ELECTROMAGNETIC PHENOMENA IN A SYSTEM MOVING WITH ANY VELOCITY LESS THAN THAT OF LIGHT

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§ 1. THE problem of determining the influence exerted on electric and optical phenomena by a translation, such as all systems have in virtue of the Earth's annual motion, admits of a comparatively simple solution, so long as only those terms need be taken into account, which are proportional to the first power of the ratio between the velocity of translation v and the velocity of light c. Cases in which quantities of the second order, i.e. of the order v^2/c^2, may be perceptible, present more difficulties. The first example of this kind is Michelson's well-known interference-experiment, the negative result of which has led Fitzgerald and myself to the conclusion that the dimensions of solid bodies are slightly altered by their motion through the ether.

Some new experiments, in which a second order effect was sought for, have recently been published. Rayleigh * and Brace † have examined the question whether the Earth's motion may cause a body to become doubly refracting. At first sight this might be expected, if the just mentioned change of dimensions is admitted. Both physicists, however, have obtained a negative result.

In the second place Trouton and Noble ‡ have endeavoured to detect a turning couple acting on a charged condenser, the plates of which make a certain angle with the direction of translation. The theory of electrons, unless it be modified by some new hypothesis, would undoubtedly require the

* Rayleigh, Phil. Mag. (6), 4, 1902, p. 678.
† Brace, Phil. Mag. (6), 7, 1904, p. 317.
‡ Trouton and Noble, Phil. Trans. Roy. Soc. Lond., A 202, 1903, p. 165.

existence of such a couple. In order to see this, it will suffice to consider a condenser with ether as dielectric. It may be shown that in every electrostatic system, moving with a velocity **v**,* there is a certain amount of "electromagnetic momentum." If we represent this, in direction and magnitude, by a vector **G**, the couple in question will be determined by the vector product †

$$[\mathbf{G}\,.\,\mathbf{v}] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if the axis of z is chosen perpendicular to the condenser plates, the velocity **v** having any direction we like; and if U is the energy of the condenser, calculated in the ordinary way, the components of **G** are given ‡ by the following formulæ, which are exact up to the first order,

$$G_x = \frac{2U}{c^2} v_x, \quad G_y = \frac{2U}{c^2} v_y, \quad G_z = 0.$$

Substituting these values in (1), we get for the components of the couple, up to terms of the second order,

$$\frac{2U}{c^2} v_y v_z, \quad -\frac{2U}{c^2} v_x v_z, \quad 0.$$

These expressions show that the axis of the couple lies in the plane of the plates, perpendicular to the translation. If a is the angle between the velocity and the normal to the plates, the moment of the couple will be $U(v/c)^2 \sin 2a$; it tends to turn the condenser into such a position that the plates are parallel to the Earth's motion.

In the apparatus of Trouton and Noble the condenser was fixed to the beam of a torsion-balance, sufficiently delicate to be deflected by a couple of the above order of magnitude. No effect could however be observed.

§ 2. The experiments of which I have spoken are not the only reason for which a new examination of the problems connected with the motion of the Earth is desirable. Poincaré * has objected to the existing theory of electric and optical phenomena in moving bodies that, in order to explain Michelson's negative result, the introduction of a new hypothesis has been required, and that the same necessity may occur each time new facts will be brought to light. Surely this course of inventing special hypotheses for each new experimental result is somewhat artificial. It would be more satisfactory if it were possible to show by means of certain fundamental assumptions and without neglecting terms of one order of magnitude or another, that many electromagnetic actions are entirely independent of the motion of the system. Some years ago, I already sought to frame a theory of this kind.† I believe it is now possible to treat the subject with a better result. The only restriction as regards the velocity will be that it be less than that of light.

§ 3. I shall start from the fundamental equations of the theory of electrons.‡ Let **D** be the dielectric displacement in the ether, **H** the magnetic force, ρ the volume-density of the charge of an electron, **v** the velocity of a point of such a particle, and **F** the ponderomotive force, i.e. the force, reckoned per unit charge, which is exerted by the ether on a volume-element of an electron. Then, if we use a fixed system of co-ordinates,

$$\left.\begin{aligned} \operatorname{div} \mathbf{D} &= \rho, \ \operatorname{div} \mathbf{H} = 0, \\ \operatorname{curl} \mathbf{H} &= \frac{1}{c}\left(\frac{\partial \mathbf{D}}{\partial t} + \rho \mathbf{v}\right), \\ \operatorname{curl} \mathbf{D} &= -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \\ \mathbf{F} &= \mathbf{D} + \frac{1}{c}[\mathbf{v}\,.\,\mathbf{H}]. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

I shall now suppose that the system as a whole moves in the direction of x with a constant velocity v, and I shall denote by **u** any velocity which a point of an electron may have in addition to this, so that

$$v_x = v + u_x, \quad v_y = u_y, \quad v_z = u_z.$$

* A vector will be denoted by a Clarendon letter, its magnitude by the corresponding Latin letter.

† See my article: "Weiterbildung der Maxwell'schen Theorie. Electronentheorie," Mathem. Encyclopädie, V, 14, § 21, a. (This article will be quoted as "M.E.")

‡ "M.E.," § 56, c.

* Poincaré, Rapports du Congrès de physique de 1900, Paris, 1, pp. 22, 23.

† Lorentz, Zittingsverslag Akad. v. Wet., 7, 1899, p. 507; Amsterdam Proc., 1898-99, p. 427.

‡ "M.E.," § 2.

If the equations (2) are at the same time referred to axes moving with the system, they become

$$div\ \mathbf{D} = \rho, \quad div\ \mathbf{H} = 0,$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{1}{c}\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)D_x + \frac{1}{c}\rho(v + u_x),$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{1}{c}\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)D_y + \frac{1}{c}\rho u_y,$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{1}{c}\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)D_z + \frac{1}{c}\rho u_z,$$

$$\frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)H_x,$$

$$\frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x} = -\frac{1}{c}\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)H_y,$$

$$\frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y} = -\frac{1}{c}\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)H_z,$$

$$F_x = D_x + \frac{1}{c}(u_y H_z - u_z H_y),$$

$$F_y = D_y - \frac{1}{c}vH_z + \frac{1}{c}(u_z H_x - u_x H_z),$$

$$F_z = D_z + \frac{1}{c}vH_y + \frac{1}{c}(u_x H_y - u_y H_x).$$

§ 4. We shall further transform these formulæ by a change of variables. Putting

$$\frac{c^2}{c^2 - v^2} = \beta^2, \quad . \quad . \quad . \quad . \quad (3)$$

and understanding by l another numerical quantity, to be determined further on, I take as new independent variables

$$x' = \beta l x, \quad y' = l y, \quad z' = l z, \quad . \quad . \quad . \quad (4)$$

$$t' = \frac{l}{\beta}t - \beta l \frac{v}{c^2}x, \quad . \quad . \quad . \quad . \quad (5)$$

and I define two new vectors $\mathbf{D}'$ and $\mathbf{H}'$ by the formulæ

$$D'_x = \frac{1}{l^2}D_x, \quad D'_y = \frac{\beta}{l^2}\left(D_y - \frac{v}{c}H_z\right), \quad D'_z = \frac{\beta}{l^2}\left(D_z + \frac{v}{c}H_y\right),$$

$$H'_x = \frac{1}{l^2}H_x, \quad H'_y = \frac{\beta}{l^2}\left(H_y + \frac{v}{c}D\right), \quad H'_z = \frac{\beta}{l^2}\left(H_z - \frac{v}{c}D_y\right),$$

for which, on account of (3), we may also write

$$\left.\begin{array}{l} D_x = l^2 D'_x, \quad D_y = \beta l^2\left(D'_y + \frac{v}{c}H'_z\right), \quad D_z = \beta l^2\left(D'_z - \frac{v}{c}H'_y\right) \\ H_x = l^2 H'_x, \quad H_y = \beta l^2\left(H'_y - \frac{v}{c}D'_z\right), \quad H_z = \beta l^2\left(H'_z + \frac{v}{c}D'_y\right) \end{array}\right\} (6)$$

As to the coefficient l, it is to be considered as a function of v, whose value is 1 for $v = 0$, and which, for small values of v, differs from unity no more than by a quantity of the second order.

The variable t' may be called the "local time"; indeed, for $\beta = 1$, $l = 1$ it becomes identical with what I formerly denoted by this name.

If, finally, we put

$$\frac{1}{\beta l^3}\rho = \rho' \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\beta^2 u_x = u'_x, \quad \beta u_y = u'_y, \quad \beta u_z = u'_z, \quad . \quad . \quad (8)$$

these latter quantities being considered as the components of a new vector $\mathbf{u}'$, the equations take the following form:—

$$\left.\begin{array}{l} \text{div}'\ \mathbf{D}' = \left(1 - \frac{v u'_x}{c^2}\right)\rho', \quad \text{div}'\ \mathbf{H}' = 0, \\ \text{curl}'\ \mathbf{H}' = \frac{1}{c}\left(\frac{\partial \mathbf{D}'}{\partial t'} + \rho'\mathbf{u}'\right), \\ \text{curl}'\ \mathbf{D}' = -\frac{1}{c}\frac{\partial \mathbf{H}'}{\partial t'}, \end{array}\right\} \quad (9)$$

$$\left.\begin{array}{l} F_x = l^2\{D'_x + \frac{1}{c}(u'_y H'_z - u'_z H'_y) + \frac{v}{c^2}(u'_y D'_y + u'_z D'_z)\}, \\ F_y = \frac{l^2}{\beta}\{D'_y + \frac{1}{c}(u'_z H'_x - u'_x H'_z) - \frac{v}{c^2}u'_x D'_y\}, \\ F_z = \frac{l^2}{\beta}\{D'_z + \frac{1}{c}(u'_x H'_y - u'_y H'_x) - \frac{v}{c^2}u'_x D'_z\}. \end{array}\right\} \quad (10)$$

The meaning of the symbols div′ and curl′ in (9) is similar to that of div and curl in (2); only, the differentiations with respect to x, y, z are to be replaced by the corresponding ones with respect to x', y', z'.

§ 5. The equations (9) lead to the conclusion that the vectors $\mathbf{D}'$ and $\mathbf{H}'$ may be represented by means of a scalar potential ϕ' and a vector potential $\mathbf{A}'$. These potentials satisfy the equations *

$$\nabla'^2\phi' - \frac{1}{c^2}\frac{\partial^2\phi'}{\partial t'^2} = -\rho' \qquad (11)$$

$$\nabla'^2\mathbf{A}' - \frac{1}{c^2}\frac{\partial^2\mathbf{A}'}{\partial t'^2} = -\frac{1}{c}\rho'\mathbf{u}', \qquad (12)$$

and in terms of them $\mathbf{D}'$ and $\mathbf{H}'$ are given by

$$\mathbf{D}' = -\frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} - \text{grad}'\,\phi' + \frac{v}{c}\text{grad}'\,A'_x \qquad (13)$$

$$\mathbf{H}' = \text{curl}'\,\mathbf{A}' \qquad (14)$$

The symbol ∇'^2 is an abbreviation for $\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$,

and grad′ ϕ' denotes a vector whose components are

$$\frac{\partial\phi'}{\partial x'},\quad \frac{\partial\phi'}{\partial y'},\quad \frac{\partial\phi'}{\partial z'}.$$

The expression grad′ A'_x has a similar meaning.

In order to obtain the solution of (11) and (12) in a simple form, we may take x', y', z' as the co-ordinates of a point P′ in a space S′, and ascribe to this point, for each value of t', the values of ρ', $\mathbf{u}'$, ϕ', $\mathbf{A}'$, belonging to the corresponding point P (x, y, z) of the electromagnetic system. For a definite value t' of the fourth independent variable, the potentials ϕ' and $\mathbf{A}'$ at the point P of the system or at the corresponding point P′ of the space S′, are given by †

$$\phi' = \frac{1}{4\pi}\int\frac{[\rho']}{r'}dS' \qquad (15)$$

$$\mathbf{A}' = \frac{1}{4\pi c}\int\frac{[\rho'\mathbf{u}']}{r'}dS' \qquad (16)$$

* "M.E.," §§ 4 and 10.

† *Ibid.*, §§ 5 and 10.

Here dS' is an element of the space S′, r' its distance from P′, and the brackets serve to denote the quantity ρ' and the vector $\rho'\mathbf{u}'$ such as they are in the element dS', for the value $t' - r'/c$ of the fourth independent variable.

Instead of (15) and (16) we may also write, taking into account (4) and (7),

$$\phi' = \frac{1}{4\pi}\int\frac{[\rho]}{r}dS \qquad (17)$$

$$\mathbf{A}' = \frac{1}{4\pi c}\int\frac{[\rho\mathbf{u}]}{r}dS, \qquad (18)$$

the integrations now extending over the electromagnetic system itself. It should be kept in mind that in these formulæ r' does not denote the distance between the element dS and the point (x, y, z) for which the calculation is to be performed. If the element lies at the point (x_1, y_1, z_1), we must take

$$r' = l\sqrt{\beta^2(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}.$$

It is also to be remembered that, if we wish to determine ϕ' and $\mathbf{A}'$ for the instant at which the local time in P is t', we must take ρ and $\rho\mathbf{u}'$, such as they are in the element dS at the instant at which the local time of that element is $t' - r'/c$.

§ 6. It will suffice for our purpose to consider two special cases. The first is that of an electrostatic system, i.e. a system having no other motion but the translation with the velocity v. In this case $\mathbf{u}' = 0$, and therefore, by (12), $\mathbf{A}' = 0$. Also, ϕ' is independent of t', so that the equations (11), (13), and (14) reduce to

$$\left.\begin{aligned}\nabla'^2\phi' &= -\rho',\\ \mathbf{D}' &= -\text{grad}'\,\phi',\\ \mathbf{H}' &= 0\end{aligned}\right\} \qquad (19)$$

After having determined the vector $\mathbf{D}'$ by means of these equations, we know also the ponderomotive force acting on electrons that belong to the system. For these the formulæ (10) become, since $\mathbf{u}' = 0$,

$$F_x = l^2 D'_x,\quad F_y = \frac{l^2}{\beta}D'_y,\quad F_z = \frac{l^2}{\beta}D'_z \qquad (20)$$

The result may be put in a simple form if we compare the moving system Σ, with which we are concerned, to another electrostatic system Σ' which remains at rest, and into which Σ is changed if the dimensions parallel to the axis of x are multiplied by βl, and the dimensions which have the direction of y or that of z, by l—a deformation for which $(\beta l, l, l)$ is an appropriate symbol. In this new system, which we may suppose to be placed in the above-mentioned space S', we shall give to the density the value ρ', determined by (7), so that the charges of corresponding elements of volume and of corresponding electrons are the same in Σ and Σ'. Then we shall obtain the forces acting on the electrons of the moving system Σ, if we first determine the corresponding forces in Σ', and next multiply their components in the direction of the axis of x by l^2, and their components perpendicular to that axis by $\frac{l^2}{\beta}$. This is conveniently expressed by the formula

$$\mathbf{F}(\Sigma) = \left(l^2, \frac{l^2}{\beta}, \frac{l^2}{\beta}\right)\mathbf{F}(\Sigma') \quad . \quad . \quad . \quad (21)$$

It is further to be remarked that, after having found $\mathbf{D}'$ by (19), we can easily calculate the electromagnetic momentum in the moving system, or rather its component in the direction of the motion. Indeed, the formula

$$\mathbf{G} = \frac{1}{c}\int[\mathbf{D} \,.\, \mathbf{H}]dS$$

shows that

$$G_x = \frac{1}{c}\int(D_y H_z - D_z H_y)dS.$$

Therefore, by (6), since $\mathbf{H}' = 0$

$$G_x = \frac{\beta^2 l^4 v}{c^2}\int(D_y'^2 + D_z'^2)dS = \frac{\beta l v}{c^2}\int(D_y'^2 + D_z'^2)dS'. \quad (22)$$

§ 7. Our second special case is that of a particle having an electric moment, i.e. a small space S, with a total charge $\int\rho dS = 0$, but with such a distribution of density that the integrals $\int\rho x dS$, $\int\rho y dS$, $\int\rho z dS$ have values differing from 0. Let ξ, μ, ζ be the co-ordinates, taken relatively to a fixed point A of the particle, which may be called its centre, and let the electric moment be defined as a vector $\mathbf{P}$ whose components are

$$P_x = \int\rho\xi dS,\ P_y = \int\rho\eta dS,\ P_z = \int\rho\zeta dS \quad . \quad . \quad (23)$$

Then

$$\frac{dP_x}{dt} = \int\rho u_x dS,\ \frac{dP_y}{dt} = \int\rho u_y dS,\ \frac{dP_z}{dt} = \int\rho u_z dS \ . \quad (24)$$

Of course, if ξ, η, ζ are treated as infinitely small, u_x, u_y, u_z must be so likewise. We shall neglect squares and products of these six quantities.

We shall now apply the equation (17) to the determination of the scalar potential ϕ' for an exterior point P (x, y, z), at a finite distance from the polarized particle, and for the instant at which the local time of this point has some definite value t'. In doing so, we shall give the symbol $[\rho]$, which, in (17), relates to the instant at which the local time in dS is $t' - r'/c$, a slightly different meaning. Distinguishing by r'_0 the value of r' for the centre A, we shall understand by $[\rho]$ the value of the density existing in the element dS at the point (ξ, η, ζ), at the instant t_0 at which the local time of A is $t' - r_0/c$.

It may be seen from (5) that this instant precedes that for which we have to take the numerator in (17) by

$$\beta^2\frac{v\xi}{c^2} + \frac{\beta(r'_0 - r')}{lc} = \beta^2\frac{v\xi}{c^2} + \frac{\beta}{lc}\left(\xi\frac{\partial r'}{\partial x} + \eta\frac{\partial r'}{\partial y} + \zeta\frac{\partial r'}{\partial z}\right)$$

units of time. In this last expression we may put for the differential coefficients their values at the point A.

In (17) we have now to replace $[\rho]$ by

$$[\rho] + \beta^2\frac{v\xi}{c^2}\left[\frac{\partial\rho}{\partial t}\right] + \frac{\beta}{lc}\left(\xi\frac{\partial r'}{\partial x} + \eta\frac{\partial r'}{\partial y} + \zeta\frac{\partial r'}{\partial z}\right)\left[\frac{\partial\rho}{\partial t}\right] \quad (25)$$

where $\left[\frac{\partial\rho}{\partial t}\right]$ relates again to the time t_0. Now, the value of t' for which the calculations are to be performed having been chosen, this time t_0 will be a function of the co-ordinates x, y, z of the exterior point P. The value of $[\rho]$ will therefore depend on these co-ordinates in such a way that

$$\frac{\partial[\rho]}{\partial x} = -\frac{\beta}{lc}\frac{\partial r'}{\partial x}\left[\frac{\partial\rho}{\partial t}\right], \text{ etc.}$$

by which (25) becomes

$$[\rho] + \beta^2\frac{v\xi}{c^2}\left[\frac{\partial\rho}{\partial t}\right] - \left(\xi\frac{\partial[\rho]}{\partial x} + \eta\frac{\partial[\rho]}{\partial y} + \zeta\frac{\partial[\rho]}{\partial z}\right).$$

Again, if henceforth we understand by r' what has above been called r'_0, the factor $\frac{1}{r'}$ must be replaced by

$$\frac{1}{r'} - \xi\frac{\partial}{\partial x}\left(\frac{1}{r'}\right) - \eta\frac{\partial}{\partial y}\left(\frac{1}{r'}\right) - \zeta\frac{\partial}{\partial z}\left(\frac{1}{r'}\right),$$

so that after all, in the integral (17), the element dS is multiplied by

$$\frac{[\rho]}{r'} + \frac{\beta^2 v\xi}{c^2 r'}\left[\frac{\partial\rho}{\partial t}\right] - \frac{\partial}{\partial x}\frac{\xi[\rho]}{r'} - \frac{\partial}{\partial y}\frac{\eta[\rho]}{r'} - \frac{\partial}{\partial z}\frac{\zeta[\rho]}{r'}.$$

This is simpler than the primitive form, because neither r', nor the time for which the quantities enclosed in brackets are to be taken, depend on x, y, z. Using (23) and remembering that $\int\rho dS = 0$, we get

$$\phi' = \frac{\beta^2 v}{4\pi c^2 r'}\left[\frac{\partial P_x}{\partial t}\right] - \frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\frac{[P_x]}{r'} + \frac{\partial}{\partial y}\frac{[P_y]}{r'} + \frac{\partial}{\partial z}\frac{[P_z]}{r'}\right\},$$

a formula in which all the enclosed quantities are to be taken for the instant at which the local time of the centre of the particle is $t' - r'/c$.

We shall conclude these calculations by introducing a new vector $\mathbf{P}'$, whose components are

$$P'_x = \beta l P_x,\ P'_y = lP_y,\ P'_z = lP_z, \quad . \quad . \quad (26)$$

passing at the same time to x', y', z', t' as independent variables. The final result is

$$\phi' = \frac{v}{4\pi c^2 r'}\frac{\partial[P'_x]}{\partial t'} - \frac{1}{4\pi}\left\{\frac{\partial}{\partial x'}\frac{[P'_x]}{r'} + \frac{\partial}{\partial y'}\frac{[P'_y]}{r'} + \frac{\partial}{\partial z'}\frac{[P'_z]}{r'}\right\}.$$

As to the formula (18) for the vector potential, its transformation is less complicated, because it contains the infinitely small vector $\mathbf{u}'$. Having regard to (8), (24), (26), and (5), I find

$$\mathbf{A}' = \frac{1}{4\pi c r'}\frac{\partial[\mathbf{P}']}{\partial t'}.$$

The field produced by the polarized particle is now wholly determined. The formula (13) leads to

$$\mathbf{D}' = -\frac{1}{4\pi c^2}\frac{\partial^2}{\partial t'^2}\frac{[\mathbf{P}']}{r'} + \frac{1}{4\pi}\,\text{grad}'\left\{\frac{\partial}{\partial x'}\frac{[P'_x]}{r'} + \frac{\partial}{\partial y'}\frac{[P'_y]}{r'} + \frac{\partial}{\partial z'}\frac{[P'_z]}{r'}\right\} \quad (27)$$

and the vector $\mathbf{H}'$ is given by (14). We may further use the equations (20), instead of the original formulæ (10), if we wish to consider the forces exerted by the polarized particle on a similar one placed at some distance. Indeed, in the second particle, as well as in the first, the velocities $\mathbf{u}$ may be held to be infinitely small.

It is to be remarked that the formulæ for a system without translation are implied in what precedes. For such a system the quantities with accents become identical to the corresponding ones without accents; also $\beta = 1$ and $l = 1$. The components of (27) are at the same time those of the electric force which is exerted by one polarized particle on another.

§ 8. Thus far we have used only the fundamental equations without any new assumptions. I shall now suppose *that the electrons, which I take to be spheres of radius R in the state of rest, have their dimensions changed by the effect of a translation, the dimensions in the direction of motion becoming βl times and those in perpendicular directions l times smaller.*

In this deformation, which may be represented by $\left(\frac{1}{\beta l}, \frac{1}{l}, \frac{1}{l}\right)$, each element of volume is understood to preserve its charge.

Our assumption amounts to saying that in an electrostatic system Σ, moving with a velocity v, all electrons are flattened ellipsoids with their smaller axes in the direction of motion. If now, in order to apply the theorem of § 6, we subject the system to the deformation $(\beta l, l, l)$, we shall have again spherical electrons of radius R. Hence, if we alter the relative position of the centres of the electrons in Σ by applying the deformation $(\beta l, l, l)$, and if, in the points thus obtained, we place the centres of electrons that remain at rest, we shall get a system, identical to the imaginary system Σ', of which we have spoken in § 6. The forces in this system and those in Σ will bear to each other the relation expressed by (21).

In the second place I shall suppose *that the forces between uncharged particles, as well as those between such particles and electrons, are influenced by a translation in quite the same way as the electric forces in an electrostatic system.* In other terms, whatever be the nature of the particles composing a ponderable body, so long as they do not move relatively to each other, we shall have between the forces acting in a system (Σ') without, and the same system (Σ) with a translation, the relation specified in (21), if, as regards the relative position of the particles, Σ' is got from Σ by the deformation $(\beta l, l, l)$, or Σ from Σ' by the deformation $\left(\frac{1}{\beta l}, \frac{1}{l}, \frac{1}{l}\right)$.

We see by this that, as soon as the resulting force is zero for a particle in Σ', the same must be true for the corresponding particle in Σ. Consequently, if, neglecting the effects of molecular motion, we suppose each particle of a solid body to be in equilibrium under the action of the attractions and repulsions exerted by its neighbours, and if we take for granted that there is but one configuration of equilibrium, we may draw the conclusion that the system Σ', if the velocity v is imparted to it, will *of itself* change into the system Σ. In other terms, the translation will *produce* the deformation $\left(\frac{1}{\beta l}, \frac{1}{l}, \frac{1}{l}\right)$.

The case of molecular motion will be considered in § 12.

It will easily be seen that the hypothesis which was formerly advanced in connexion with Michelson's experiment, is implied in what has now been said. However, the present hypothesis is more general, because the only

limitation imposed on the motion is that its velocity be less than that of light.

§ 9. We are now in a position to calculate the electromagnetic momentum of a single electron. For simplicity's sake I shall suppose the charge e to be uniformly distributed over the surface, so long as the electron remains at rest. Then a distribution of the same kind will exist in the system Σ' with which we are concerned in the last integral of (22). Hence

$$\int(\mathrm{D}'^2_y + \mathrm{D}'^2_z)d\mathrm{S}' = \frac{2}{3}\int \mathrm{D}'^2 d\mathrm{S}' = \frac{e^2}{6\pi}\int_{\mathrm{R}}^{\infty}\frac{dr}{r^2} = \frac{e^2}{6\pi \mathrm{R}},$$

and

$$\mathrm{G}_x = \frac{e^2}{6\pi c^2 \mathrm{R}}\beta l v.$$

It must be observed that the product βl is a function of v and that, for reasons of symmetry, the vector $\mathbf{G}$ has the direction of the translation. In general, representing by $\mathbf{v}$ the velocity of this motion, we have the vector equation

$$\mathbf{G} = \frac{e^2}{6\pi c^2 \mathrm{R}}\beta l \mathbf{v} \quad . \quad . \quad . \quad . \quad (28)$$

Now, every change in the motion of a system will entail a corresponding change in the electromagnetic momentum and will therefore require a certain force, which is given in direction and magnitude by

$$\mathbf{F} = \frac{d\mathbf{G}}{dt} \quad . \quad . \quad . \quad . \quad (29)$$

Strictly speaking, the formula (28) may only be applied in the case of a uniform rectilinear translation. On account of this circumstance—though (29) is always true—the theory of rapidly varying motions of an electron becomes very complicated, the more so, because the hypothesis of § 8 would imply that the direction and amount of the deformation are continually changing. It is, indeed, hardly probable that the form of the electron will be determined solely by the velocity existing at the moment considered.

Nevertheless, provided the changes in the state of motion be sufficiently slow, we shall get a satisfactory approximation by using (28) at every instant. The application of (29) to such a *quasi-stationary* translation, as it has been called by Abraham,* is a very simple matter. Let, at a certain instant, $\mathbf{a}_1$ be the acceleration in the direction of the path, and $\mathbf{a}_2$ the acceleration perpendicular to it. Then the force $\mathbf{F}$ will consist of two components, having the directions of these accelerations and which are given by

$$\mathbf{F}_1 = m_1\mathbf{a}_1 \text{ and } \mathbf{F}_2 = m_2\mathbf{a}_2,$$

if

$$m_1 = \frac{e^2}{6\pi c^2 R}\frac{d(\beta l v)}{dv} \text{ and } m_2 = \frac{e^2}{6\pi c^2 R}\beta l \,. \qquad (30)$$

Hence, in phenomena in which there is an acceleration in the direction of motion, the electron behaves as if it had a mass m_1; in those in which the acceleration is normal to the path, as if the mass were m_2. These quantities m_1 and m_2 may therefore properly be called the "longitudinal" and "transverse" electromagnetic masses of the electron. I shall suppose *that there is no other, no "true" or "material" mass.*

Since β and l differ from unity by quantities of the order v^2/c^2, we find for very small velocities

$$m_1 = m_2 = \frac{e^2}{6\pi c^2 R}.$$

This is the mass with which we are concerned, if there are small vibratory motions of the electrons in a system without translation. If, on the contrary, motions of this kind are going on in a body moving with the velocity v in the direction of the axis of x, we shall have to reckon with the mass m_1, as given by (30), if we consider the vibrations parallel to that axis, and with the mass m_2, if we treat of those that are parallel to OY or OZ. Therefore, in short terms, referring by the index Σ to a moving system and by Σ' to one that remains at rest,

$$m(\Sigma) = \left(\frac{d(\beta l v)}{dv}, \beta l, \beta l\right) m(\Sigma') \qquad (31)$$

* Abraham, Wied. Ann., 10, 1903, p. 105.

§ 10. We can now proceed to examine the influence of the Earth's motion on optical phenomena in a system of transparent bodies. In discussing this problem we shall fix our attention on the variable electric moments in the particles or "atoms" of the system. To these moments we may apply what has been said in § 7. For the sake of simplicity we shall suppose that, in each particle, the charge is concentrated in a certain number of separate electrons, and that the "elastic" forces that act on one of these, and, conjointly with the electric forces, determine its motion, have their origin within the bounds of the *same* atom.

I shall show that, if we start from any given state of motion in a system without translation, we may deduce from it a corresponding state that can exist in the same system after a translation has been imparted to it, the kind of correspondence being as specified in what follows.

(*a*) Let A'_1, A'_2, A'_3, etc., be the centres of the particles in the system without translation (Σ'); neglecting molecular motions we shall assume these points to remain at rest. The system of points A_1, A_2, A_3, etc., formed by the centres of the particles in the moving system Σ, is obtained from A'_1, A'_2, A'_3, etc., by means of a deformation $\left(\frac{1}{\beta l}, \frac{1}{l}, \frac{1}{l}\right)$. According to what has been said in § 8, the centres will of themselves take these positions A'_1, A'_2, A'_3, etc., if originally, before there was a translation, they occupied the positions A_1, A_2, A_3, etc.

We may conceive any point P′ in the space of the system Σ' to be displaced by the above deformation, so that a definite point P of Σ corresponds to it. For two corresponding points P′ and P we shall define corresponding instants, the one belonging to P′, the other to P, by stating that the true time at the first instant is equal to the local time, as determined by (5) for the point P, at the second instant. By corresponding times for two corresponding *particles* we shall understand times that may be said to correspond, if we fix our attention on the *centres* A′ and A of these particles.

(*b*) As regards the interior state of the atoms, we shall assume that the configuration of a particle A in Σ at a certain time may be derived by means of the deformation $\left(\frac{1}{\beta l}, \frac{1}{l}, \frac{1}{l}\right)$ from the configuration of the corresponding particle in Σ', such as it is at the corresponding instant. In so far as this assumption relates to the form of the electrons themselves, it is implied in the first hypothesis of § 8.

Obviously, if we start from a state really existing in the system Σ', we have now completely defined a state of the moving system Σ. The question remains, however, whether this state will likewise be a possible one.

In order to judge of this, we may remark in the first place that the electric moments which we have supposed to exist in the moving system and which we shall denote by **P**, will be certain definite functions of the co-ordinates x, y, z of the centres A of the particles, or, as we shall say, of the co-ordinates of the particles themselves, and of the time t. The equations which express the relations between **P** on one hand and x, y, z, t on the other, may be replaced by other equations containing the vectors **P**$'$ defined by (26) and the quantities x', y', z', t' defined by (4) and (5). Now, by the above assumptions a and b, if in a particle A of the moving system, whose co-ordinates are x, y, z, we find an electric moment **P** at the time t, or at the local time t', the vector **P**$'$ given by (26) will be the moment which exists in the other system at the true time t' in a particle whose co-ordinates are x', y', z'. It appears in this way that the equations between **P**$'$, x', y', z', t' are the same for both systems, the difference being only this, that for the system Σ' without translation these symbols indicate the moment, the co-ordinates, and the true time, whereas their meaning is different for the moving system, **P**$'$, x', y', z', t' being here related to the moment **P**, the co-ordinates x, y, z and the general time t in the manner expressed by (26), (4), and (5).

It has already been stated that the equation (27) applies to both systems. The vector **D**$'$ will therefore be the same in Σ' and Σ, provided we always compare corresponding places and times. However, this vector has not the same meaning in the two cases. In Σ' it represents the electric force, in Σ it is related to this force in the way expressed by (20). We may therefore conclude that the ponderomotive forces acting, in Σ and in Σ', on corresponding particles at corresponding instants, bear to each other the relation determined by (21). In virtue of our assumption (b), taken in connexion with the second hypothesis of § 8, the same relation will exist between the "elastic" forces; consequently, the formula (21) may also be regarded as indicating the relation between the total forces, acting on corresponding electrons, at corresponding instants.

It is clear that the state we have supposed to exist in the moving system will really be possible if, in Σ and Σ', the products of the mass m and the acceleration of an electron are to each other in the same relation as the forces, i.e. if

$$m\mathbf{a}(\Sigma) = \left(l^2, \frac{l^2}{\beta}, \frac{l^2}{\beta}\right) m\mathbf{a}(\Sigma') \quad . \quad . \quad . \quad (32)$$

Now, we have for the accelerations

$$\mathbf{a}(\Sigma) = \left(\frac{l}{\beta^3}, \frac{l}{\beta^2}, \frac{l}{\beta^2}\right)\mathbf{a}(\Sigma') \quad . \quad . \quad . \quad (33)$$

as may be deduced from (4) and (5), and combining this with (32), we find for the masses

$$m(\Sigma) = (\beta^3 l, \beta l, \beta l) m(\Sigma').$$

If this is compared with (31), it appears that, whatever be the value of l, the condition is always satisfied, as regards the masses with which we have to reckon when we consider vibrations perpendicular to the translation. The only condition we have to impose on l is therefore

$$\frac{d(\beta l v)}{dv} = \beta^3 l.$$

But, on account of (3),

$$\frac{d(\beta v)}{dv} = \beta^3,$$

so that we must put

$$\frac{dl}{dv} = 0, \; l = \text{const.}$$

The value of the constant must be unity, because we know already that, for $v = 0$, $l = 1$.

We are therefore led to suppose *that the influence of a translation on the dimensions (of the separate electrons and of a ponderable body as a whole) is confined to those that have the direction of the motion, these becoming* β *times smaller than they are in the state of rest.* If this hypothesis is added to those we have already made, we may be sure that two states, the one in the moving system, the other in the same system while at rest, corresponding as stated above, may both be possible. Moreover, this correspondence is not limited to the electric moments of the particles. In corresponding points that are situated either in the ether between the particles, or in that surrounding the ponderable bodies, we shall find at corresponding times the same vector **D**′ and, as is easily shown, the same vector **H**′. We may sum up by saying: If, in the system without translation, there is a state of motion in which, at a definite place, the components of **P**, **D**, and **H** are certain functions of the time, then the same system after it has been put in motion (and thereby deformed) can be the seat of a state of motion in which, at the corresponding place, the components of **P**′, **D**′, and **H**′ are the same functions of the local time.

There is one point which requires further consideration. The values of the masses m_1 and m_2 having been deduced from the theory of quasi-stationary motion, the question arises, whether we are justified in reckoning with them in the case of the rapid vibrations of light. Now it is found on closer examination that the motion of an electron may be treated as quasi-stationary if it changes very little during the time a light-wave takes to travel over a distance equal to the diameter. This condition is fulfilled in optical phenomena, because the diameter of an electron is extremely small in comparison with the wave-length.

§ 11. It is easily seen that the proposed theory can account for a large number of facts.

Let us take in the first place the case of a system without translation, in some parts of which we have continually **P** = 0, **D** = 0, **H** = 0. Then, in the corresponding state for the moving system, we shall have in corresponding parts (or, as we may say, in the same parts of the deformed system) **P**′ = 0, **D**′ = 0, **H**′ = 0. These equations implying **P** = 0,

D = 0, **H** = 0, as is seen by (26) and (6), it appears that those parts which are dark while the system is at rest, will remain so after it has been put in motion. It will therefore be impossible to detect an influence of the Earth's motion on any optical experiment, made with a terrestrial source of light, in which the geometrical distribution of light and darkness is observed. Many experiments on interference and diffraction belong to this class.

In the second place, if, in two points of a system, rays of light of the same state of polarization are propagated in the same direction, the ratio between the amplitudes in these points may be shown not to be altered by a translation. The latter remark applies to those experiments in which the intensities in adjacent parts of the field of view are compared.

The above conclusions confirm the results which I formerly obtained by a similar train of reasoning, in which, however, the terms of the second order were neglected. They also contain an explanation of Michelson's negative result, more general than the one previously given, and of a somewhat different form; and they show why Rayleigh and Brace could find no signs of double refraction produced by the motion of the Earth.

As to the experiments of Trouton and Noble, their negative result becomes at once clear, if we admit the hypotheses of § 8. It may be inferred from these and from our last assumption (§ 10) that the only effect of the translation must have been a contraction of the whole system of electrons and other particles constituting the charged condenser and the beam and thread of the torsion-balance. Such a contraction does not give rise to a sensible change of direction.

It need hardly be said that the present theory is put forward with all due reserve. Though it seems to me that it can account for all well-established facts, it leads to some consequences that cannot as yet be put to the test of experiment. One of these is that the result of Michelson's experiment must remain negative, if the interfering rays of light are made to travel through some ponderable transparent body.

Our assumption about the contraction of the electrons cannot in itself be pronounced to be either plausible or inadmissible. What we know about the nature of electrons is very little, and the only means of pushing our way farther will be to test such hypotheses as I have here made. Of course, there will be difficulties, e.g. as soon as we come to consider the rotation of electrons. Perhaps we shall have to suppose that in those phenomena in which, if there is no translation, spherical electrons rotate about a diameter, the points of the electrons in the moving system will describe elliptic paths, corresponding, in the manner specified in § 10, to the circular paths described in the other case.

§ 12. There remain to be said a few words about molecular motion. We may conceive that bodies in which this has a sensible influence or even predominates, undergo the same deformation as the systems of particles of constant relative position of which alone we have spoken till now. Indeed, in two systems of molecules Σ' and Σ, the first without and the second with a translation, we may imagine molecular motions corresponding to each other in such a way that, if a particle in Σ' has a certain position at a definite instant, a particle in Σ occupies at the corresponding instant the corresponding position. This being assumed, we may use the relation (33) between the accelerations in all those cases in which the velocity of molecular motion is very small as compared with v. In these cases the molecular forces may be taken to be determined by the relative positions, independently of the velocities of molecular motion. If, finally, we suppose these forces to be limited to such small distances that, for particles acting on each other, the difference of local times may be neglected, one of the particles, together with those which lie in its sphere of attraction or repulsion, will form a system which undergoes the often mentioned deformation. In virtue of the second hypothesis of § 8 we may therefore apply to the resulting molecular force acting on a particle, the equation (21). Consequently, the proper relation between the forces and the accelerations will exist in the two cases, if we suppose *that the masses of all particles are influenced by a translation to the same degree as the electromagnetic masses of the electrons.*

§ 13. The values (30), which I have found for the longitudinal and transverse masses of an electron, expressed in terms of its velocity, are not the same as those that had been previously obtained by Abraham. The ground for this difference is to be sought solely in the circumstance that, in his theory, the electrons are treated as spheres of invariable dimensions. Now, as regards the transverse mass, the results of Abraham have been confirmed in a most remarkable way by Kaufmann's measurements of the deflexion of radium-rays in electric and magnetic fields. Therefore, if there is not to be a most serious objection to the theory I have now proposed, it must be possible to show that those measurements agree with my values nearly as well as with those of Abraham.

I shall begin by discussing two of the series of measurements published by Kaufmann* in 1902. From each series he has deduced two quantities η and ζ, the "reduced" electric and magnetic deflexions, which are related as follows to the ratio $\gamma = v/c$:—

$$\gamma = k_1 \frac{\zeta}{\eta}, \quad \psi(\gamma) = \frac{\eta}{k_2 \zeta^2} \quad . \quad . \quad . \quad (34)$$

Here $\psi(\gamma)$ is such a function, that the transverse mass is given by

$$m_2 = \frac{3}{4} \cdot \frac{e^2}{6\pi c^2 R} \psi(\gamma), \quad . \quad . \quad . \quad (35)$$

whereas k_1 and k_2 are constant in each series.

It appears from the second of the formulæ (30) that my theory leads likewise to an equation of the form (35); only Abraham's function $\psi(\gamma)$ must be replaced by

$$\frac{4}{3}\beta = \frac{4}{3}(1 - \gamma^2)^{-1/2}.$$

Hence, my theory requires that, if we substitute this value for $\psi(\gamma)$ in (34), these equations shall still hold. Of course, in seeking to obtain a good agreement, we shall be justified in giving to k_1 and k other values than those of Kaufmann, and in taking for every measurement a proper value of the velocity v, or of the ratio γ. Writing sk_1, $\frac{3}{4} k'_2$

* Kaufmann, Physik. Zeitschr., 4, 1902, p. 55.

and γ' for the new values, we may put (34) in the form

$$\gamma' = sk_1\frac{\zeta}{\eta} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and

$$(1 - \gamma'^2)^{-1/2} = \frac{\eta}{k'_2\zeta^2} \quad . \quad . \quad . \quad . \quad (37)$$

Kaufmann has tested his equations by choosing for k_1 such a value that, calculating γ and k_2 by means of (34), he obtained values for this latter number which, as well as might be, remained constant in each series. This constancy was the proof of a sufficient agreement.

I have followed a similar method, using, however, some of the numbers calculated by Kaufmann. I have computed for each measurement the value of the expression

$$k'_2 = (1 - \gamma'^2)^{1/2}\psi(\gamma)k_2, \quad . \quad . \quad . \quad (38)$$

that may be got from (37) combined with the second of the equations (34). The values of $\psi(\gamma)$ and k_2 have been taken from Kaufmann's tables, and for γ' I have substituted the value he has found for γ, multiplied by s, the latter coefficient being chosen with a view to obtaining a good constancy of (38). The results are contained in the tables on opposite page, corresponding to the Tables III and IV in Kaufmann's paper.

The constancy of k'_2 is seen to come out no less satisfactorily than that of k_2, the more so as in each case the value of s has been determined by means of only two measurements. The coefficient has been so chosen that for these two observations, which were in Table III the first and the last but one, and in Table IV the first and the last, the values of k'_2 should be proportional to those of k_2.

I shall next consider two series from a later publication by Kaufmann,* which have been calculated by Runge † by means of the method of least squares, the coefficients k_1 and k_2 having been determined in such a way that the values of η, calculated, for each observed ζ, from Kaufmann's equations (34), agree as closely as may be with the observed values of η.

* Kaufmann, Gött. Nachr. Math. phys. Kl., 1903, p. 90.
† Runge, *ibid.*, p. 326.

III. $s = 0{\cdot}933$.

γ.	$\psi(\gamma)$.	k_2.	γ'.	k'_2.
0·851	2·147	1·721	0·794	2·246
0·766	1·86	1·736	0·715	2·258
0·727	1·78	1·725	0·678	2·256
0·6615	1·66	1·727	0·617	2·256
0·6075	1·595	1 655	0·567	2·175

IV. $s = 0{\cdot}954$.

γ.	$\psi(\gamma)$.	k_2.	γ'.	k'_2.
0·963	3·28	8·12	0·919	10·36
0·949	2·86	7·99	0·905	9·70
0·933	2·73	7·46	0·890	9·28
0·883	2·31	8·32	0·842	10·36
0·860	2·195	8·09	0·820	10·15
0·830	2·06	8·13	0·792	10·23
0·801	1·96	8·13	0·764	10·28
0·777	1·89	8·04	0·741	10·20
0·752	1·83	8·02	0·717	10·22
0·732	1·785	7·97	0·698	10·18

I have determined by the same condition, likewise using the method of least squares, the constants a and b in the formula

$$\eta^2 = a\zeta^2 + b\zeta^4,$$

which may be deduced from my equations (36) and (37). Knowing a and b, I find γ for each measurement by means of the relation

$$\gamma = \sqrt{a}\frac{\zeta}{\eta}.$$

For two plates on which Kaufmann had measured the electric and magnetic deflexions, the results are as follows (p. 34), the deflexions being given in centimetres.

I have not found time for calculating the other tables in Kaufmann's paper. As they begin, like the table for Plate 15 (next page) with a rather large negative difference between the values of η which have been deduced from the observations and calculated by Runge, we may expect a satisfactory agreement with my formulæ.

Plate No. 15. $a = 0{\cdot}06489$, $b = 0{\cdot}3039$.

ζ	η					γ	
	Observed.	Calculated by R.	Diff.	Calculated by L.	Diff.	Calculated by R.	Calculated by L.
0·1495	0·0388	0·0404	− 16	0·0400	− 12	0·987	0·951
0·199	0·0548	0·0550	− 2	0·0552	− 4	0·964	0·918
0·2475	0·0716	0·0710	+ 6	0·0715	+ 1	0·930	0·881
0·296	0·0896	0·0887	+ 9	0·0895	+ 1	0·889	0·842
0 3435	0·1080	0·1081	− 1	0·1090	− 10	0·847	0·803
0·391	0·1290	0·1297	− 7	0·1305	− 15	0·804	0·763
0·437	0·1524	0·1527	− 3	0·1532	− 8	0·763	0·727
0·4 25	0·1788	0·1777	+ 11	0·1777	+ 11	0·724	0·692
0·5265	0·2033	0·2039	− 6	0·2033	0	0·688	0·660

Plate No. 19. $a = 0{\cdot}05867$, $b = 0{\cdot}2591$.

ζ	η					γ	
	Observed.	Calculated by R.	Diff.	Calculated by L.	Diff.	Calculated by R.	Calculated by L.
0·1495	0·0404	0·0388	+ 16	0·0379	+ 25	0·990	0·954
0·199	0·0529	0·0527	+ 2	0·0522	+ 7	0·969	0·923
0·247	0·0678	0·0675	+ 3	0·0674	+ 4	0·939	0·888
0·296	0·0834	0·0842	− 8	0·0844	− 10	0·902	0·849
0·3435	0·1019	0·1022	− 3	0·1026	− 7	0·862	0·811
0·391	0·1219	0·1222	− 3	0·1226	− 7	0·822	0·773
0·437	0·1429	0·1434	− 5	0·1437	− 8	0·782	0·736
0·4825	0·1660	0·1665	− 5	0·1664	− 4	0·744	0·702
0·5265	0·1916	0·1906	+ 10	0·1902	+ 14	0·709	0·671