Finite-dimensional inequivalent irreducible projective representations of the Lie algebra of the rotation group[1]

$$[J_i, J_j] = i\,\epsilon_{ijk} J_k \qquad\qquad i,j,k = \{1,2,3\} = \{x,y,z\} \tag{0.1}$$

The squared modulus is given by

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{0.2}$$

and from eq. (0.1) it can be easily shown that

$$[J^2, J_i] = 0 \tag{0.3}$$

A standard representations $\vec{J}^{(s)}$ and a standard basis $|s,m\rangle$ are obtained by diagonalizing simultaneously J^2 and J_z

$$\left(J^{(s)}\right)^2 |s,m\rangle = s(s+1)\,|s,m\rangle \tag{0.4}$$

$$J_z^{(s)} |s,m\rangle = m\,|s,m\rangle \tag{0.5}$$

with

$$s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}, \ldots \tag{0.6}$$

$$m = -s,\ -s+1,\ -s+2, \ldots, s-2,\ s-1,\ s \tag{0.7}$$

By defining

$$J^{\pm} = J_x \pm i J_y \tag{0.8}$$

it can be easily shown that

$$\langle s', m'|J^{\pm}|s,m\rangle = \sqrt{(s\mp m)(s\pm m+1)}\,\delta_{s,s'}\,\delta_{m',m\pm 1} \tag{0.9}$$

and by inverting eq. (0.8) we can compute

$$\langle s', m'|J_x|s,m\rangle \qquad \text{and} \qquad \langle s', m'|J_y|s,m\rangle \tag{0.10}$$

The representations in this basis are given in the following.

1 $s = 0$

$$\vec{J}^{(0)} = \vec{0} \tag{1.1}$$

[1] http://virgilio.mib.infn.it/~oleari

2 $s = 1/2$

$$J_x^{\left(\frac{1}{2}\right)} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \equiv \frac{\sigma_x}{2} \qquad J_y^{\left(\frac{1}{2}\right)} = \frac{i}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \equiv \frac{\sigma_y}{2} \qquad J_z^{\left(\frac{1}{2}\right)} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv \frac{\sigma_z}{2} \tag{2.1}$$

$$\left(J^{\left(\frac{1}{2}\right)}\right)^2 = \frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.2}$$

3 $s = 1$

$$J_x^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad J_y^{(1)} = \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad J_z^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.1}$$

$$\left(J^{(1)}\right)^2 = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

4 $s = \frac{3}{2}$

$$J_x^{\left(\frac{3}{2}\right)} = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad J_y^{\left(\frac{3}{2}\right)} = i\begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \tag{4.1}$$

$$J_z^{\left(\frac{3}{2}\right)} = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \qquad \left(J^{\left(\frac{3}{2}\right)}\right)^2 = \frac{15}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

5 $s = 2$

$$J_x^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{\sqrt{6}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 & \frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad J_y^{(2)} = i \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\frac{\sqrt{6}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 & -\frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.1}$$

$$J_z^{(2)} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad \left(J^{(2)}\right)^2 = 6 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.2}$$