$\boldsymbol{P}$, $\boldsymbol{C}$ and $\boldsymbol{T}$

Paolo Nason, lecture notes

Parity

Parity flips the sign of position and momentum. It thus leaves angular momentum invariant (i.e. it commutes with the rotation operator), and, for consistency, it must also leave the spin invariant. By applying the parity operator twice we should get back the original state, which means

$$P^2 = e^{i\phi}, \tag{1}$$

where ϕ is a constant phase. The phase must be constant, since if we have two different states $|a\rangle$ and $|b\rangle$ such that

$$P^2|a\rangle = e^{i\phi_a}|a\rangle, \quad P^2|b\rangle = e^{i\phi_b}|b\rangle, \text{ then } P^2[|a\rangle + |b\rangle] \neq e^{i\phi}[|a\rangle + |b\rangle] \text{ for any } \phi. \tag{2}$$

We can always redefine $P \to e^{-i\phi/2}P$, so that $P^2 = 1$, that we will assume from now on. In non-relativistic quantum mechanics, a single particle with spin s is represented by a ket $|x, s_z\rangle$. If we define

$$P|0, s\rangle = p_s|0, s\rangle, \text{ with } p_s = \pm 1, \tag{3}$$

using the rotation operator, we have

$$R|0, s\rangle = \sum_{m=-s,s} A^R_{s,m}|0, m\rangle, \tag{4}$$

where A is the matrix representation of the rotation group for the spin s. Applying the parity operator we get

$$PR|0, s\rangle = RP|0, s\rangle = p_s R|0, s\rangle = \sum_{m=-s,s} A^R_{s,m} p_m|0, m\rangle, \tag{5}$$

that, together with equation (4), and with the assumption of the irreducibility of the spin representation, implies $p_s = p_m$ for all m. We thus define $p = p_m$ to be the intrinsic parity of the particle.

Intrinsic parity of the pion

There is freedom in the assignment of an intrinsic parity to the particles. If an assignment exists such that the Hamiltonian is parity invariant, our system has parity symmetry. Of course, intrinsic parity plays a rôle only in reactions where the number of particles of a given species changes. If there are other symmetries in our system, parity should commute with them. Thus, we assign the same parity to the particles belonging to the same symmetry multiplet.

In the 1954 paper "*Absorption of Negative Pions in Deuterium: Parity of the Pion*", William Chinowsky and Jack Steinberger demonstrated that the pion has negative parity. They studied the decay of an "atom" made from a deuterium nucleus d and a π^- in a state with zero orbital angular momentum into two neutrons n

$$d\pi^- \to nn. \tag{6}$$

The π^- mass is 139.57 MeV, and its lifetime is $2.603 10^{-8}$ seconds. We can expect a relativistic π^- to travel of the order of 3×2.6 meters before it decays, not including the γ factor. At a speed of the order of $\alpha = 1/137$ it can certainly travel several orbits around the deuteron nucleus, i.e. it has the time to form a mesic atom. The charged π beams are easily produced in proton nucleus collisions (they are in fact the most abundant specie being produced).

The deuteron has spin one. It is made up by a proton and a neutron in an S wave state, their spins being aligned. If we assume that proton and neutron, being related by isospin symmetry, have the same parity, their total intrinsic parity is the same in the initial and final state, and is thus irrelevant. Forgetting thus about the proton and neutron intrinsic parity, the parity of the initial state is just given by the intrinsic parity of the pion. In fact, both the neutron-proton and the pion are in an S wave state, so their spatial wavefunction is invariant under parity.

The final state must have spin 1. If the spin wavefunction of the $2n$ system is antisymmetric (spin zero), the spatial wave function should be symmetric, corresponding to even L and thus even J. Thus, the spin state must be symmetric, since the initial total angular momentum, being one, is odd. Then L must be odd by Fermi statistics, which yields odd parity. One then concludes that the parity of the π^- is negative.

Parity violation

The $K^\pm$ has a mass of 493.7 MeV, and a lifetime of $1.24 10^{-8}$ seconds. It is produced in proton nucleus collisions. The following decays are observed:

	mode	Branching fraction
K^+	$\pi^+\pi^0$	$(20.66 \pm 0.08)\%$
K^+	$\pi^+\pi^0\pi^0$	$(1.761 \pm 0.022)\%$
K^+	$\pi^+\pi^+\pi^-$	$(5.59 \pm 0.04)\%$

(7)

Assuming that the K^+ has spin zero, the final state amplitude must be independent on the final state momenta in the two body decay. It must therefore be even under parity, since the π^+ and π^0 have the same intrinsic parity. In the 3-body decay, the amplitude must be a function of rotational invariants. Calling $\vec{k}_1$ and $\vec{k}_2$ the momenta of two pions (the third one is $-\vec{k}_1 - \vec{k}_2$), the only invariants we can form are bilinear in $\vec{k}_1$ and $\vec{k}_2$. The amplitude $A(\vec{k}_1, \vec{k}_2)$ is thus invariant under parity:

$$A(\vec{k}_1, \vec{k}_2) = A(-\vec{k}_1, -\vec{k}_2). \tag{8}$$

Since the intrinsic parity of the pion is -1, it follows that the final state must have negative parity. Thus, parity must be violated in the K^+ decay.

In the early 1950's, parity was considered an exact symmetry. Thus people believed that the particles decaying into two and three pions were two different particles, and were called τ and θ at that time. However, as experiments were refined, it looked like the τ and θ had exactly the same mass and lifetime, and the same production characteristics. In other words, they looked like the same particle. This was named the τ-θ puzzle. The puzzle was better pinned down by the work of Dalitz (1954), who established that the particle decaying into 3 pions had spin zero. We notice that the $\pi^+\pi^+\pi^-$ decay should be easily studied in a spectrometer. A detector capable of measuring the $\pi^\pm$ trajectories in a magnetic field can extract their momenta. The study is perfomed by creating a Dalitz plot of the decay. The three body phase space for the decay is given by

$$\begin{aligned} d\Phi_3 &= \frac{d^3\vec{k}_1}{2k_1^0(2\pi)^3}\frac{d^3\vec{k}_2}{2k_2^0(2\pi)^3}\frac{d^3\vec{k}_3}{2k_3^0(2\pi)^3}(2\pi)^4\delta(q-k_1-k_2-k_3) \\ &= \frac{d^3\vec{k}_1}{2k_1^0(2\pi)^3}\frac{d^3\vec{k}_2}{2k_2^0(2\pi)^3}\frac{\pi}{k_3^0}\delta(M-k_1^0-k_2^0-k_3^0) \\ &= \frac{dk_1^0\,k_1}{2(2\pi)^3}d\Omega_1\frac{dk_2^0\,k_2}{2(2\pi)^3}d\Omega_2\frac{\pi}{k_3^0}\delta(E_3-k_3^0), \end{aligned} \tag{9}$$

where $E_3 = M - k_1^0 - k_2^0$. We rewrite

$$\frac{1}{k_3^0}\delta(E_3 - k_3^0) = 2\delta(E_3^2 - k_3^{0 2}), \tag{10}$$

and use

$$k_3^{0 2} = m_3^2 + (k_1^2 + k_2^2 + 2\cos\theta_{12}\, k_1 k_2). \tag{11}$$

Taking the Ω_2 angular integral relative to particle 1, we get $d\Omega_2 = d\varphi_{12}\, d\cos\theta_{12}$, and perform the $d\cos\theta_{12}$ integral with the delta function, which yields

$$\frac{\pi}{4(2\pi)^6}\, d\Omega_1\, d\varphi_{12}\, dk_1^0\, dk_2^0, \tag{12}$$

thus the phase space is uniform in k_1^0, k_2^0. A point in phase space is represented as a point in the so called Dallitz plot, shown in figure

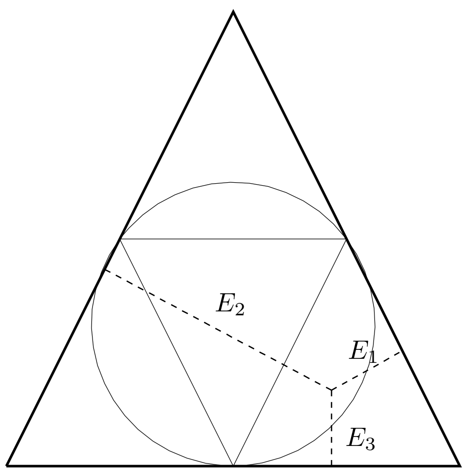

Figure 1. Dallitz plane. The inscribed triangle gives the available phase space in the case of massless partincles. The inscibed circle gives the available phase space in the case of non-relativistic particles with the same mass.

In order for the integral to be non vanishing, we must have

$$m_3^2 + (k_1^2 + k_2^2 + 2\, k_1 k_2) \geqslant E_3^2 \geqslant m_3^2 + (k_1^2 + k_2^2 - 2\, k_1 k_2) \tag{13}$$

corresponding to the triangular inequality

$$k_1 + k_2 \geqslant k_3 \geqslant |k_1 - k_2|. \tag{14}$$

The boundary of the region can also be written as

$$(k_3^2 - k_1^2 - k_2^2)^2 = 4k_1^2 k_2^2. \tag{15}$$

that in the non-relativistic limit is bilinear in the energies, and thus repsesents a conic section. In the case of three bodies with equal masses, because of 120^o rotation symmetry, it must correspond to a circle. It is represented by the circle inscribed in the large triangle in fig. 1. In the ultrarelativistic limit we have $E_i = k_i$, the triangular inequality becomes linear in the energies, and it describes the inside of the upsidedown triangle in the figure. Notice that the boundary of the triangle always correspond to $\cos\theta_{12} = \pm 1$, that is to say, the momenta of the particles lie all on the same line. In particular, if, for example, $E_3 = 0$, the configuration of the system corresponds to particles 1 and 2 having the same energy and opposite momenta, and particle 3 being at rest. In this case, the angular momentum of the system is that of the 1-2 system. Being a two particle system, the parity of its wavefunction is equal to $(-)^{L_{12}} = (-)^J$. The parity of the wavefunction in the two body decay must satisfies the same property, so that the parity in the two and three body decay is opposite by intrinsic parity only. Thus, if the $E_3 = 0$ region of the Dallitz plot is populated by events, the K^+ can decay into two systems with opposite parity, and thus parity is violated. In fact, the parity of the wavefunction is the same in two and three body decays (the latter at $E_3 = 0$), while the intrinsic parity is opposite. The same line of reasoning can be applied to the point at the maximum value of E_3. In this case $E_1 = E_2$, and particle 1-2 have equal momenta. The system is thus equivalent to particle 3 recoiling against particles 1-2 at relative rest, and thus is equivalent to a 2-body system with the parity of the wavefunction equal to $(-)^{L_{3(12)}} = (-)^J$.

Historically, because of the two different decay modes of the K^+, since parity was always assume to hold, it was inferred that there were two particles, named at that time τ and θ, with similar mass, and opposite parity, one decaying into the 2-pion system, and the other decaying into 3 pions. It was however found that also the lifetime of the two particles were very similar, a result in apparent contradiction with the widely different phase space for 2 or 3 pion decays. This was the so called τ-θ puzzle. In order to solve this puzzle Yang and Lee argued that parity may be violated. They realized that there was no experimental evidence of parity invariance in weak interactions, and thus proposed experiments to test for parity invariance in β decay.

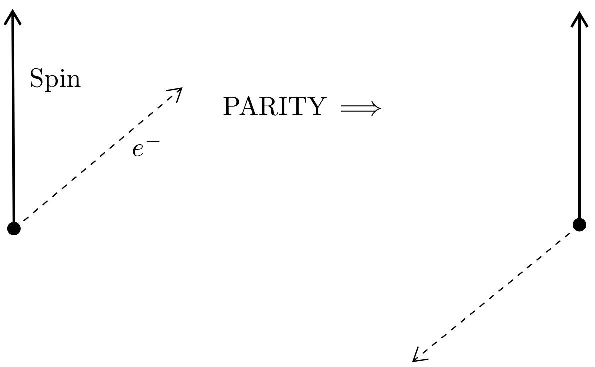

Figure 2. Parity in the beta decay of a polarized nucleus. The two decay configurations in the figure should have the same rate if parity is conserved.

The experiment of Madsame Wu (Chien Shiung Wu) with Ambler, Hayward, Hobble and Hudson, in 1957 showed that in the beta decay of Cobalt 60, the electron comes preferably antialign with the initial spin of the nucleus.

Parity for Dirac fields

Parity changes a left-handed spinor into a right handed one. Since the Dirac field is made up of a left-handed and a right handed spinor, parity should exchange them, while also changing $x \to -x$. We thus assume

$$P\psi(x)P^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi(-x) = \gamma^0\psi(-x). \tag{16}$$

We now study the change of a generic fermion bilinear under parity. We have

$$P\bar{\psi}(x)\Gamma\psi(x)P^\dagger = (\gamma^0\psi(-x))^\dagger\gamma^0\Gamma\gamma^0\psi(-x) = \bar{\psi}(-x)\gamma^0\Gamma\gamma^0\psi(-x). \tag{17}$$

Thus, the $\bar{\psi}\,\psi$ bilinear is even, while $\bar{\psi}\,\gamma^5\psi$ is odd under parity. Furthermore, $\bar{\psi}\,\gamma^\mu\psi$ changes sign in all its space components

$$P\bar{\psi}(x)\gamma^\mu\psi(x)P^\dagger = (-)^\mu\,\bar{\psi}(-x)\gamma^\mu\psi(-x). \tag{18}$$

The axial current transforms with the opposite sign

$$P\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)P^\dagger = -(-)^\mu\,\bar{\psi}(-x)\gamma^\mu\gamma^5\,\psi(-x), \tag{19}$$

while the antisymmetric tensor

$$P\bar{\psi}(x)\,[\gamma^\mu,\gamma^\nu]\psi(x)P^\dagger = (-)^\mu(-)^\nu\,\bar{\psi}(-x)\,[\gamma^\mu,\gamma^\nu]\psi(-x). \tag{20}$$

Notice that the Lagrangian remains invariant

$$L = \int d^3x\,\bar{\psi}\,(i\partial\!\!\!/ - m)\psi, \tag{21}$$

the minus sign in the spatial components of the gamma matrices being compensated by the transformation $\partial_x \to -\partial_{-x}$, before the overall relabeling $x \to -x$. If the electromagnetic interaction is included $e\,\bar{\psi}A\!\!\!/\psi$, invariance requires that the space part of the electromagnetic vector changes sign, as must be the case.

Charge conjugation

By charge conjugation, we mean changing sign to all the charges of the system. The charge of a quantum field is related to a phase transformation of the field. Thus, charge conjugation should send a quantum field int its hermitean conjugate. A Dirac spinor is composed by a left handed spinor and a right handed spinor with the same charge. Their Hermitean conjugate changes their behaviour under Lorents transformation. In order for the transformed Dirac field to conserve its Lorents transformation properties, charge conjugation should also exchange the upper and lower spinor, as follows

$$\chi_\alpha \to (\zeta_{\dot{\alpha}})^*, \quad \zeta^{\dot{\beta}} \to (\chi^\beta)^*. \tag{22}$$

So

$$C\psi(x)C^\dagger = \begin{pmatrix} 0 & \epsilon_{\alpha\beta} \\ \epsilon^{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix}\psi^*(x) = -i\gamma^2\psi^*(x). \tag{23}$$

Let us find the effect of C on a bilinear $\bar{\psi}_a \Gamma \psi_b$, where Γ is a combination of gamma matrices. We have

$$C\,\bar{\psi}_a \Gamma \psi_b\, C^\dagger = C\,(\gamma^0 \psi_a)^\dagger\, \Gamma \psi_b\, C^\dagger = (\gamma^0(-i\gamma^2)\psi_a^*)^\dagger\, \Gamma(-i\gamma^2)\psi_b^*, \tag{24}$$

where we have used $CA^\dagger C^\dagger = (CAC^\dagger)^\dagger$. This equals

$$C\,\bar{\psi}_a \Gamma \psi_b\, C^\dagger = \psi_a^T(-i\gamma^2)\gamma^0\Gamma(-i\gamma^2)\psi_b^* = \psi_a^T\gamma^2\gamma^0\Gamma\gamma^0\gamma^2\gamma^0\psi_b^*, \tag{25}$$

where we have used the property $(\gamma^0)^\dagger = \gamma^0$, $\vec{\gamma}^{\,\dagger} = -\vec{\gamma}$, together with $(\gamma^0)^2 = 1$ and the anticommutation properties of the γ. Transposing eq. (25) we get

$$C\,\bar{\psi}_a \Gamma \psi_b\, C^\dagger = -\bar{\psi}_b \gamma^2\gamma^0\Gamma^T\gamma^0\gamma^2\psi_a. \tag{26}$$

Here we have used the fact that γ^0 is real, so $(\gamma^0)^T = \gamma^0$, and γ^2 is imaginary, so

$$(\gamma^2)^T = -(\gamma^2)^\dagger = 1, \tag{27}$$

and we have added a $-$ sign, because transposition changes the order of two fermionic operators. We get immediately:

$$C\bar{\psi}_a\psi_b\, C^\dagger = \bar{\psi}_b\psi_c,\ \ C\bar{\psi}_a\gamma^5\psi_b\, C^\dagger = \bar{\psi}_b\gamma^5\psi_a, \tag{28}$$

which follows from the fact that γ^5 is symmetric and anticommutes with $\gamma^0\gamma^2$. Also

$$\gamma^2\gamma^0(\gamma^\mu)^T\gamma^0\gamma^2 = \gamma^\mu. \tag{29}$$

This is because transposition changes sign to γ^1 and γ^3, and leaves γ^0 and γ^2 unchanged. Taking γ^0 and γ^2 across γ^1 and γ^3 changes the sign two more time, and γ^2 squared yields another -1, so γ^1 and γ^3 remain the same. On the other hand, γ^0 and γ^2 change sign two times, one for taking across γ^2 and γ^0 respectively, and the other from γ^2 squared. Thus

$$C\bar{\psi}_a\gamma^\mu\psi_b\, C^\dagger = -\bar{\psi}_b\gamma^\mu\psi_c\,. \tag{30}$$

We also have

$$C\bar{\psi}_a\gamma^\mu\gamma^5\psi_b\, C^\dagger = \bar{\psi}_b\gamma^\mu\gamma^5\psi_c\,. \tag{31}$$

The sign difference from the previous case is due to the fact that

$$(\gamma^\mu\gamma^5)^T = \gamma^5(\gamma^\mu)^T, \tag{32}$$

and from the reordering γ^5. Also

$$C\bar{\psi}_a[\gamma^\mu,\gamma^\nu]\psi_b\, C^\dagger = -\bar{\psi}_b[\gamma^\mu,\gamma^\nu]\psi_c\,, \tag{33}$$

since transposition changes the order of $\gamma^\mu\gamma^\nu$.

The action

$$\int d^4x\,\bar{\psi}\,(i\partial\!\!\!/ - m)\psi \tag{34}$$

is also invariant, since the change of sign of the γ^μ bilinear is compensated by the change of sign due to partial integration, needed to have the derivative to act on the right field after transposition. Observe that, introducing electromagnetic interaction, we must have $CA_\mu C^\dagger = -A_\mu$ in order for charge conjugation to be preserved.

Parity of charge conjugate spinors

Notice that, denoting with $\psi_c = C\psi C^\dagger$, we have

$$P\psi_c(x)P^\dagger = P(-i\gamma^2)\psi^*(x)P^\dagger = -i\gamma^2\,\gamma^0\psi^*(-x) = -\gamma^0(-i\gamma^2)\psi^*(-x) = -\gamma^0\psi_c(-x). \tag{35}$$

This is often stated by saying that fermion and antifermion fields have opposite intrinsic parity.

Positronium

The electron-positron bound state, can be safely treated as a non-relativistic system. The ground state must be an eigenstate of parity and charge conjugation. Parity requires that the angular momentum is either even or odd. Since charge conjugation exchanges the electron and the positron, is is also affected by the angular momentum being even or odd, but also by the spin wavefunction being either symmetric or antisymmetric. Thus, the states must have definite spin symmetry properties, if they have to be eigenstates of both parity and charge conjugation.

The S wave ground state may have the particle antiparticle pair in the spin singlet or triplet state. That is to say, calling $|s_1,\, s_2\rangle$ the electron-positron spin wavefunction, it can be in the combination

$$\frac{1}{\sqrt{2}}[|+,-\rangle - |-,+\rangle], \tag{36}$$

or in a combination of the three symmetric states

$$|+,+\rangle,\quad \frac{1}{\sqrt{2}}[|+,-\rangle + |-,+\rangle], |-,-\rangle. \tag{37}$$

Parity does not change the spin. Thus, in both cases, the parity of the system is given by its intrinsic parity, which is -1. On the other hand, charge conjugation interchanges the electron and the positron. The exchange of two fermions must bring about a change of sign. Furthermore, the antisymmetric spin state also changes sign in the spin wavefunction. As a consequence of this, for positronium decay into n photons, for the spin 1 state (orthopositronium) n must be odd, while for the spin 0 state (parapositronium) n must be even.

Notice that the π^0 is also a pseudoscalar. Its decay into two photons let us conclude that it must be even under C. Its phenomenological coupling to the nucleons has the form

$$\pi^0\bar{\psi}\, i\gamma^5\psi, \tag{38}$$

that preserves parity. It also preserves C if π^0 is assigned a positive C.

It is instructive to ask ourselves wether a 2γ decay of the parapositronium (or of the π^0) is allowed by parity conservation. In order to do this, we should study the parity of the 2γ system. This is not a non-relativistic system, and thus cannot be dealt with with by separating the spin and spatial (or momentum) wavefunction. The trick it to use helicity eigenstates. Helicity is in fact a rotational invariant, and it is also invariant by longitudinal boosts in the direction of motion of the particle. This is a consequence of the fact that the Lorentz boost generator along a direction commute with the generator of rotations around the same direction. Calling q the momentum of a particle along the z direction, we can introduce its helicity eigenstates

$$|(0,0,q),\lambda\rangle, \tag{39}$$

where $\lambda = -s, ..., s$, s being the spin of the particle, and λ its spin component along the z axis in its rest frame. We have

$$e^{-iJ_z\theta}|(0,0,q),\lambda\rangle = e^{-i\lambda\theta}|(0,0,q),\lambda\rangle, \tag{40}$$

which is immediately proved using the fact that boosts and rotations along z commute. We can define our helicity basis by rotating our basic vector

$$|q,\theta,\phi,\lambda\rangle = |(q\sin\theta\cos\phi, q\sin\theta\sin\phi, q\cos\theta),\lambda\rangle = e^{-iJ_z\phi}e^{-iJ_y\theta}e^{iJ_z\phi}|\vec{q_z},\lambda\rangle. \tag{41}$$

Since the helicity commutes with the angular momentum, we can also find angular momentum representations with fixed helicity. Their matrix elements with our momentum helicity states are easily obtained

$$\langle J,M,\lambda|q,\theta,\phi,\lambda\rangle = \langle J,M,\lambda|e^{-iJ_z\phi}e^{-iJ_y\theta}e^{iJ_z\phi}|q,\lambda\rangle = \sum_{M'} D^J_{M,M'}(\theta,\phi)\langle J,M',\lambda|q,\lambda\rangle \tag{42}$$

But M' must equal λ, so we get

$$\langle J,M,\lambda|q,\theta,\phi,\lambda\rangle = D^J_{M,\lambda}(\theta,\phi)\langle J,\lambda,\lambda|q,\lambda\rangle. \tag{43}$$

We do not bother here to find the normalization of the states. The important thing is that the wavefunction is given by rotation matrices for a representation of spin J.

The same reasoning can be applied to a system of two particle in a two-body decay, the two particle traveling in opposite direction. In this case we would heave found $D^J_{M,\lambda_1-\lambda_2}$ for the rotation matrices.

Now we go back to our two photon system, from the decay of a spin zero particle. Since $J=0$, we must have $D^J_{M,\lambda_1-\lambda_2} = \delta_{M,0}\,\delta_{\lambda_1,\lambda_2}$. Thus, the wave function in terms of helicity eigenstates has no spatial dependence, and the helicity of the two photons must be equal. We thus have two available states, that satisfy naturally Bose statistics:

$$|+,+\rangle \quad \text{and} \quad |-,-\rangle. \tag{44}$$

Parity flips the helicities. So, we can construct a negative parity state:

$$|+,+\rangle - |-,-\rangle. \tag{45}$$

Thus, the two photon system in positronium decay do not have definite polarization. The constitute a typical example of an untangled state. If, at any distance from the decay point, the helicity of one photon is measured, one immediately knows the outcome of an eventual helicity measurement performed on the photon in the opposite direction.

Time reversal

Time reversal must be described by an antiunitary operator. It leaves the space coordinate invariant and it flips the sign of momentum. It thus also flip the sign of angular momentum. It must be antiunitary, since otherwise the sign of the commutator $[q,p]=i$, under $q\to q$ and $p\to -p$ would change. In non-relativistic quantum mechanics of spinless particles, time reversal is easily represented by the complex conjugation operator K. For particles with spin, the operator K is no longer enough, since it does not flip all the spin components. In fact, let us consider a particle of spin j. The phase convention for the eigenstates of s_z is such that

$$s^-|m\rangle = N^-_m|m-1\rangle, \qquad s^+|m\rangle = N^+_m|m+1\rangle, \tag{46}$$

where $N_m^{\pm}$ are real, positive normalization factor. Thus, $s^{\pm}$ and s_z are all represented by real matrices in this basis. Since

$$s^{\pm} = \frac{1}{\sqrt{2}}[s_x \pm i\, s_y], \;\; s_x = \frac{1}{\sqrt{2}}\left(s^+ + s^-\right), \;\; s_y = \frac{-i}{\sqrt{2}}\left(s^+ - s^-\right) \tag{47}$$

it follows that s_x is also real, while s_y is purely immaginary. Thus, K changes sign to s_y only, but leaves s_x and s_z unchanged. We thus define the time reversal operator as

$$T = \exp[-i\pi\, s_y]K. \tag{48}$$

so that, by a π rotation around the y axis also s_x and s_z change sign. Observe that we have

$$T^2 = \exp[-i\pi\, s_y]K\exp[-i\pi\, s_y]K = \exp[-i2\pi s_y], \tag{49}$$

which is equal to -1 for odd spin, and 1 for even spin. In the case of spin 1/2, we have

$$\exp[-i\pi s_y] = -i\, s_y, \tag{50}$$

that is easily verify by checking its correctness for the ± 1 eigenvectors.

In relativistic theories, time reversal should flip the sign of momentum, but not the helicity. A single Weil spinor should thus transform as

$$\chi(k) \to \epsilon\chi(-k). \tag{51}$$

The need of the ϵ antisymmetric two by two matrix is needed to preserve the Hamiltonian

$$H = -\int \frac{d^3k}{(2\pi)^3}\,\chi^*(k)\,\vec{\sigma}\cdot\vec{k}\,\chi(k), \tag{52}$$

that, because of antiunitarity, goes into

$$THT^{\dagger} = -\int \frac{d^3k}{(2\pi)^3}\,\chi^*(-k)\epsilon^T\vec{\sigma}^*\cdot\vec{k}\,\epsilon\chi(-k) = -\int \frac{d^3k}{(2\pi)^3}\,\chi^*(-k)\,\vec{\sigma}\cdot(-\vec{k}\,)\chi(-k) = H. \tag{53}$$

Thus, as in the case of non-relativistic quantum mechanics, $\epsilon = i\sigma_y$ has the role, together with complex conjugation, of inverting the spin.

For a Dirac field we can write

$$T\psi(x)T^{\dagger} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\psi(x) = -\gamma^1\gamma^3\psi(x). \tag{54}$$

Its action on a generic fermionic bilinear is

$$T\bar{\psi}\,\Gamma\psi T^{\dagger} = T\psi^{\dagger}\gamma^0\Gamma\psi T^{\dagger} = T\psi^{\dagger}T^{\dagger}\,\gamma^0\Gamma^*\,T\psi T^{\dagger} = (-\gamma^1\gamma^3\psi)^{\dagger}\,\gamma^0\Gamma^*(-\gamma^1\gamma^3)\psi, \tag{55}$$

where we have used the relation $UA^{\dagger}U^{\dagger} = (UAU^{\dagger})^{\dagger}$. The use of this relation with antiunitary operators is easily shown to be consistent:

$$(u, UA^{\dagger}U^{\dagger}v) = (U^{\dagger}u, A^{\dagger}U^{\dagger}v)^* = (AU^{\dagger}u, U^{\dagger}v)^* = (UAU^{\dagger}u, v). \tag{56}$$

Thus

$$T\bar{\psi}\,\Gamma\psi T^{\dagger} = \bar{\psi}\,\gamma^3\gamma^1\Gamma^*\gamma^1\gamma^3\psi. \tag{57}$$

From this it follows immediately that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$ are even under T. Furthermore

$$T\bar{\psi}\gamma^\mu\psi T^\dagger = (-)^\mu \bar{\psi}\gamma^\mu\psi\,, \tag{58}$$

where $(-)^\mu$ is a short hand notation for: 1 for $\mu = 0$, -1 for $\mu = 1, 2, 3$. We also have

$$T\bar{\psi}\gamma^\mu\gamma^5\psi T^\dagger = (-)^\mu \bar{\psi}\gamma^\mu\gamma^5\psi\,, \tag{59}$$

and

$$T\bar{\psi}\,[\gamma^\mu, \gamma^\nu]\psi T^\dagger = (-1)^\mu (-1)^\nu\,\bar{\psi}\,[\gamma^\mu, \gamma^\nu]\psi\,. \tag{60}$$

The invariance of the Lagrangian

$$T\Big[\int \bar{\psi}\,(i\not{\partial} - m)\psi d^3x\Big]T^\dagger = \int \bar{\psi}\,(i\not{\partial} - m)\psi d^3x, \tag{61}$$

follows from the above properties, and from the change $\partial^0 \to -\partial^0$. Introducing the electromagnetic interaction, we infer that $A^\mu \to A^\mu(-)^\mu$ under time reversal, consistent with the fact that the electric field should not change, while the magnetic field should.

CPT properties of fermion bilinears

With $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$:

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CP	$+1$	-1	$-(-1)^\mu$	$-(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$(-1)^\mu$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

It is intended that, whenever C is involved, we have

$$\bar{\psi}_a\Gamma\psi_b \to \bar{\psi}_b\Gamma'\psi_a\,. \tag{62}$$