Maximum k_T in initial state radiation

Consider the underlying Born kinematics $\bar{x}_1$, $\bar{x}_2$, s_b, where s_b is the s invariant of the underlying Born. We work in the underlying Born rest frame, where

$$P_1\bar{x}_1 = P_2\bar{x}_2,\ s_b = (P_1\bar{x}_1 + P_2\bar{x}_2)^2 = 4P_1\bar{x}_1 P_2\bar{x}_2\,. \tag{1}$$

In the ISR configuration, the final state system with squared mass s_b acquires a transverse momentum k_T, and a massless parton is radiated with transverse momentum k_T. Energy momentum balance requires that

$$\begin{aligned} \sqrt{s_b + k_T^2} + \sqrt{k_L^2 + k_T^2} &= x_1 P_1 + x_2 P_2, & (2)\\ k_L &= x_1 P_1 - x_2 P_2\,. & (3) \end{aligned}$$

We solve these conditions as follows:

$$\sqrt{k_L^2 + k_T^2} = x_1 P_1 + x_2 P_2 - \sqrt{s_b + k_T^2}, \tag{4}$$

so we must require

$$x_1 P_1 + x_2 P_2 - \sqrt{s_b + k_T^2} > 0, \tag{5}$$

then we can square to get (setting $m_T = \sqrt{s_b + k_T^2}$)

$$k_L^2 + k_T^2 = (x_1 P_1 + x_2 P_2)^2 + s_b + k_T^2 - 2m_T(x_1 P_1 + x_2 P_2), \tag{6}$$

that, using eq. 3 yields

$$m_T = \frac{4x_1 x_2 P_1 P_2 + s_b}{2(x_1 P_1 + x_2 P_2)}\,. \tag{7}$$

Defining $y_1 = x_1/\bar{x}_1$, $y_2 = x_2/\bar{x}_2$, we get

$$\frac{m_T}{\sqrt{s_b}} = \frac{1 + y_1 y_2}{y_1 + y_2}\,. \tag{8}$$

The constraint in these variables yields

$$y_1 + y_2 \geqslant 2\frac{m_T}{\sqrt{s_b}} = 2\frac{1 + y_1 y_2}{y_1 + y_2}\,, \tag{9}$$

which immediately yields

$$y_1^2 + y_2^2 \geqslant 2\,. \tag{10}$$

A further constraint on y arises because $m_T/\sqrt{s_b} \geqslant 1$. This yields

$$\frac{1 + y_1 y_2}{y_1 + y_2} > 0 \Longrightarrow (1 - y_1)(1 - y_2) \geqslant 0\,. \tag{11}$$

that together with eq. 10 implies that both y_1 and y_2 must be larger than 1. Furthermore

$$\frac{\partial}{\partial y_1}\frac{1 + y_1 y_2}{y_1 + y_2} = \frac{y_2^2 - 1}{(y_1 + y_2)^2},\ \ \frac{\partial}{\partial y_2}\frac{1 + y_2 y_1}{y_2 + y_1} = \frac{y_1^2 - 1}{(y_1 + y_2)^2}\,; \tag{12}$$

thus m_T has positive derivatives with respect to y_1 and y_2, which means that its maximum is where both y_1 and y_2 reach thei maxima, respectively $1/\bar{x}_1$ and $1/\bar{x}_2$. Thus

$$\frac{m_T^{\max}}{\sqrt{s_b}} = \frac{\bar{x}_1\bar{x}_2 + 1}{\bar{x}_1 + \bar{x}_2} \Longrightarrow k_{T\,\max}^2 = s_b\left(1 - \left[\frac{\bar{x}_1\bar{x}_2 + 1}{\bar{x}_1 + \bar{x}_2}\right]^2\right) = s_b\frac{(1 - x_1^2)(1 - x_2^2)}{(\bar{x}_1 + \bar{x}_2)^2}\,. \tag{13}$$