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Lecture 2 : Chiral anomalies in QCD and their applications

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34 Anomlies in QCD and Similar Non-Abelian Gauge Theories

In this section we will discuss QCD and non-Abelian gauge theories at large which are self-consistent, i.e. free of internal anomalies. In particular, dealing with chiral theories we should follow strict rules in constructing the matter sector (see Section 22.1.1). Nevertheless, these theories have external anomalies: the scale anomaly and and those in the divergence of external axial currents.† The latter are also referred to as the chiral (a.k.a triangle, a.k.a Adler–Bell–Jackiw [8]) anomalies. We will analyze and derive the chiral and scale anomalies using QCD as a showcase. More exactly, we will assume that the theory under consideration has the gauge group SU(N) and contains N_f massless quarks (Dirac fields in the fundamental representation). In this section it will be convenient to write the action in the canonic normalization,

$$S = \int d^4x \left\{ -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu\, a} + \sum_{f=1}^{N_f} \bar\psi_f \, i\not{D} \, \psi^f \right\} . \tag{34.1}$$

We will start from examining the classical symmetries of the above action.

Global symmetries of QCD

In addition to the scale (implying, in fact, full conformal) invariance of the action of which I will speak later, (34.1) has the following symmetry

$$\mathrm{U}(1)_V \times \mathrm{U}(1)_A \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \tag{34.2}$$

acting in the matter sector. The vector U(1) corresponds to the baryon number conservation, with the current

$$j^B_\mu = \frac{1}{3} \bar\psi_f \, \gamma_\mu \, \psi^f \, . \tag{34.3}$$

The axial U(1) corresponds to the overall chiral phase rotation

$$\psi^f_L \to e^{i\alpha} \psi^f_L \, , \qquad \psi^f_R \to e^{-i\alpha} \psi^f_R \, , \qquad \psi_{L,R} = \frac{1}{2}(1 \mp \gamma^5)\psi \, . \tag{34.4}$$

† By external I mean such currents which are not coupled to the gauge fields of the theory under consideration.

The axial current generated by (34.4) is

$$j^\mu_A = \bar\psi_f\,\gamma^\mu\,\gamma^5\,\psi^f\,. \tag{34.5}$$

Singlet and nonsinglet axial currents

Finally, the last two factors in (34.2) reflect the invariance of the action with regards to the chiral flavor rotations

$$\psi^f_L \to U^f_g\,\psi^g_L\,, \qquad \psi^f_R \to \tilde U^f_g\,\psi^g_R\,, \tag{34.6}$$

where U and $\tilde U$ are arbitrary (independent) matrices from $\mathrm{SU}(N_f)$. Equation (34.6) implies conservation of the following vector and axial currents:

$$j^a_\mu = \bar\psi\,\gamma_\mu\,T^a\,\psi\,, \quad \text{and} \quad j^{5\,a}_\mu = \bar\psi\,\gamma^\mu\,\gamma^5\,T^a\,\psi\,. \tag{34.7}$$

Here T^a's are the generators of the *flavor* $\mathrm{SU}(N_f)$ in the fundamental representation. These generators act in the flavor space, i.e. ψ is a column of ψ^f's while the matrices T^a act on this column.

At the quantum level (i.e. including loops with a regularization) the fate of the above symmetries is different. The vector U(1) invariance generated by (34.3) stays a valid anomaly-free symmetry at the quantum level.† The same is true with regards to the vector $\mathrm{SU}(N_f)$ currents, they are conserved. The axial currents are anomalous. One should distinguish, though, between the singlet current (34.5) and the $\mathrm{SU}(N_f)$ currents $j^{5\,a}_\mu = \bar\psi_f\,\gamma^\mu\,\gamma^5\,T^a\,\psi^f$. The former is anomalous in QCD *per se.* The latter become anomalous only upon introduction of appropriate external vector currents. As we will see later, this circumstance is in one-to-one correspondence with the spontaneous breaking of the axial $\mathrm{SU}(N_f)$ symmetry in QCD, accompanied by the emergence of N_f^2-1 Goldstone bosons. The vector $\mathrm{SU}(N_f)$ symmetry is realized linearly.

In the weakly coupled Schwinger model considered in Section 33.1 we could take both routes (and we actually did) to derive the chiral anomaly: infrared and ultraviolet. The first route is closed in QCD, since this theory is strongly coupled in the infrared domain which invalidates any conclusions based on the Feynman graph calculations Neither quarks nor gluons are relevant in the infrared. However, the second route is open, and we will follow it in the subsequent sections. We will limit ourselves to one-loop analysis. Higher loops, where present, generally speaking, lie outside the scope of this book. The only exception is a class of supersymmetric gauge theories, to be considered in Part II (Section 59).

† I hasten to make a reservation. This statement is valid in vector-like theories. As we already know from Section 23, this is not true in the chiral models such as the Standard Model. You should remember that for the time being we discuss QCD.

34.1 Chiral anomaly in the singlet axial current

Differentiating (34.5) naively we get $\partial_\mu j^\mu_A = \bar\psi_f \overleftarrow{\not{D}}\, \gamma^5\psi^f - \bar\psi_f\gamma^5\not{D}\psi^f = 0$ by virtue the equation of motion $\not{D}\,\psi^f = 0$. Experience gained in the Schwinger model teaches us, however, that the axial current conservation will not hold upon switching on a gauge-symmetry respecting regularization. To make the calculation of the anomaly reliable we must exploit only the Green functions at short distances. This means that we must focus directly on $\partial_\mu j^\mu_A$ exploiting one of appropriate ultraviolet regularizations. The following demonstration will be based on the Schwinger and Pauli–Villars regularizations. ‡

34.1.1 The Schwinger regularization

In this regularization we ε-split the current,

$$j^{A,\,R}_\mu(x) = \bar\psi_f(x+\varepsilon)\,\gamma_\mu\,\gamma^5\left\{\exp\left[\int_{x-\varepsilon}^{x+\varepsilon} ig\,A_\rho(y)\,dy^\rho\right]\right\}\psi^f(x-\varepsilon)\,. \quad (34.8)$$

Here the superscript R marks the regularized current while $A_\rho \equiv A^a_\rho T^a$. The ε parameter must be set to zero at the very end. The exponent is necessary to ensure gauge invariance of the regularized current $j^{A,\,R}_\mu$ after the split

Look through Section 33.3

$$\bar\psi_f(x+\varepsilon)\,\psi^f(x-\varepsilon)\,. \quad (34.9)$$

Next, we differentiate over x using the equations of motion above. Expanding in ε and keeping terms $O(\varepsilon)$ we arrive at

$$\begin{aligned}\partial^\mu j^{A,\,R}_\mu \;&=\; \bar\psi_f(x+\varepsilon)\left\{-ig\not{A}(x+\varepsilon)\,\gamma^5 - \gamma^5\,ig\not{A}(x-\varepsilon)\right.\\ &+\; \left. ig\,\gamma^\mu\gamma^5\varepsilon^\beta(0)\,G_{\mu\beta}(0)\right\}\psi^f(x-\varepsilon)\,. \end{aligned} \quad (34.10)$$

The third term in the braces in (34.10) contains the gluon-field strength tensor and results from differentiation of the exponential factor. The gluon 4-potential A_μ and the field strength tensor $G_{\mu\beta}$ are treated as background fields. For convenience I impose the Fock–Schwinger gauge condition on the background field, $y^\mu A_\mu(y) = 0$ (for a pedagogical course on this gauge and its uses see [7]).† In this gauge $A_\mu(y) = \frac{1}{2}y^\rho\,G_{\rho\mu}(0) + \ldots$. Now, we contract the quark lines (34.9) to form the quark Green's function $S(x-\varepsilon,\,x+\varepsilon)$ in the background field and then get ‡

Chiral anomaly

‡ The widely used dimensional regularization is awkward and inappropriate in those problems in which γ^5 is involved.

† This gauge condition is not obligatory, of course. Although it is convenient, one can work in any other gauge; the final result is gauge independent.

$$
\begin{aligned}
\partial^\mu j_\mu^{A,\,R} &= -ig\,N_f\,\mathrm{Tr}_{\mathrm{C,L}}\left\{-2i\,\varepsilon^\rho\,G_{\rho\mu}(0)\,\gamma^\mu\gamma^5\,S(x-\varepsilon,\,x+\varepsilon)\right\} \\
&= -N_f\,\frac{g^2}{2}\,G_{\rho\mu}(0)^a\,\tilde{G}_{\alpha\phi}(0)^a\,\frac{\varepsilon^\rho\varepsilon^\alpha}{\varepsilon^2}\,\frac{1}{8\pi^2}\,\mathrm{Tr}_{\mathrm{L}}\left(\gamma^\mu\gamma^5\gamma^\phi\gamma^5\right) \\
&= \frac{N_f\,g^2}{16\pi^2}\left\{G^{\alpha\beta\,a}\,\tilde{G}^a_{\alpha\beta}\right\}_{\mathrm{bckgd}}, \qquad (34.11)
\end{aligned}
$$

where

$$
\tilde{G}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\rho\mu}G^{\rho\mu}\,, \qquad (34.12)
$$

and the subscripts C and L mark the traces over the color and Lorentz indices, respectively. The most crucial point is that the Green's function $S(x-\varepsilon,\,x+\varepsilon)$ is used *only* at very short distances $\sim \varepsilon \to 0$, where it is reliably known in the form of an expansion in the background field. We need only the first nontrivial term in this expansion (the Fock–Schwinger gauge),

$$
S(x,\,y) = \frac{1}{2\pi^2}\frac{\not{r}}{(r^2)^2} - \frac{1}{8\pi^2}\frac{r^\alpha}{r^2}\,g\,\tilde{G}_{\alpha\phi}(0)\,\gamma^\phi\gamma^5 + ..., \qquad r = x-y\,. \qquad (34.13)
$$

In passing from the second to the third line in Eq. (34.11) I averaged over the angular orientations of the four-vector ε.

34.1.2 The Pauli–Villars regularization

Parallelizing our two-dimensional studies in Section 33.7 we introduce the Pauli–Villars fermion regulators R with mass M_R to be send to infinity at the very end. Then the regularized singlet axial current takes the form

$$
j_\mu^{A,\,R} = \bar{\psi}_f\,\gamma_\mu\gamma^5\,\psi^f + \bar{R}_f\,\gamma_\mu\gamma^5\,R^f\,. \qquad (34.14)
$$

Since the current is now regularized, its divergence can be calculated according to the equations of motion,

$$
\partial^\mu\,j_\mu^{A,\,R} = 2i\,M_R\,\bar{R}_f\gamma^5 R^f\,. \qquad (34.15)
$$

As was expected, the result contains only the regulator term. Our next task is to project it onto "our" sector of the theory in the limit $M_R \to \infty$. In this limit only the two-gluon operator will survive, as depicted in the triangle diagram of Fig. 34.1. This diagram can be calculated either by the standard Feynman graph technique, or using the background field method [7], which

‡ If you have difficulties in reproducing (34.11), a step-by-step derivation can be found on p. 609 in [7].

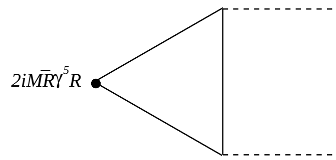

Fig. 34.1. Diagrammatic representation of the triangle anomaly. The solid and dashed lines denote the regulator and gluon fields, respectively.

is quite straightforward in the case at hand,

$$
\begin{aligned}
2i\,M_R\,\bar{R}_f\gamma^5 R^f \;&\to\; 2i\,M_R N_f \mathrm{Tr}_{\mathrm{C,L}}\left(\gamma^5\,\frac{i}{i\not{\mathcal{D}}-M_R}\right)\\
&\to\; -2\,M_R N_f \mathrm{Tr}_{\mathrm{C,L}}\left[\gamma^5\,\frac{1}{(i\mathcal{D})^2-M_R^2+\frac{ig}{2}\,G_{\mu\nu}\sigma^{\mu\nu}}\,(i\not{\mathcal{D}}+M_R)\right].
\end{aligned}
\tag{34.16}
$$

Here I omitted an extra minus sign which would have been necessary if it were an ordinary fermion loop. Given that the triangle loop in Fig. 34.1 is that of the regulator fields, the extra minus sign must not be inserted. The term $i\not{\mathcal{D}}$ in the last brackets can be dropped because of the trace with γ^5. Remembering that $M_R\to\infty$ one can expand the denominator in $G\sigma$. The zeroth-order term in this expansion vanishes for the same reason. The term $O(G\sigma)$ vanishes because of the color trace. The term $O((G\sigma)^2)$ does not vanish, while all higher order terms are suppressed by positive powers of $1/M_R$ and disappear in the limit $M_R\to\infty$. In this way we arrive at

$$
\begin{aligned}
2i\,M_R\,\bar{R}_f\gamma^5 R^f \;&\to\; \frac{M_R^2\,g^2}{2}\,N_f\,\mathrm{Tr}_{\mathrm{C}}\,(G_{\mu\nu}G_{\alpha\beta})\,\mathrm{Tr}_{\mathrm{L}}\left(\gamma^5\sigma^{\mu\nu}\sigma^{\alpha\beta}\right)\\
&\times\;\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{(p^2-M_R^2)^3}\,.
\end{aligned}
\tag{34.17}
$$

which, in turn, implies that

$$
\partial^\mu\, j_\mu^A = N_f\,\frac{g^2}{16\pi^2}\;G^{\alpha\beta\,a}\,\tilde{G}^a_{\alpha\beta}\,,
\tag{34.18}
$$

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in full accord with the result (34.11) obtained in the Schwinger regularization. Characteristic distances saturating the triangle loop in Fig. 34.1 are of the order of $M_R^{-1}\to 0$ at $M_R\to\infty$.

34.1.3 The chiral anomaly for generic fermions

What changes occur in the chiral anomaly if instead of the fundamental representation we consider fermions in some other representation R? The answer to this question is simple. If we inspect derivations in Sections 34.1.1 and 34.1.2 we will observe that the result for the anomalous divergence of the axial current is proportional to $\mathrm{Tr}T^aT^b$. For the fundamental representation in SU(N)

$$\mathrm{Tr}T^aT^b = \frac{1}{2}\delta^{ab}\,.$$

In the general case

$$\mathrm{Tr}T^aT^b = T(R)\delta^{ab}\,,$$

See Eq. (56.4) and Table 10.3

where $T(R)$ is (one half) of the Dynkin index for the given representation. Thus, if we have N_f massless Dirac fermions in the representation R, then Eq. (34.18) must be replaced by the following formula:

$$\partial^\mu\left(\bar\psi_f\gamma_\mu\gamma^5\psi_f\right) = N_f\,\frac{T(R)\,g^2}{8\pi^2}\;G^{\alpha\beta\,a}\,\tilde G^a_{\alpha\beta}\,. \tag{34.19}$$

For instance, for the adjoint representation in SU(N) one has T(adj)$=N$. Note that for the real representations, such as the adjoint, one can consider not only Dirac fermions, but Majorana fermions as well. Each Majorana fermion counts as $N_f=\frac{1}{2}$. The same is true with regards to the Weyl fermions with which one has to deal in chiral Yang–Mills theories.

34.2 Introducing external currents

What does that mean? Assume that we study QCD. Then our dynamical gauge bosons are gluons. However, typically, we have a number of color-singlet conserved *vector* currents that can be "gauged" too. These vector currents correspond to *global* symmetries. One can couple these currents to "external" *nondynamical* gauge bosons. One can think of them as of gauge bosons of a weakly coupled theory whose dynamics can be ignored. The axial currents which were anomaly-free can (and typically will) acquire anomalies with regards to these external nondynamical gauge bosons.

For example, the currents j^a_μ given in Eq. (34.7) are conserved. Gauging the global SU$(N_f)_V$ symmetry we introduce auxiliary vector bosons $A^{\mu\,a}$ with the coupling $j^a_\mu A^{\mu\,a}$. Now, the divergence of $j^{5,\,a}_\mu$ which was anomaly-free in QCD *per se*, will acquire the $F\tilde F$ term, with F's built from the above auxiliary vector bosons $A^{\mu\,a}$.

To further illustrate the point in the most graphic way, let us assume

$N_f = 2$. Then ψ is a two-component column in the flavor space, while three generator matrices are in fact the Pauli matrices (up to a normalizing factor 1/2). The background gauge fields are $A^{\mu\,1,2,3}$, or, alternatively, $A^{\mu\,3}$ and $A^{\mu\,\pm}$. The current j^B_μ in (34.3) is conserved too. Therefore, we can also introduce an external field A_μ with the coupling $A_\mu\,\bar\psi_f\gamma^\mu\psi_f$. Another possible alternative is to gauge the electromagnetic interaction, in addition to $A^{\mu\,a}$. Then we will have a photon (which is an external gauge boson with regards to QCD) interacting with the current $\frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d$. The latter current is a linear combination of the isotriplet and isosinglet,

$$j^{\rm em}_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d = \frac{1}{2}\left(\bar u\gamma_\mu u - \bar d\gamma_\mu d\right) + \frac{1}{6}\left(\bar u\gamma_\mu u + \bar d\gamma_\mu d\right)\,. \quad (34.20)$$

To distinguish the photon field from other external gauge bosons, temporarily (in this section) we will denote it by $\mathcal{A}^\mu$. Then the interaction takes the form $e\mathcal{A}^\mu\,j^{\rm em}_\mu$.

It is instructive to study this simple example further and to derive the anomaly in the $j^{5,\,a}_\mu$ currents. Keeping in mind a particularly important application (to be discussed shortly) we will limit ourselves to the neutral component, which we will denote by a^μ,

$$a_\mu \equiv j^{5,\,(a=3)}_\mu = \frac{1}{2}\left(\bar u\gamma_\mu\gamma^5 u - \bar d\gamma_\mu\gamma^5 d\right). \quad (34.21)$$

Third component (in the isospace) of the flavor axial current defined in (34.7)

We will have to analyze the same graph as previously (Fig. 34.1), with the regulator fields for the u and d quarks. They carry exactly the same quantum numbers of those of the u and d quarks. The only difference is that the regulator loop, as usual, has the opposite sign.† It is obvious that the current a^μ is anomaly-free in QCD *per se* since the triangle loops with the u and d quark regulators exactly cancel each other. Including the external photons with the interaction $e\mathcal{A}^\mu\,j^{\rm em}_\mu$, which obviously distinguishes between u and d, will ruin the cancellation.

In fact, we do not have to do the full computation anew. All we have to do is to reevaluate the diagram in Fig 34.1 with the external gluons replaced by photons. Starting from Eq. (34.18) derived in Section 34.1.2 we must take into account the difference in the vertex factors in this triangle graph. First, we will deal with the color factors, $N = 3$. While in (34.18), with the gluon background field, we used ${\rm Tr}_{\rm C}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab}$, in the case of the photon background field we replace this by ${\rm Tr}_{\rm C}\,1 = N$. Next, in the u loop we replace $g \to Q_u\,e$ and in the d loop $g \to Q_d\,e$. (Here $Q_u = 2/3$ and

† This is, certainly, in addition to the requirement of taking the regulator masses in the limit $M_R = \infty$ at the very end.

$Q_d = -1/3$.) As a result,

$$N_f\, g^2 \to \frac{1}{2}\left(Q_u^2 - Q_d^2\right) e^2\,, \tag{34.22}$$

where the factor $\frac{1}{2}$ is due to $\frac{1}{2}$ in the definition (34.21). Assembling all factors together we arrive at

$$\partial_\mu a^\mu = \frac{\alpha}{4\pi}\, N\left(Q_u^2 - Q_d^2\right) \mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} \tag{34.23}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$. Generalization to other external currents is straightforward.

Studying anomalies in the presence of external currents provides us with a precious tool for uncovering subtle aspects of strong dynamics at large distances, as we will see momentarily.

34.3 Longitudinal part of the current

Under certain circumstances one can reconstruct from (34.23) the longitudinal part of the current [9, 10]. Let us separate the longitudinal and transverse parts of a^μ,

$$a^\mu \equiv a^\mu_{||} + a^\mu_\perp\,, \qquad \partial_\mu a^\mu_\perp = 0\,. \tag{34.24}$$

It is clear that (34.23), viewed as an equation for the current, says nothing about $a^\mu_\perp$. However, it imposes a constraint on $a^\mu_{||}$, which allows one to unambiguously determine $a^\mu_{||}$ under appropriate kinematical conditions. Namely, assume that the photons in (34.23) are produced with the momenta $k^{(1)}$ and $k^{(2)}$ and are on mass shell, i.e.

$$\left[k^{(1)}\right]^2 = 0\,, \quad \left[k^{(2)}\right]^2 = 0\,. \tag{34.25}$$

The total momentum transferred from the current a^μ to the pair of photons is $q_\mu = k^{(1)}_\mu + k^{(2)}_\mu$ (Fig. 34.2). Then

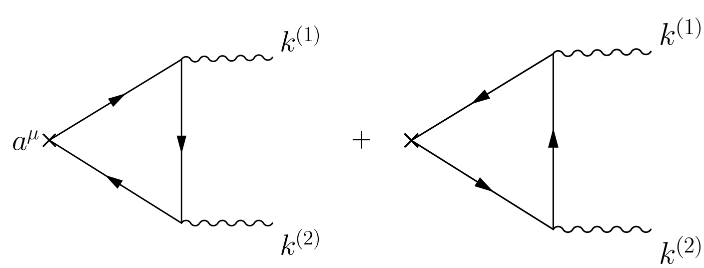

Fig. 34.2. Anomaly in a^μ.

$$\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} \longrightarrow -2\times 2\times \varepsilon^{\mu\nu\alpha\beta}\, k_\mu^{(1)}\epsilon_\nu^{(1)}\, k_\alpha^{(2)}\epsilon_\beta^{(2)}\,. \tag{34.26}$$

Here $\epsilon_\mu^{(1,2)}$ is the polarization vector of the first or second photon. The first factor of 2 in (34.26) comes from combinatorics: one can produce the first photon either from the first $\mathcal{F}_{\mu\nu}$ tensor or the second. Gauge invariance with regards to the external photons is built in in our regularization.

The statement following from (34.23) *and* (34.25) is as follows [9, 10]: for on-mass-shell photons the two-photon matrix element of $a_{||}^\mu$ is determined unambiguously,

$$\langle 0|\, a_{||}^\mu\, |2\gamma\rangle = i\,\frac{q^\mu}{q^2}\,\frac{\alpha}{\pi}\,N\left(Q_u^2 - Q_d^2\right)\varepsilon^{\mu\nu\alpha\beta}\, k_\mu^{(1)}\epsilon_\nu^{(1)}\, k_\alpha^{(2)}\epsilon_\beta^{(2)}\,. \tag{34.27}$$

This result is exact and is valid for any value of q^2, in particular, at $q^2\to 0$. I would like to emphasize the emergence of the pole $1/q^2$, with far-reaching physical consequences. Note that the gluon anomaly in the singlet axial current (see Eq. (34.19)) does not imply the existence of the pole in $a_{||}^\mu$ at $q^2\to 0$ because one cannot put gluons on shell — the condition (34.25) crucial for the derivation of (34.27) cannot be met.

That (34.27) is the solution to (34.23) is obvious. The fact that it is the *only* possible solution is less obvious. I refer the reader to [9, 10] for a comprehensive proof.

Exercises

34.1 Consider two-dimensional CP(1) model with fermions presented in Section 55.3.4. Find the anomaly in the divergence of the axial current $\bar\psi\gamma^\mu\gamma^5\psi$. Can it be called the triangle anomaly?

35 't Hooft Matching and its Physical Implications

In this section we will turn to physical consequences. We will start from a general interpretation of the pole in (34.27) and similar anomalous relations for other currents, formulate the 't Hooft matching condition, prove (at large N) spontaneous breaking of the global $\mathrm{SU}(N_f)_A$ symmetry and, finally, calculate the $\pi^0\to 2\gamma$ decay width.

35.1 Infrared matching

Poles do not appear in physical amplitudes for no reason. In fact, the only way an amplitude can acquire a pole is through massless particles in the spectrum of the theory which are coupled to the external currents under consideration. There are two possible scenarios: (i) the global axial symmetry is spontaneously broken (it would be more exact to say that it is realized nonlinearly); (ii) linear realization with massless spin-1/2 fermions.

In the first case massless Goldstone bosons appear in the physical spectrum. They must be coupled to $j_\mu^{5,a}$ and external vector gauge bosons. Equation (34.27) or similar equations for other currents present a constraint on the product of the Goldstone boson couplings which can always be met.

The second scenario is more subtle and, apparently, is rather exotic. It is true that the triangle loop (Fig. 34.2) with massless spin-1/2 fermions yields $\frac{q^\mu}{q^2}$ in the longitudinal part $a_{||}^\mu$ of the axial current [9, 10]. However, not only the kinematic factor $\frac{q^\mu}{q^2}$ is exactly predicted by anomaly, the coefficient in front this factor is known *exactly* too. For instance, in the example of Section 34.3, this coefficient is $\frac{\alpha}{\pi} N \left(Q_u^2 - Q_d^2\right)$. For the chiral symmetry to remain unbroken, the massless spin-1/2 (composite) fermions that might be potential contributors to the triangle loop must exactly reproduce this coefficient, which, generally speaking, is a highly nontrivial requirement. The search for massless spin-1/2 fermions which could match the coefficient in front of $\frac{q^\mu}{q^2}$ in $a_{||}^\mu$ is the celebrated 't Hooft matching procedure [10]

Needless to say, if free massless N-colored quarks existed in the spectrum of asymptotical states, they would automatically provide the required matching.† Alas ... quark confinement implies the absence of quarks in the physical spectrum. The only spin-1/2 fermions we deal with in QCD are composite baryons.

35.2 Spontaneous breaking of the axial symmetry

Let us see whether or not we can match (34.27) with the baryon contribution. We will put $N = 3$, as in our world, and consider first $N_f = 2$. Then the lowest-lying spin-1/2 baryons are proton and neutron (p and n), with the electric charges charges $Q_p = 1$ and $Q_n = 0$, respectively. Hence, only p contributes in the triangle loop in Fig. 34.2. If it were massless, it would generate a formula repeating (34.27) with the substitution

$$N\left(Q_u^2 - Q_d^2\right) \to Q_p^2 \,. \tag{35.1}$$

† In all theories strongly coupled in the infrared the only proper way of obtaining $a_{||}^\mu$ in the form (34.27) is the ultraviolet derivation through the external anomaly. However, if we pretend to forget all correct things about QCD and just blindly calculate the triangle loop of Fig. 34.2 with *noninteracting* massless quarks, we would get exactly the same formula. I hasten to add that this coincidence acquires a meaning only in the context of the 't Hooft matching. Feynman diagrams, in particular, that in Fig. 34.2, saturated in the infrared, have no meaning whatsoever in QCD-like theories.

The right- and left-hand sides in Eq. (35.1) are equal! Thus, in this particular case the 't Hooft matching does not rule out the linearly realized axial SU(2) symmetry with massless baryons p and n. This may be an accidental coincidence, though. Therefore, let us not make hasty conclusions and try to examine the stability of the above matching.

To this end we add the third quark, s, keeping intact the axial current to be analyzed, see (34.21). The electromagnetic current (34.20) acquires the additional term $-\frac{1}{3}\bar{s}\gamma_\mu s$. The anomaly-based prediction (34.27) remains intact.

In the theory with u, d and s quarks the lowest-lying spin-1/2 baryons form the baryon octet

$$B = (p,\, n,\, \Sigma^\pm,\, \Lambda, \Sigma^0,\, \Xi^-,\, \Xi^0). \tag{35.2}$$

If both the vector and axial SU(3) flavor symmetries are realized linearly, the baryon-baryon-photon coupling constants and the constants $\langle B|a^\mu|B\rangle$ at zero momentum transfer are unambiguously determined from the baryon quantum numbers (for instance, $\langle\Sigma^+|a^\mu|\Sigma^+\rangle = \bar{\Sigma}\gamma^\mu\gamma^5\Sigma$). Calculating the triangle diagram of Fig. 34.2 (more exactly, its longitudinal part) we find that the baryon octet does *not* contribute there due to cancellations: the proton contribution (the quark content uud) is canceled by that of Ξ^- (the quark content ssd) while the Σ^- contribution (the quark content dds) is canceled by Σ^+ (the quark content uus). Other baryons from (35.2) are neutral and decouple from the photon. The absence of matching seemingly tells us that that global $\mathrm{SU}(3)_A$ symmetry must be spontaneously broken.

Although the above argument is suggestive, it it is still inconclusive. It tacitly assumes that baryons with other quantum numbers, e.g. $J^P = \frac{1}{2}^-$, are irrelevant in the calculation of $a^\mu_{||}$, which need not be the case. How can one prove that the combined contribution of all baryons cannot be equal to (34.27)?

To answer this question let us explore the N dependence in Eq. (34.27). The anomaly-based calculation naturally produces the factor N on the right-hand side. At the same time, saturating the triangle loop by baryons at large N the linear dependence on N cannot be obtained [11]: each baryon loop is suppressed exponentially, as e^{-N}, since each baryon consists of N quarks. This observation proves that the global $\mathrm{SU}(N_f)_A$ symmetry must be spontaneously broken at least in the multicolor limit. As a result, $N_f^2 - 1$ massless Goldstone bosons (pions) emerge in the spectrum. Note that this argument is inapplicable to the singlet axial current (see the remark at the end of Section 34.3). The singlet pseudoscalar meson need not be massless.

Consult Section 38

Caveat: To my mind, the above assertion of the exponential suppression *of the baryon loops has the status of a "physical proof" rather than mathematical theorem. It is intuitively natural, indeed. However, in the absence of the full dynamical solution of Yang–Mills theories at strong coupling, one cannot completely rule out exotic scenarios in which the loop expansion in* $1/N$ *(implying* e^{-N} *for baryons) is invalid, see [12]. I do think that this expansion is valid in QCD per se. Doubts remain concerning models with more contrived fermion sectors. Note that in two dimensions examples of baryons defying formal* $1/N$ *expansion for are known.*

35.3 Predicting the $\pi^0 \to 2\gamma$ decay rate

If the global $\mathrm{SU}(N_f)_A$ symmetry is realized nonlinearly, through the Goldstone bosons (which for two flavors are called pions), saturation of the anomaly-based formula (34.27) is trivial (Fig. 35.1). The pole in $a^\mu_{||}$ is due to the pion contribution. The constraint (34.27) provides us with a relation

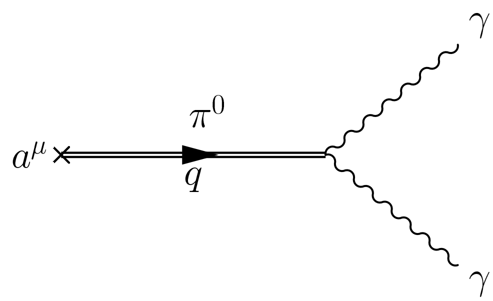

Fig. 35.1. The pion saturation of the anomaly.

between the $a^\mu \to \pi^0$ amplitude and the $\pi^0 \to 2\gamma$ coupling constant. The result is known from the 1960s. For completeness I will recall its derivation.

The $\pi^0 \to \gamma\gamma$ amplitude can be parametrized as

$$A(\pi^0 \to 2\gamma) = F_{\pi 2\gamma}\, \mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} \to -4\, F_{\pi 2\gamma}\, k^{(1)}_\mu \epsilon^{(1)}_\nu\, k^{(2)}_\alpha \epsilon^{(2)}_\beta\, \varepsilon^{\mu\nu\alpha\beta} \tag{35.3}$$

where I use the same notation as in Sections 34.2 and 34.3. Moreover, the amplitude $\langle 0|a^\mu|\pi^0\rangle$ is parametrized by the constant f_π playing the central role in the pion physics,

Pion constant f_π

$$\langle 0|a^\mu|\pi^0\rangle = \frac{1}{\sqrt{2}} i\, f_\pi\, q_\mu\,, \qquad f_\pi \approx 130 \text{ MeV}\,. \tag{35.4}$$

Then the pion contribution to the matrix element in the left-hand side of

Eq. (34.27) is

$$\langle 0|\, a_{||}^{\mu}\, |2\gamma\rangle = i\,\frac{q^{\mu}}{q^2}\,\frac{f_\pi}{\sqrt{2}}\,4\,F_{\pi 2\gamma}\,\varepsilon^{\mu\nu\alpha\beta}\,k_\mu^{(1)}\epsilon_\nu^{(1)}\,k_\alpha^{(2)}\epsilon_\beta^{(2)}\,. \tag{35.5}$$

Comparing with (34.27) we arrive at the following formula:

$$F_{\pi 2\gamma} = \frac{N}{2\sqrt{2}}\,\frac{1}{f_\pi}\,\frac{\alpha}{\pi}\left(Q_u^2 - Q_d^2\right) \to \frac{1}{2\sqrt{2}\,f_\pi}\,\frac{\alpha}{\pi}\,. \tag{35.6}$$

This is in good agreement with experiment.

Before the advent of QCD people did not know about color; the factor $N = 3$ was omitted from the prediction (35.6). In fact, the analysis of the $\pi^0 \to \gamma\gamma$ decay was one of a very few quantitative proofs of existence of color in the early 1970s.

Exercises

35.1 Assume the number of colors to be large, and you try to saturate the triangle graph in Fig. 34.2 by baryons. What N_c dependence would you expect?

36 Scale anomaly

In this section I will briefly discuss the scale anomaly in Yang–Mills theories. For simplicity I will limit myself to pure Yang–Mills, i.e. without matter,

$$S = \int d^4x\;\frac{-1}{4g_0^2}\,G_{\mu\nu}^a\,G^{\mu\nu\,a}\,, \tag{36.1}$$

where the subscript 0 marks the bare coupling constant. At the classical level the action (36.1) is obviously invariant under the scale transformations

$$x \to \lambda^{-1}\,x\,, \qquad A_\mu^a \to \lambda\,A_\mu^a\,, \tag{36.2}$$

where λ is an arbitrary real number. Barring subtleties (see Appendix 1A), the scale invariance of the theory with any local Lorentz invariant Lagrangian implies the full conformal symmetry [13]. Roughly speaking, scale-invariant theories contain only *dimensionless* constants in the Lagrangian (otherwise, the action would not be invariant under the scale transformations). Then, the conformal invariance of the action is quite clear, at least, at the intuitive level.

Look through Appendix 1A

The scale transformations are generated by the current [13]

$$j_\nu^D = x^\mu\,\theta_{\mu\nu}\,, \tag{36.3}$$

where $\theta_{\mu\nu}$ is the symmetric and conserved energy-momentum tensor of the theory under consideration. For instance, in pure Yang–Mills theory (36.1)

$$\theta_{\mu\nu} = -\frac{1}{g^2}\left(G^a_{\mu\alpha}\, G^{\alpha\, a}_{\nu} - \frac{1}{4} g_{\mu\nu}\, G^a_{\alpha\beta}\, G^{\alpha\beta\, a}\right). \tag{36.4}$$

The classical scale invariance of (36.1) implies that the current j^D_ν is conserved, $\partial^\nu j^D_\nu = 0$. Indeed,

$$\partial^\nu j^D_\nu = \theta^\mu_\mu\,, \tag{36.5}$$

and the trace of the energy-momentum tensor (36.4) obviously vanishes, $\theta^\mu_\mu = 0$.

The vanishing of θ^μ_μ is valid only at the *classical* level. At the quantum level θ^μ_μ acquires an anomalous part. I will derive this (scale) anomaly at one loop. Unlike the chiral anomaly, we do not have to deal with γ^5 here; therefore, the simplest derivation is based on dimensional regularization. Namely, instead of considering the action (36.1) in four dimensions, we will consider it in $4-\epsilon$ dimensions where $\epsilon \to 0$ at the very end. In $4-\epsilon$ dimensions $\int d^{4-\epsilon}x\, G^{\,2}_{\mu\nu}$ is not scale invariant. A change of $\int d^{4-\epsilon}x\, G^{\,2}_{\mu\nu}$ under the scale transformation is proportional to ϵ, of course. One should not forget, however, that $1/g_0^2$, being expressed in terms of the renormalized coupling, also depends on ϵ. The latter dependence contains $1/\epsilon$. As a result, in the limit $\epsilon \to 0$, a finite term giving us noninvariance of (36.1) remains.

Concretely,

$$\begin{aligned} \delta\, S &= \int d^{4-\epsilon}x \left\{-\frac{1}{4}\left(\frac{1}{g^2} + \frac{\beta_0}{8\pi^2}\,\frac{1}{\epsilon}\right)(\lambda^\epsilon - 1)\, G^a_{\mu\nu} G^{\mu\nu\, a}\right\} \\ &\to \int d^4x\ \ln\lambda \left(-\frac{\beta_0}{32\pi^2}\; G^a_{\mu\nu} G^{\mu\nu\, a}\right) \end{aligned} \tag{36.6}$$

where $\beta_0 = \frac{11\,N}{3}$ is the first coefficient of the β function, cf. Eq. (3.8). Equation (36.6) immediately leads us to the conclusion that [14]

$$\theta^\mu_\mu = -\frac{\beta_0}{32\pi^2}\, G^a_{\mu\nu} G^{\mu\nu\, a}\,. \tag{36.7}$$

Anomaly in θ^μ_μ

This expression for θ^μ_μ remains valid even in the presence of massless fermions, although the value of β_0 changes, of course.

The scale anomaly formula (36.7) expresses the fact that, although the classical Yang–Mills action contains only dimensionless constants, a dynamical scale parameter Λ of dimension of mass is generated at the quantum level (this phenomenon is referred to as *dimensional transmutation*). All hadronic masses are proportional to Λ. The expectation value of $G_{\mu\nu}^2$ over a given hadron is proportional to the mass of this hadron [15] (in the chiral limit).