Soft Integrals

The phase space in the soft limit always factorizes as

$$d\Phi^{n+1} = d\Phi^n \frac{d^{d-1}l}{2l^0(2\pi)^{d-1}}\,. \tag{1}$$

We write now

$$d^{d-1}l = dl_1\,dl_2\,d^{d-3}l_\perp = dl_1\,dl_2\,dl_\perp\; l_\perp^{-2\epsilon}\,\Omega^{1-2\epsilon}, \tag{2}$$

where we have used $d = 4-2\epsilon$, and Ω^n is the solid angle in n dimension. From

$$\Omega^n = \frac{n\,\pi^{n/2}}{\Gamma(1+n/2)} = \frac{\pi^{n/2}\,2^n\,\Gamma\!\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma(n)} \Longrightarrow \Omega^{1-2\epsilon} = 2\frac{(4\pi)^{-\epsilon}\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{3}$$

Turning eq. (2) into polar coordinates we get

$$\frac{d^{d-1}l}{2l^0(2\pi)^{d-1}} = \frac{\pi^\epsilon\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{1}{(2\pi)^3}l^{1-2\epsilon}\,dl\;d\cos\theta\,d\phi\,(\sin\theta\sin\phi)^{-2\epsilon}\,. \tag{4}$$

Observe that $l_\perp$ is positive. Having defined

$$l_1 = \cos\theta,\;\; l_2 = \sin\theta\cos\phi,\;\; l_\perp = \sin\theta\sin\phi\,, \tag{5}$$

this means that $0<\phi<\pi$, and that only even quantities can be integrated in this way. In other words, $l_\perp$ should not be confused with l_3 (l_3 is no longer available at this stage). Inserting

$$l = \xi\frac{\sqrt{s}}{2}, \tag{6}$$

and multiplying by $g^2 = 4\pi\alpha\mu^{2\epsilon}$, we get

$$g^2\frac{d^{d-1}l}{2l^0(2\pi)^{d-1}} = \left[\frac{(4\pi)^\epsilon\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\right] s^{-\epsilon}\,\frac{\alpha\mu^{2\epsilon}}{2\pi}\,\frac{s}{4\pi}\,\xi^{1-2\epsilon}\,d\xi\;d\cos\theta\,d\phi\,(\sin\theta\sin\phi)^{-2\epsilon}\,. \tag{7}$$

This is to be multiplied by $\xi^{-2}[\xi^2 M^2]$, with $[\xi^2 M^2]$ having a finite limit as $\xi\to 0$. The ξ integration is performed by separating first

$$\xi^{-1-2\epsilon} = -\frac{\xi_c^{-2\epsilon}}{2\epsilon}\delta(\xi) + \left(\frac{1}{\xi}\right)_{\xi_c} - 2\epsilon\left(\frac{\log\xi}{\xi}\right)_{\xi_c}, \tag{8}$$

where the δ term yields the soft contribution, which is then given by

$$-\frac{1}{2\epsilon}\left[\frac{(4\pi)^\epsilon\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{\alpha}{2\pi}\,s^{-\epsilon}\,\xi_c^{-2\epsilon}\frac{\alpha\mu^{2\epsilon}}{2\pi}\frac{s}{4\pi}\int d\cos\theta\,d\phi\,(\sin\theta\sin\phi)^{-2\epsilon}\,[\xi^2 M^2], \tag{9}$$

or, collecting the normalization factor of (2.93) of FNR2006

$$\mathcal{N} = \frac{\alpha}{2\pi}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{Q^2}\right)^\epsilon,$$

we consider the integral

$$\mathcal{N}\left[1-\frac{\pi^2}{6}\epsilon^2\right]\left(\frac{Q^2}{s\;\xi_c^2}\right)^{\epsilon}\left(\frac{-1}{2\epsilon}\right)\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\left[\frac{s\xi^2}{4}M^2\right]. \tag{10}$$

We begin with an iconal factor for massless particles

$$I(k_1,k_2,l)=\frac{k_1\cdot k_2}{k_1\cdot l\;\;k_2\cdot l}\,, \tag{11}$$

and define

$$I(k_1,k_2)=\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\left[\frac{s\xi^2}{4}\frac{k_1\cdot k_2}{k_1\cdot l\;\;k_2\cdot l}\right], \tag{12}$$

$$I(k_1,k_2)=\frac{1}{\epsilon}I_d(k_1,k_2)+I_0(k_1,k_2)+\epsilon I_\epsilon(k_1,k_2)$$

We first expand

$$I(k_1,k_2,l)=\frac{k_1\cdot k_2}{k_1\cdot l\;\;(k_1+k_2)\cdot l}+\frac{k_1\cdot k_2}{k_2\cdot l\;\;(k_1+k_2)\cdot l}\,. \tag{13}$$

and define

$$I(k,m)=\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\left[\frac{s\xi^2}{4}\frac{k\cdot m}{k\cdot l\;\;m\cdot l}\right], \tag{14}$$

so that

$$I(k_1,k_1+k_2)+I(k_2,k_1+k_2)=\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\left[\frac{s\xi^2}{4}\frac{k_1\cdot k_2}{k_1\cdot l\;\;k_2\cdot l}\right]. \tag{15}$$

We have:

$$I(k,m)=\frac{1}{\epsilon}I_d(k,m)+I_0(k,m)+\epsilon I_\epsilon(k,m)\,. \tag{16}$$

We separate out the collinear component

$$\frac{k\cdot m}{k\cdot l\;m\cdot l}=\left[\frac{k\cdot m}{k\cdot l\;m\cdot l}-\frac{n\cdot k}{k\cdot l\;\;n\cdot l}\right]+\frac{n\cdot k}{k\cdot l\;\;n\cdot l}\,, \tag{17}$$

where the term in square bracket has no collinear singularities. Assuming n along the time direction, we have:

$$\frac{s\xi^2}{4}\;\frac{n\cdot k_1}{k\cdot l\;\;n\cdot l}=\frac{1}{1-\cos\theta}\,, \tag{18}$$

and

$$\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\,\frac{1}{1-\cos\theta}=\frac{-1}{\epsilon}\,, \tag{19}$$

so

$$I_d(k,m)=-1\,. \tag{20}$$

The remaining integral has no collinear singularities. Defining:We find

$$\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\,\frac{s\xi^2}{4}\left[\frac{k\cdot m}{k\cdot l\,m\cdot l}-\frac{k^0}{k\cdot l\;\,l^0}\right]=I_0(k,m)+\epsilon I_\varepsilon(k,m) \tag{21}$$

(with $k^2=0$ and $m^2>0$), one finds

$$I(k,m)=I_0(k,m)+\epsilon I_\varepsilon(k,m)\,, \tag{22}$$

were, defining $\hat{k}=k/k^0$, and $\hat{m}=m/m^0$, we have

$$I_0(k,m)=\log\frac{(\hat{k}\cdot\hat{m})^2}{\hat{m}^2}\,, \tag{23}$$

$$I_\epsilon(k,m)=-2\left[\frac{1}{4}\log^2\frac{1-\beta}{1+\beta}+\log\frac{\hat{k}\cdot\hat{m}}{1+\beta}\log\frac{\hat{k}\cdot\hat{m}}{1-\beta}+\mathrm{Li}_2\left(1-\frac{\hat{k}\cdot\hat{m}}{1+\beta}\right)+\mathrm{Li}_2\left(1-\frac{\hat{k}\cdot\hat{m}}{1-\beta}\right)\right], \tag{24}$$

with $\beta=\sqrt{1-m^2}$.
we have

$$\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\left[\frac{s\xi^2}{4}\frac{k_1\cdot k_2}{k_1\cdot l\ k_2\cdot l}\right]=I(k_1,k_1+k_2)+I(k_2,k_1+k_2) \tag{25}$$

So:

$$\left[1-\frac{\pi^2}{6}\epsilon^2\right]\left(\frac{Q^2}{s\,\xi_c^2}\right)^\epsilon\left(\frac{-1}{2\epsilon}\right)\int d\cos\theta\,\frac{d\phi}{\pi}\,(\sin\theta\sin\phi)^{-2\epsilon}\left[\frac{s\xi^2}{4}\frac{k_1\cdot k_2}{k_1\cdot l\,k_2\cdot l}\right]=\frac{A}{\epsilon^2}+\frac{B}{\epsilon}+C\,, \tag{26}$$

with

$$A = 1 \tag{27}$$

$$B = -\frac{1}{2}[I_0(k_1,k_1+k_2)+I_0(k_2,k_1+k_2)]+\log\frac{Q^2}{s\,\xi_c^2} \tag{28}$$

$$\begin{aligned}C = {} & -\frac{\pi^2}{6}+\frac{1}{2}\log^2\frac{Q^2}{s\,\xi_c^2}-\frac{1}{2}[I_0(k_1,k_1+k_2)+I_0(k_2,k_1+k_2)]\log\frac{Q^2}{s\,\xi_c^2}+\\ & -\frac{1}{2}[I_\epsilon(k_1,k_1+k_2)+I_\epsilon(k_2,k_1+k_2)]\,.\end{aligned} \tag{29}$$

In case k_1 is massless and k_2 is not, we get instead

$$A = \frac{1}{2} \tag{30}$$

$$B = -\frac{1}{2}\,I_0(k_1,k_2)+\frac{1}{2}\log\frac{Q^2}{s\,\xi_c^2} \tag{31}$$

$$C = -\frac{\pi^2}{12}+\frac{1}{4}\log^2\frac{Q^2}{s\,\xi_c^2}-\frac{1}{2}I_0(k_1,k_2)\log\frac{Q^2}{s\,\xi_c^2}-\frac{1}{2}I_\epsilon(k_1,k_2)\,. \tag{32}$$

In case both k_1 and k_2 are massive, we instead define

$$I(k_1,k_2)=I_0(k_1,k_2)+\epsilon I_\varepsilon(k_1,k_2)\,, \tag{33}$$

$$I_0(k_1,k_2)=\int d\cos\theta\,\frac{d\phi}{\pi}\left[\frac{s\xi^2}{4}\frac{k_1\cdot k_2}{k_1\cdot l\,k_2\cdot l}\right]=\log\frac{4(k_1\cdot k_2)^2}{k_1^2\,k_2^2}, \tag{34}$$

$$I_\epsilon(k_1,k_2)=-2\int d\cos\theta\,\frac{d\phi}{\pi}\log[\sin\theta\sin\phi]\left[\frac{s\xi^2}{4}\frac{k_1\cdot k_2}{k_1\cdot l\,k_2\cdot l}\right]. \tag{35}$$

and get

$$A = 0 \tag{36}$$

$$B = -\frac{1}{2}\,I_0(k_1,k_2) \tag{37}$$

$$C = -\frac{1}{2}I_0(k_1,k_2)\log\frac{Q^2}{s\,\xi_c^2}-\frac{1}{2}I_\epsilon(k_1,k_2)\,. \tag{38}$$