero contributions. In particular, the calculation of the diagram 3 contribution with account of eq. (57) gives

$$W^{2\mathrm{PI}}_{\mathrm{R,one\text{-}loop}}(\mathrm{C}_\gamma;\, g_{\mathrm{R}},\, \mu) = -\frac{\alpha_{\mathrm{s}}}{2\pi} C_{\mathrm{F}}\left[(\gamma - i\pi)\coth\gamma - 1\right]\ln\frac{\mu^2}{\Lambda^2} \tag{59a}$$

and, correspondingly,

$$\Gamma^{\mathrm{one\text{-}loop}}_{\mathrm{cusp}}(\gamma,\, g_{\mathrm{R}}) = \frac{\alpha_{\mathrm{s}}}{\pi} C_{\mathrm{F}}\left[(\gamma - i\pi)\coth\gamma - 1\right]. \tag{59b}$$

Note, that formally these relations can be obtained from eqs. (34), (36) by using eqs. (58) and (32). Let us now find the region of the momentum space of the gluons responsible for the imaginary term in eq. (59). To this end we calculate it in two ways.

Consider the frame where $p_{\mathrm{T}} = q_{\mathrm{T}} = 0$, $p^+ > p^-$, $q^- > q^+$ (with $k^{\pm} = \sqrt{\tfrac{1}{2}}\,(k^0 \pm k^3)$ being the light one variables). Then the integral contributing to eq. (59) is

$$\mathcal{M} = -ig^2 C_{\mathrm{F}} \int \frac{\mathrm{d}k^+\,\mathrm{d}k^-\,\mathrm{d}k_{\mathrm{T}}^{n-2}}{(2\pi)^n}$$
$$\times \frac{(pq)}{\left(2k^+k^- - k_{\mathrm{T}}^2 - \Lambda^2\right)\left(k^+p^- + k^-p^+ - i0\right)\left(k^+q^- + k^-q^+ + i0\right)} . \tag{60}$$

The position of the poles in the complex k^- plane is shown in fig. 5 with the numbers representing the denominator factors of eq. (60). Taking the residue of the pole (3) for $k^+ \leqslant 0$ and of (2) for $k^+ > 0$ and using the identity $1/(x \pm i0) = P(1/x)$

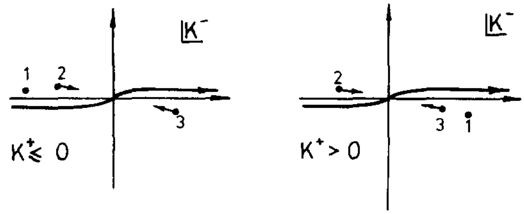


Fig. 5. Position of poles related to the gluon propagators of eq. (60) in the complex K^- plane in the Glauber regime.

$\mp i\pi\delta(x)$ to calculate the integral over k^+ it is easy to find that

$$\mathscr{M} = \tfrac{1}{2}iC_F\alpha_s\frac{(pq)}{p^+q^- - p^-q^+}\int_0^\infty \frac{d^{n-2}k_T}{k_T^2+\Lambda^2} + \cdots = \tfrac{1}{2}iC_F\alpha_s \coth\gamma\int_0^\infty \frac{d^{n-2}k_T}{k_T^2+\Lambda^2} + \cdots, \tag{61}$$

where the dots stand for the real part of the integral. Thus, the imaginary part of eq. (59) is formed by the region where the gluon momentum is most transverse

$$k_T \gg k^+ \sim k^- \to 0,$$

i.e. by the Glauber regime for the gluons [19] in which their emission does not change the virtuality of one-dimensional fermions. As $k_\mu \to 0$, the poles of eq. (60) in this regime move as indicated by arrows in fig. 5 and the integration contour is eventually pinched in the origin. As a result, there appears the singular imaginary part of eq. (61).

It is also instructive to calculate eq. (60) in the α-representation (see, e.g., ref. [3]) in which

$$\mathscr{M} = i\alpha_s C_F\int\frac{\Pi_\sigma d\alpha_\sigma}{\alpha_1^{n/2}}\exp\left[-\frac{i}{\alpha_1}(p\alpha_2 - q\alpha_3)^2 - \varepsilon(\alpha_1+\alpha_2+\alpha_3)\right] \tag{62}$$

and the main contribution to the integral originates from the region in the α-parameter space where the exponential form

$$A(\alpha, p, q) = \frac{p^2}{\alpha_1}\left(\alpha_2 - \frac{Q^2}{p^2}\alpha_3\right)\left(\alpha_2 - \frac{q^2}{Q^2}\alpha_3\right)$$

vanishes [16]. The requirement $A = 0$ defines planes

$$S_1:\ \alpha_2 = \frac{Q^2}{p^2}\alpha_3, \qquad S_2:\ \alpha_2 = \frac{q^2}{Q^2}\alpha_3$$

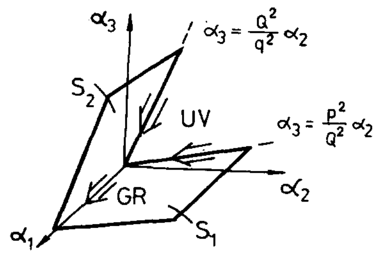


Fig. 6. Pinch hyperplanes S_1, S_2 in the α-parameter space containing the ultraviolet (UV) and the Glauber (GR) regimes of the gluon momenta.

illustrated in fig. 6. According to ref. [20] in the regions S_1 and S_2 the "pinch regime" is realized producing a regular contribution of each separate hyperplane of eq. (62). However, in our case there exists also the line $\alpha_2 = \alpha_3 = 0$ common both for S_1 and S_2 where the "intensity" of the pinch singularity is higher and the corresponding integration gives eqs. (61), (59) possessing the divergences both in UV $(\alpha_1, \alpha_2, \alpha_3 \to 0$ on $S_1, S_2)$ and in IR-regions $(\alpha_1 \to \infty$ on $S_1, S_2)$. Thus, the Glauber regime of the gluon momenta corresponds to a pinch regime in the α-parameter space.

Note now that in the allowed region $\alpha_i \geqslant 0$ the pinch hyperplanes S_1 and S_2 of fig. 6 (as well as the imaginary contributions to eq. (59)) disappear if $p^2, q^2 < 0$, $(pq) > 0$ if any two points on the contour shown in fig. 2 are in the space-like region and the T-ordering in eq. (1) can be omitted. This means, in particular, that eq. (59) as a function of the vectors p, q has a singularity on the light cone.

6. Conclusions

In the present paper we studied the renormalization properties of the cusp singularities of the contour averages. Incorporating some properties of the $K_\alpha R$-procedure subtracting the divergences due to the cusp singularities we established the general form of the PT series for the cusp anomalous dimension in the limit of large minkowskian cusp angles. The two-loop contribution to the cusp anomalous dimension was explicitly calculated and its connection to the nonleading IR behaviour of the quark form factor was demonstrated. We observed also that there exist two sources of nonanalyticity of the results obtained with respect to the cusp angle γ: first, for $\gamma_M = i\pi$ $(\gamma_E = \pi)$ there appears a linear divergence and second, the Glauber gluons in the Minkowski space give a nonzero contribution to the cusp singularity.

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Note added in proof

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