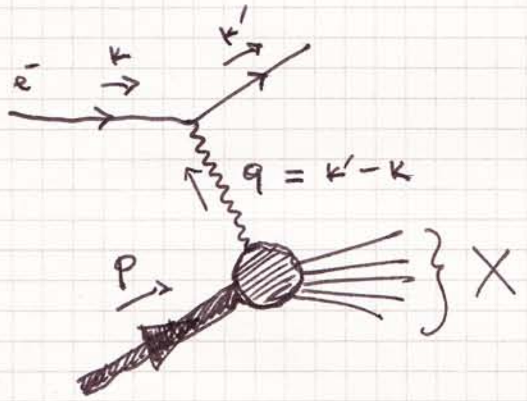
Deep inelastic scattering

The process in which an electron scatters off a proton where the energy and the momentum transfer is large enough to break the proton to fragments

$$e^- + P \to e^- + X$$

Only the electron is detected in the final state!



$$P^2 = m^2$$

$$\nu \equiv P \cdot q = m \cdot (E_{k'} - E_k)$$ @ the rest frame of the proton.

$$x \equiv \frac{Q^2}{2\nu} \qquad Q^2 \equiv -q^2$$

$$y = \frac{q \cdot p}{k \cdot p} = \frac{E'_k - E_k}{E_k}$$ @ rest frame of the proton

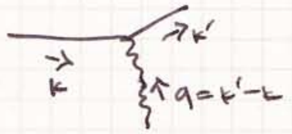
The differential cross-section for the process is

$$d\sigma = \frac{1}{\text{flux}} [dk'] [dX] \sum_{\substack{\text{spins}\\ \text{colors}}} \overline{|M|}^2 (2\pi)^4 \delta^4 (P + k - k' - X)$$

where $[dk'] = \frac{d^3 k'}{(2\pi)^3 2E_{k'}}$ $\quad [dX] = \prod_j \frac{d^3 p_j}{(2\pi)^3 2E_j}$ where the sum extends to all particles that are undetected.

→ We want to separate the contributions in $\overline{|M|}^2$ that are due to the (known and easy) lepton-photon interaction from those of the (unknown and difficult) proton-photon interaction.

The matrix element piece of the lepton-photon interaction is

just 

$= \bar{u}_{\lambda'}(k')\, \gamma^\mu\, u_\lambda(k) \;\dfrac{(-g_{\mu\nu})}{q^2}\,(ie)$

$$\equiv \frac{\tilde{L}_\nu}{q^2}\,(ie)$$

The hadronic piece is 

$\equiv \tilde{W}^\nu \; ie\, e_q$ where e_q is the charge of the proton constituent that actually interacts. The four-vector W^ν includes all the information on how exactly the proton-photon interaction takes place.

Squaring, we get $\quad d\sigma = \dfrac{1}{\text{flux}} \dfrac{(e^2 e_q)^2}{Q^4} \, L_{\mu\nu}\, W^{\mu\nu} \,[dk'] \cdot 4\pi$

$$\text{with} \quad W^{\mu\nu} = \frac{1}{2\cdot 4\pi} \int [dX] \sum_\lambda W^\mu W^{\nu *} (2\pi)^4 \delta^4(X-q-P) \quad (*)$$

It is easy to show that $\quad L_{\mu\nu} = \dfrac{1}{2}\, 4\,(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k\cdot k')$

→ Thanks to the QED Ward identity (which ensures charge conservation) we know that the general form of the hadronic tensor, $W^{\mu\nu}$, has to be

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right) W_2 \quad (**)$$

Contracting $L_{\mu\nu}$ with $W^{\mu\nu}$ and substituting back in the equation for $d\sigma$ we get :

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi}{x}\,\frac{\alpha^2 e_q^2}{Q^4}\left[xy^2 F_1 + \left(1-y-\frac{(xy\,m)^2}{Q^2}\right)F_2\right]$$

with $F_1 \equiv \frac{W_1}{x}$ $\qquad F_2 \equiv \frac{\nu W_2}{x}$

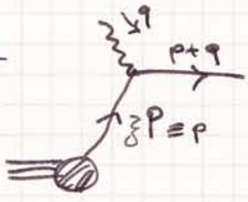
The functions F_1, F_2 are called the <u>structure functions</u> of the proton, and contain all information on the hadronic piece.

→ The Bjorken limit is the kinematical limit

$$\left.\begin{array}{l} Q^2 \to \infty \\ \nu \to \infty \end{array}\right\} \quad x = \frac{Q^2}{2\nu} \quad \text{fixed}$$

The structure functions F_1, F_2 in general depend both on x and on Q^2. It is an important experimental observation that as the Bjorken limit is approached, $F_1(x, Q^2)$, $F_2(x, Q^2)$ depend only on x and <u>not</u> on Q^2. This behavior is called "Bjorken scaling".

→ The "naive parton model" was historically the first attempt to construct a theory that predicts Bjorken scaling. It assumes that of all the constituents of the proton, only one quark, carrying a fraction of the proton momentum, ξ, interacts with the probing photon.

Then the hadronic piece is just 

$iee_q\, \bar{u}(p+q)\gamma_\nu u(q)$

and it's easy to show that $\sum_{\text{spins}} w_\mu w_\nu^* = 4e^2 e_q^2 \left[(p+q)^\mu p^\nu + p^\mu (p+q)^\nu - g^{\mu\nu}\, p\cdot(p+q) \right]$

(eq. **)

Note: From the general form of $W^{\mu\nu}$, it is clear that we can define two scalar quantities $W = -g_{\mu\nu} W^{\mu\nu}$ and $WP = P_\mu P_\nu W^{\mu\nu}$

Then
$$W = 3W_1 - BW_2$$
$$WP = -BW_1 + B^2 W_2$$
with $B = P^2 - \frac{(P\cdot q)^2}{q^2}$

from which
$$W_1 = \frac{1}{2}\left(W + \frac{WP}{B}\right)$$
$$BW_2 = \frac{1}{2}\left(W + 3\frac{WP}{B}\right)$$

From $W^{\mu\nu}$'s definition (eq. *) we can see that, for the naive parton model,

$$W = \frac{1}{2\cdot 4\pi} \int [dX] \sum_\lambda \left(-w^\mu w_\mu^* \right) (2\pi)^4 \delta^4 (X - q - p)$$

parton → p, Proton → P

with $[dX] = \frac{d^3 p'}{(2\pi)^3 2E_p'} = \frac{d^4p'\,\delta(p'^2)}{(2\pi)^3}$, $p' = p+q = \xi P + q$ $\quad (p = \xi P)$

We have $\sum_\lambda -w^\mu w_\mu^* = 8e^2 e_q^2\, \xi\, \nu$ (<u>assuming massless quarks!</u>)

The phase-space integral is performed with the help of $\delta^4(p' - q - p)$ and, since momentum is conserved, $p'^2 = (p+q)^2$
$$= (\xi - x)\, 2P\cdot q$$

So $\delta(p'^2) = \delta(\nu(\xi - x)) = \frac{1}{\nu}\delta(\xi - x)$ and we get

$$W = \frac{1}{8\pi} \frac{(2\pi)^4}{(2\pi)^3} \frac{1}{\nu} \delta(\xi - x) \; 8 e^2 e_q^2 \xi \nu$$

$$= e^2 e_q^2 \; \xi \, \delta(\xi - x)$$

For W_P we get immediately zero since $p^2 = 0$.

This means $\quad W_1 = \frac{1}{2} W = B W_2$

and $\quad B = p^2 - \frac{(p\cdot q)^2}{q^2} = -\frac{p\cdot q}{q^2} \cdot p\cdot q = \frac{2 p\cdot q}{Q^2} \frac{p\cdot q}{2} = \frac{\nu}{2x}$

So $\quad W_1 = \frac{\nu W_2}{2x}$ and for the structure functions

$$F_1 = \frac{F_2}{2x} \quad \text{or} \quad \boxed{F_2 = 2x F_1} \quad \text{Callan-Gross relation.}$$

Actually $\quad F_1 = \frac{1}{2} \frac{W}{\cancel{x}} = \frac{e_q^2}{2} \delta(x - \xi)$

$$F_2 = 2x F_1 = x \, e_q^2 \, \delta(x - \xi).$$

As promised, $F_{1,2}$ are independent of Q^2 in the naive parton model, hence, Bjorken Scaling is predicted!

$F_{1,2}$ are <u>partonic</u> structure functions: we have assumed that the scattering takes place with a single quark of momentum $p = \xi P$. In reality we have to assume that there is a probability distribution $q(\xi)$ for the quark to have a fraction between ξ and $\xi + d\xi$ of the proton momentum, and sum over all values of ξ from zero to one:

$$F_2^p(x) = \int_0^1 d\xi\; q(\xi)\; F_2(x) = x\, e_q^2\, q(x)$$

Moreover, the proton has many different quarks, so one has to sum over all possibilities

$$F_2^p(x) = x \sum_{q_i} q_i(x)\, e_{q_i}^2$$

→ Measuring F_1, F_2 from DIS experiments gives information on $q_i(x)$.

The proton has two u-quarks and one d-quark, plus a "sea" of u, d, s, c and $\bar{u}, \bar{d}, \bar{s}, \bar{c}$ quarks and maybe a $b, \bar{b}$

The sum of the up-type distributions

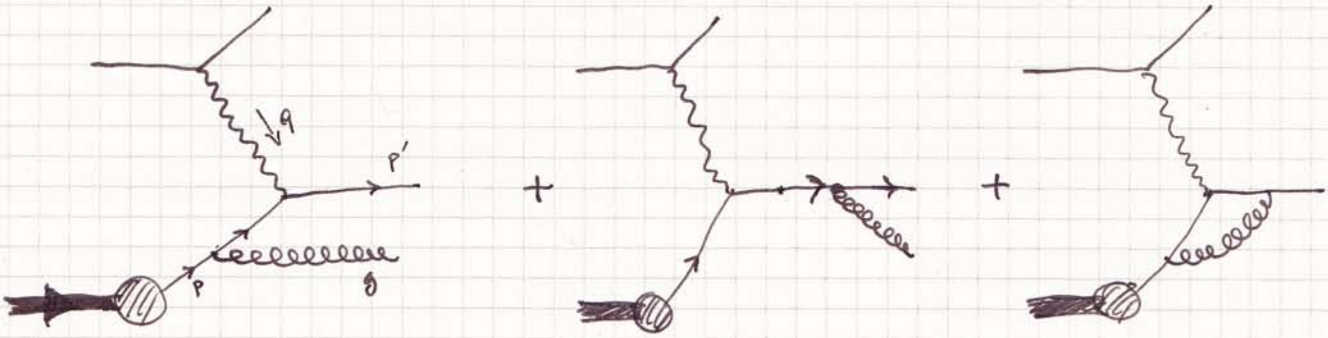
$$\int_0^1 dx\;\; u(x) = 2 \quad , \quad \int_0^1 dx\;\; d(x) = 1$$

whereas the sum of the <u>momentum</u> distributions (<u>note</u> that if $P(\xi)$ is the quark distribution, $\int P(\xi)\, d\xi$ measures the average number of the quark, whereas $\int \xi \cdot P(\xi)\, d\xi$ measures the average momentum for this quark type) should be equal to one. However

$$\int_0^1 dx \;\; x\; (u + d + \text{sea}) \approx 0.5 \qquad \text{experimentally.}$$

This suggests strongly that there is another, missing, constituent of the proton!

The above derivation and its consequences are practically the Leading Order approximation to the DIS cross-section. At Next-to-Leading order one gets further contributions

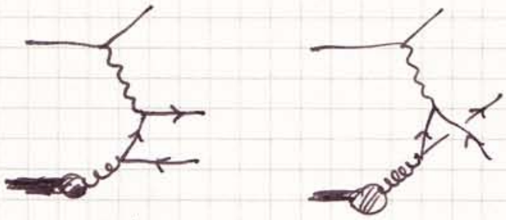


(a) initial state gluon emission

(b) final state gluon emission

(c) virtual contribution

There are also



(d)

(e)

Let us look at the contribution from the square of (a)+(b)

We have $-g^{\mu\nu} \sum W_\mu^{(a+b)} W_\nu^{(a+b)*} = 8\, e^2 e_q^2\, g_s^2 \; Tr(t^a t^a) \left[\frac{g \cdot p'}{g \cdot p} + \frac{g \cdot p}{g \cdot p'} + Q^2 \frac{p \cdot p'}{g \cdot p \; g \cdot p'} \right]$

We immediately realise that we have to face the singularity structure:

$$g \cdot p \rightarrow 0 \quad \Rightarrow \quad E_g \cdot E_p \cdot (1 - \cos\theta_{gp}) \rightarrow 0 \qquad \text{(initial state singul.)}$$

$$g \cdot p' \rightarrow 0 \quad \Rightarrow \quad E_g \cdot E_{p'} \, (1 - \cos\theta_{g \cdot p'}) \rightarrow 0 \qquad \text{(final state singul.)}$$

We'll focus in the initial state singularity

The singularity is aproached when

$E_g \to 0$, with $|\cos\theta| < 1$ soft

$\cos\theta_{gq} \to 0$ with $E_g > 0$ collinear

$E_g \to 0$ and $\cos\theta_{gq} \to 0$ soft-collinear.

Assuming that the initial quark carries a fraction ξ of the proton momentum, we can write the "center-of-momentum" energy for the process as

$$s = (p+q)^2 = 2p\cdot q + q^2 = 2\xi P\cdot q - Q^2 = 2\xi P\cdot q \frac{Q^2}{Q^2} - Q^2 = Q^2\left(\frac{\xi}{x} - 1\right)$$

We define $z = \frac{x}{\xi}$ and have $s = Q^2 \frac{1-z}{z}$

Note that when the gluon is soft $E_g = |\vec{P}_g| \to 0$, $s = (p+q)^2 = (p'+g)^2 = 0$ and $z \to 1$, $x \to \xi$. These are tree-level kinematics. We therefore expect no problem from the cancellation of soft singularities between initial-state real emission and virtual diagrams (here (c)).

The same is true for final state singularities, both collinear and soft, (diagram (b)) since then $s = (p'+g)^2 = 2p'\cdot g \to 0$ and again $z \to 1$, $x \to \xi$.

The problem appears at the initial state collinear singularity, i.e when the transverse momentum of the gluon is approaching zero.

Let's see the scalar part of $W^{\mu\nu}$

$$W^{(a+b)} = -g_{\mu\nu} W^{(a+b)\,\mu\nu} = \frac{1}{2\cdot 4\pi} \int [dX] \; \sum W_{\mu}^{(a+b)} W^{\mu\,(a+b)*} (2\pi)^4 \, \delta^4(X-q-p)$$

with $[dX] = [dp'] \, [dg]$

when we try to integrate over the gluon momenta, writting

$$\frac{d^3 g}{(2\pi)^3 2E_g} = \frac{dg_z \; d(g_T^2) \, d\theta}{(2\pi)^3 \; 4E_g}$$

we get $$W^{(a+b)} \sim e_q^2 \frac{\alpha_s}{2\pi} C_F \int_0^{\frac{Q^2(1-z)}{4z}} \frac{dg_T^2}{g_T^2} \cdot \ldots$$

If we cut off the lower edge of the integral (thus regulating it) we immediately have

$$W^{(a+b)} \sim e_q^2 \frac{\alpha_s}{2\pi} C_F P_{qq}(z) \log \frac{Q^2}{k^2} + \text{finite}$$

where $P_{qq}(z)$ is the coefficient multiplying the logarithmicly divergent term.

Instead of that, we use dimensional regularization to find

$$W^{(a+b)} = e_q^2 \frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi}{Q^2}\right)^{\varepsilon} (1-\varepsilon) \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \cdot \Big\{$$

$$\left(\frac{2}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{7}{2}\right) \delta(1-z) - \frac{1}{\varepsilon} \frac{1+z^2}{(1-z)_+} +$$

$$+ (1+z^2) \left[\frac{\log(1-z)}{1-z}\right]_+ - \frac{1+z^2}{1-z} \log z - \frac{3}{2} \frac{1}{(1-z)_+} + 3 - z + \mathcal{O}(\varepsilon) \Big\}$$

Calculating the virtual piece (c) one gets

$$W^{(c)} = e_q^2 (1-\varepsilon) \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{4\pi}{Q^2}\right)^{\varepsilon} \frac{\alpha_s}{4\pi} C_F \cdot (-1) \cdot \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 + \frac{\pi^2}{3} \right) \delta(1-z)$$

So upon adding the contributions one gets

$$W = e_q^2 \frac{\alpha_s}{2\pi} C_F (1-\varepsilon) \left\{ \frac{1}{\varepsilon} \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right) \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{4\pi}{Q^2}\right)^{\varepsilon} + \text{finite} \right\}.$$

The coefficient $P_{qq}^{(0)}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]$ of the collinear divergence is called the Altarelli - Parisi splitting function.

* Note: diagrams (d) and (e) contribute to another splitting function, $P_{qg}^{(0)}$, ~~a~~relevant for the subprocess $e^- g \to e^- q\bar{q}$ (instead of $e^- q \to e^- q g$). We will see ~~these~~ it at ~~the~~ the next lecture.

* Note: The splitting function is a distribution that multiplies the remaining singularity in the structure functions (and hence in the DIS cross-section) <u>after</u> soft$^{(*)}$ and final-state soft/collinear poles are cancelled. The remaining, initial state collinear singularity and its treatment will be the topic of the next lecture.

(*) note that there are no $1/\varepsilon^2$ poles in the sum of virtual (c) plus real (a+b). This is completely generic at NLO QCD.