M. Shifman, Lecture 2

21.15 Global Anomaly

In Section 21.10 we discussed the massless quark effects in the presence of instantons. In particular, a formula for counting the number of the fermion zero modes was derived. An inspection of this formula leads one to a perplexing question. Indeed, let us assume that, instead of QCD, we deal with an SU(2) theory with one massless left-handed Weyl fermion transforming as a doublet with respect to SU(2). So far only the Dirac fermions were considered; one Dirac fermion is equivalent to *two* Weyl fermions. Now we want to consider a chiral theory. Before the advent of the instantons this theory was believed to be perfectly well-defined. It has no internal anomalies, see Chapter 8. Moreover, in perturbation theory, order by order, one encounters no reasons to make the theory sick. And yet, this theory is pathological. Analysis of the instanton-induced effects helps us reveal the pathology.

Indeed, following the line of reasoning in Section 21.10 in the SU(2) theory with a *single* massless left-handed Weyl fermion we would immediately discover that the instanton-induced fermion vertex of the 't Hooft type must be *linear* in the fermion field. Indeed, in the instanton transition with one Dirac fermion $\Delta Q_5 = 2$,† but the Weyl fermion = 1/2 of the Dirac fermion, and hence $\Delta Q_5 = 1$!

It was obvious to many that something was unusual in this theory. The intuitive feeling of pathology was formalized by Witten who showed [45] that this theory is ill-defined because of the *global anomaly.* Such theory is mathematically inconsistent. It simply does not exist.

One of the possible proofs of the global anomaly is based on the fermion level restructuring in the instanton transition. The key elements are the following: (i) the vacuum-to-vacuum amplitude in the theory with one Weyl fermion is proportional to $\sqrt{\det(i\mathcal{D})}$; (ii) only one of the fermion levels changes its positions with regards to the sea level (I mean the Dirac sea, of course) when $\mathcal{K} = n$ goes in $\mathcal{K} = n + 1$, as opposed to one *pair* in the case

† Weyl fermion's contribution to the chiral anomaly is 1/2 of that of the fermion

of the Dirac fermion, Fig. 21.3. This forces the partition function to vanish making all correlation functions ill-defined. For further details see Ref. [45].

Exercises

21.1 Generalize our derivation of the antiinstanton field in the spinorial notation, see Eq. (21.36), for instantons. Hint: treat the indices of the color matrix as dotted.

21.2 Prove Eq. (21.76) through a direct calculation using definitions and results presented in Sections 20.2, 21.3, and 21.5.

Solution:

As a warm up exercise let us determine the vector $\hat{v}$. Since any rotation matrix M can be written as $M = \exp(i\omega^a\tau^a/2) = \cos\frac{\omega}{2} + i\,\vec{n}\vec{\tau}\sin\frac{\omega}{2}$ (here $\omega = |\vec{\omega}|$ and $\vec{n}$ is the unit vector in the direction of $\vec{\omega}$), we determine that

$$\vec{\hat{v}} = \vec{n}\,\sin\frac{\omega}{2}\,, \qquad \hat{v}_4 = -\cos\frac{\omega}{2}\,,$$

implying that $\hat{v}^2 = 1$. Let us choose the reference frame in which $\vec{R} = 0$ and only $R_4 \neq 0$. One can always do that. Then

$$\eta_{a\alpha\beta}\,\bar{\eta}_{b\alpha\gamma}\,R_\beta\,R_\gamma = -\delta_{ab}\,R_4^2\,.$$

One should also use the facts that

$$O^{ab} = \frac{1}{2}\,\mathrm{Tr}\left\{\tau^b\,\hat{v}_\mu\tau^+_\mu\,\tau^a\,\hat{v}_\nu\tau^-_\nu\right\}$$

and

$$\tau^a\,\tau^+_\mu\,\tau^a = -\tau^+_\mu + s_\mu\,, \qquad s_\mu = \begin{cases} 0\,, & \text{for } \mu = 1,2,3, \\ -4\,i\,, & \text{for } \mu = 4 \end{cases}\,.$$

Now, assembling all these expressions one arrives at

$$O^{ab}\,\eta_{a\alpha\beta}\,\bar{\eta}_{b\alpha\gamma}\,R_\beta\,R_\gamma = \hat{v}^2R_4^2 - 4\hat{v}_4^2R_4^2 \to \hat{v}^2R^2 - 4\,(\hat{v}R)^2\,.$$

21.3 Explicitly calculate the integral in (21.26). Find the instanton field in the $A_0 = 0$ gauge for arbitrary values of τ.

21.4 Verify that the expression (21.107) is indeed a solution of Eq. (21.105).

21.5 Verify Eq. (21.128).

[redacted]