Chiral symmetry breaking and the Banks–Casher relation on the lattice

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Based on: L. G. and M. Lüscher JHEP 0903 (2009) 013 [arXiv:0812.3638]

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Outline

- Status of QCD dynamical simulations in the CLS
- Banks-Casher relation
- Renormalization of the spectral density
- Spectral density in ChPT
- Spectral projectors and extraction of Σ
- First numerical results
- Conclusions and Outlook

Coordinated Lattice Simulations (CLS) [https://twiki.cern.ch/twiki/bin/view/CLS/WebHome]

Several European groups coordinate their efforts to generate configuration ensembles with $N_{\rm f} = 2$ dynamical quarks degenerate in mass

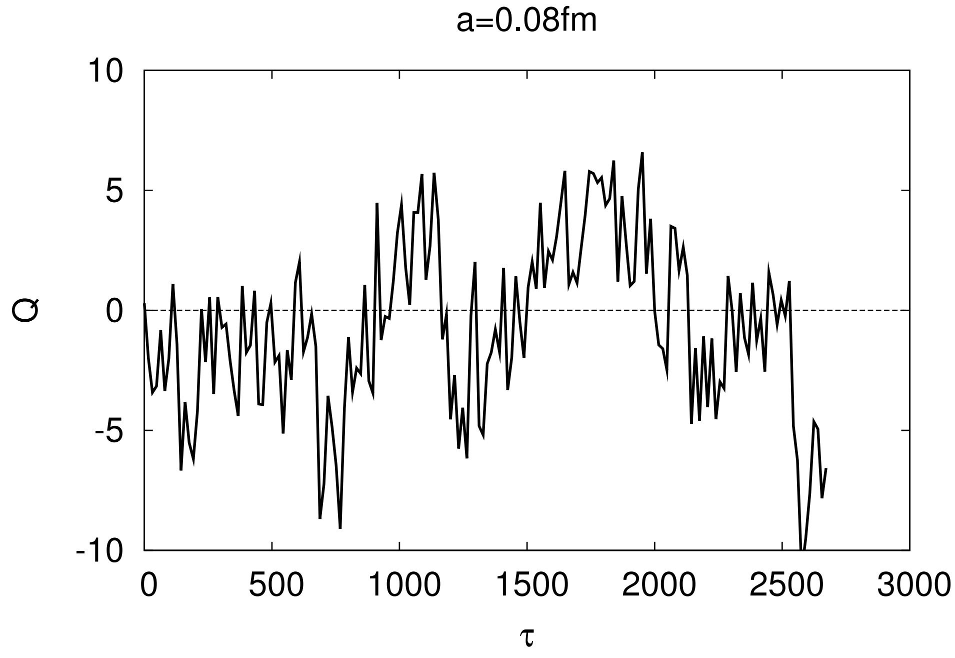
Gluon action: Wilson
Quark action: $O(a)$-improved Wilson
Algorithm : DD-HMC

β	V	a[fm]	L[fm]	Masses	Status
5.3	48×24^3	0.08	1.9	6	Completed
5.5	64×32^3	0.06	1.9	6	Completed
5.7	96×48^3	0.04	1.9	3	Running
5.3	64×32^3	0.08	2.5	6	Completed
5.5	96×48^3	0.06	2.5	4	Running
5.7	128×64^3	0.04	2.5		
5.3	96×48^3	0.08	3.8		
5.5	128×64^3	0.06	3.8		
...	...	...	...	...	...

Groups:
Berlin (U. Wolff)
CERN (L. G., M. Lüscher)
DESY-Zeuthen (R. Sommer)
Madrid (C. Pena)
Mainz (H. Wittig)
Rome (R. Petronzio)
Valencia (P. Hernández)

Physics:
Spontaneous symmetry breaking
Fundamental parameters
Light-light mesons
Heavy-light mesons
Baryons
Weak matrix elements
.....

History plot for the topological charge [Schaefer, Sommer and Virotta 09]



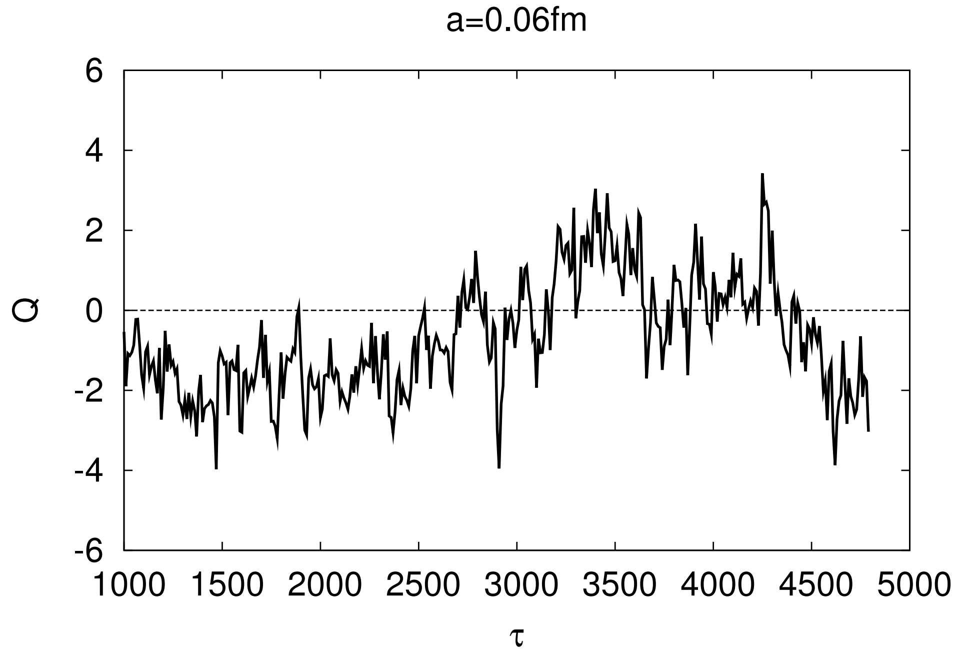
- Topological charge defined as (HYP smeared links)

$$Q = \frac{1}{16\pi^2} a^4 \sum_x \mathrm{Tr}\left[F(x)\tilde{F}(x)\right]$$

  cooling definition gives similar results

- At $\beta = 5.3$ ($a \sim 0.8$ fm) long autocorrelations, but quite decent sampling of all sectors

History plot for the topological charge [Schaefer, Sommer and Virotta 09]



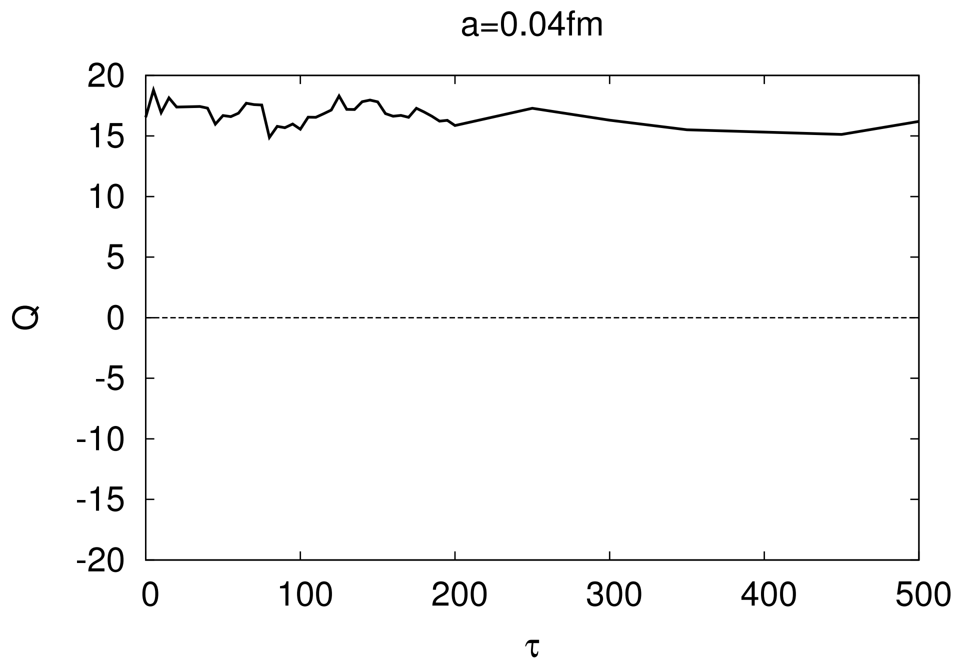
- Topological charge defined as (HYP smeared links)

$$Q = \frac{1}{16\pi^2} a^4 \sum_x \mathrm{Tr}\Big[F(x)\tilde{F}(x)\Big]$$

  cooling definition gives similar results

- At $\beta = 5.5$ ($a \sim 0.6$ fm) the auto-correlation get worse and . . .

History plot for the topological charge [Schaefer, Sommer and Virotta 09]



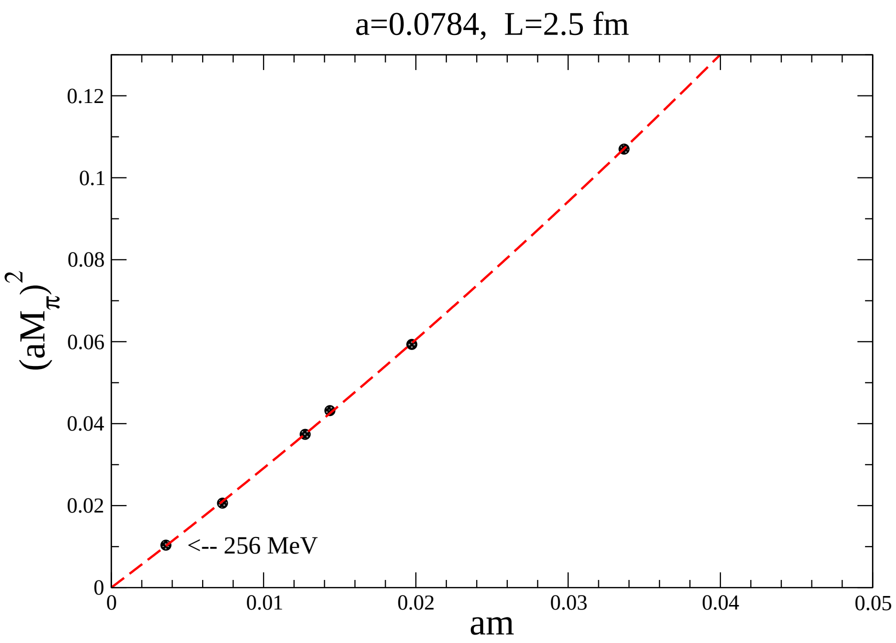
- Topological charge defined as (HYP smeared links)

$$Q = \frac{1}{16\pi^2} a^4 \sum_x \mathrm{Tr}\left[F(x)\tilde{F}(x)\right]$$

  cooling definition gives similar results

- At $\beta = 5.7$ ($a \sim 0.4$ fm) charge does not move considerably. Possible solution proposed [Luscher 09]. Trivializing maps being investigated [Luscher, Palombi in progress]

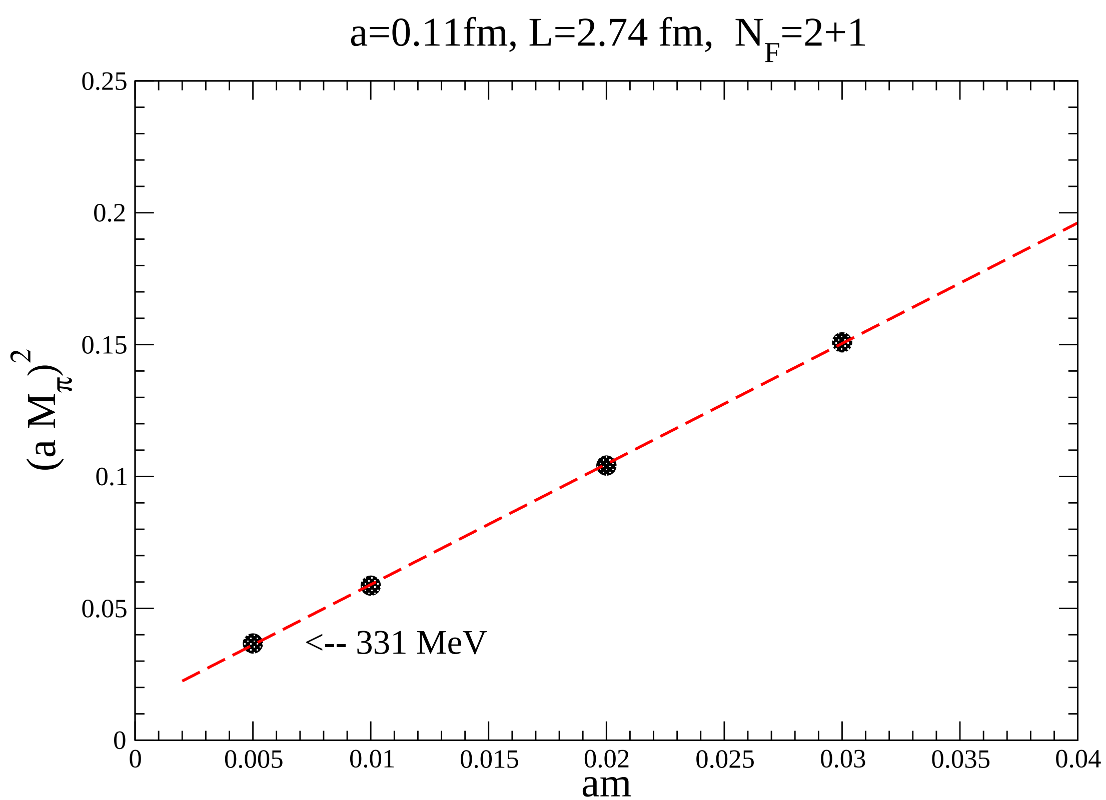
Pion Mass with $O(a)$-improved Wilson quarks [Del Debbio et al. 06-08]



$$(aM_\pi)^2 = 2.80(3) \times am + 11.3(12) \times (am)^2 \qquad \chi^2/\mathrm{dof} = 0.9$$

- A striking linearity observed: at the smallest masses non-linear correction is 1 - 3%. SSB as expected. Similar results from the other collaborations.
- It is time for a precise quantitative test of spontaneous symmetry breaking in QCD The GMOR relation is maybe the simplest to start with

Pion Mass with Domain-Wall fermions ($N_F = 2+1$) [C. Allton et al. RBC and UKQCD Coll. 07-08]



- A striking linearity observed also in this case
- .....

Banks-Casher relation [Banks, Casher 80; Leutwyler, Smilga 92; Shuryak, Verbaarschot 93]

- The spectral density of the Dirac operator (two degenerate flavours of mass m) is

$$\rho(\lambda, m) \equiv \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle$$

where $\langle \dots \rangle$ indicates the path-integral average, and for each gauge configuration

$$D\psi_k = i\bar{\lambda}_k \psi_k \ , \qquad \bar{\lambda}_k = \lambda_k + \mathcal{O}(a^2)$$

- $\rho(\lambda, m)$ has a well defined infinite volume limit, χ-sym. implies $\rho(\lambda, m) = \rho(-\lambda, m)$
- The chiral condensate is given by

$$\Sigma(m_v, m) \equiv -\langle \bar{\psi}\psi \rangle = \int_{-\infty}^{\infty} d\lambda \, \frac{\rho(\lambda, m)}{i\lambda + m_v} \ , \qquad \frac{\Sigma(0,0)}{\pi} = \lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m)$$

- A condensate is generated if eigenvalues condensate near zero with spacing $\propto 1/V$

Renormalization and continuum limit of the spectral density (I) [Del Debbio at al. 06; L.G., Lüscher 08]

- Instead of the spectral density, consider the spectral sum

$$\sigma_k(m_v, m) = \langle \mathrm{Tr}\left[(D^\dagger D + m_v^2)^{-k}\right]\rangle$$
$$= V\int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m_v^2)^k}$$

  ∗ Convergent integral if $m_v^2 > 0$ and $k \geq 3$

  ∗ The relation $\rho(\lambda, m) \leftrightarrow \sigma_k(m_v, m)$ invertible for every k

- It is enough to study the renormalization and continuum limit of the spectral sum σ_k

- If we introduce $2k$ valence Ginsparg–Wilson quarks, the spectral sum is given by

$$\sigma_k(m_v, m) = -a^{8k} \sum_{x_1,\dots,x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \dots P_{2k\,1}(x_{2k}) \rangle \ , \qquad P_{ij} = \bar{\psi}_i \gamma_5 \psi_j$$

thanks to the fact that the (modulo) square of the quark propagator is

$$(D^\dagger D + m_v^2)^{-1}$$

- An overall factor $(1/Z_m)^{2k}$ renormalizes the correlation function when the pseudoscalar densities are inserted at physical distance

- At short distances ($x_1 \to x_2$) the flavour structure implies

$$P_{12}(x_1) P_{23}(x_2) \sim C(x_1 - x_2) S_{13}(x_1) \qquad S_{13} = \bar{\psi}_1 \psi_3$$

where by power counting $C(x)$ diverges like $|x|^{-3}$. This singularity is integrable. Analogous argument for all other short-distance singularities. No extra contact terms are needed to renormalize σ_k

Renormalization and continuum limit of the spectral density (III) [Del Debbio et al. 06; L.G., Lüscher 08]

- Once the the gauge coupling and the mass are renormalized, the spectral sum

$$\sigma_{k\mathrm{R}}(m_{v\mathrm{R}}, m_\mathrm{R}) = Z_m^{-2k} \sigma_k \left(\frac{m_{v\mathrm{R}}}{Z_m}, \frac{m_\mathrm{R}}{Z_m} \right)$$

  is ultraviolet finite. In particular no extra UV power divergences need to be subtracted

- The renormalized density reads

$$\rho_\mathrm{R}(\lambda_\mathrm{R}, m_\mathrm{R}) = Z_m^{-1} \rho \left(\frac{\lambda_\mathrm{R}}{Z_m}, \frac{m_\mathrm{R}}{Z_m} \right)$$

- For Wilson fermions similar derivation but with twisted-mass valence quarks

- If we introduce k doublets of twisted-mass valence quarks with Dirac operator

$$D_{\rm tm} = D_{m_v} + i\mu_v\gamma_5\tau^3 \; , \qquad P^{\pm}_{ij} = \bar\psi_i\gamma_5\tau^{\pm}\psi_j$$

the spectral sum is given by

$$\sigma_k(\mu_v, m) = -a^{8k}\sum_{x_1,\dots,x_{2k}} \Big\langle P^{+}_{12}(x_1)P^{-}_{23}(x_2)\dots P^{+}_{(2k-1)(2k)}(x_{2k-1})P^{-}_{(2k)1}(x_{2k})\Big\rangle$$

thanks to the fact that for tm quarks the (modulo) square of the quark propagator is

$$(D^{\dagger}_{m_v}D_{m_v} + \mu_v^2)^{-1}$$

- As before the short-distance singularities are integrable ($x_1 \rightarrow x_2$)

$$P^{+}_{12}(x_1)P^{-}_{23}(x_2) \sim \mathcal{C}(x_1 - x_2)S^{\pm}_{13}(x_1) \; , \qquad S^{\pm}_{13} = \bar\psi_1\tau^{+}\tau^{-}\psi_3$$

since by power counting $\mathcal{C}(x)$ diverges like $|x|^{-3}$

Extracting Σ from the spectral density

- The dynamical properties of the theory are encoded in $\rho(\lambda, m)$
- The power divergences in the chiral condensate

$$\Sigma(z,m) \equiv \int_{-\infty}^{\infty} d\lambda \, \frac{\rho(\lambda,m)}{i\lambda + z}$$

do not originate from the spectral density itself but from the particular integral taken

- Note that if $\rho(\lambda, m)$ is extracted from

$$2\pi\rho(\lambda,m) = \lim_{\epsilon\to 0} \Big[\Sigma(i\lambda+\epsilon,m) + \Sigma(-i\lambda+\epsilon,m)\Big]$$

the divergences cancel out on the r.h.s.

- By choosing a different probe function, it is possible to extract Σ from integrals of the spectral density which are not plagued by power divergences [L.G., S. Necco 07]

- The average number of eigenstates of D with $|\lambda| < \Lambda$

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda\, \rho(\lambda, m)$$

is maybe the simplest integral to consider

- In the infinite volume limit it scales with V if SSB is at work
- The continuum limit can be taken for any value of Λ and m, i.e.

$$\nu_{\rm R}(M_{\rm R}, m_{\rm R}) = \nu(M, m)$$

even without chiral symmetry implemented

- $O(a)$ improvement automatic

Spectral density in the chiral effective theory (I) [Smilga, Stern 93; Osborn et al. 99; L.G., Lüscher 08]

- Standard chiral effective theory supplemented with a valence quark

$$SU(2)_L \otimes SU(2)_R \to SU(2)_V \quad \Longrightarrow \quad SU(3|1)_L \otimes SU(3|1)_R \to SU(3|1)_V$$

- The unitary fields are given by

$$U = \exp\left\{\frac{2i}{F}\Phi\right\} , \qquad \Phi = \sum_a \phi^a T^a$$

and the leading order Lagrangian is [$M = \text{diag}(m, m, m_v, m_v)$]

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left\{\text{Str}\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{2\Sigma}{F^2}\,\text{Str}\left[M U^\dagger + M^\dagger U\right]\right\}$$

- At the NLO the relevant term is given by

$$\mathcal{L}^{(4)} = -\frac{4\Sigma^2}{F^4} L_6\,\text{Str}\left[U^\dagger M + M^\dagger U\right]\text{Str}\left[U^\dagger M + M^\dagger U\right] + \dots$$

Spectral density in the chiral effective theory (II) [Smilga, Stern 93; Osborn et al. 99; L.G., Lüscher 08]

- If the condensate is analytically continued in the valence quark mass

$$\Sigma(z,m) \equiv -\langle\bar{\psi}\psi\rangle = \int_{-\infty}^{\infty} d\lambda\, \frac{\rho(\lambda,m)}{i\lambda+z}$$

then the spectral density can be computed as

$$2\pi\rho(\lambda,m) = \lim_{\epsilon\to 0}\Big[\Sigma(i\lambda+\epsilon,m)+\Sigma(-i\lambda+\epsilon,m)\Big]$$

- In the infinite volume limit an at the NLO in ChPT

$$\begin{aligned}\rho^{\rm nlo}(\lambda,m) \quad &= \quad \frac{\Sigma}{\pi}\Big\{1+\frac{\Sigma}{(4\pi)^2F^4}\Big[m\left(3\,\bar{l}_6-1\right)+2\lambda\arctan\Big(\frac{\lambda}{m}\Big)-\pi|\lambda| \\ &- \quad 2\,m\ln\left(\frac{\Sigma\sqrt{\lambda^2+m^2}}{F^2\mu^2}\right)-m\ln\left(\frac{2\Sigma|\lambda|}{F^2\mu^2}\right)\Big]\Big\}\end{aligned}$$

where $\bar{l}_6 = (1024\pi^2)\hat{L}_6/3+\ldots$

Spectral Projectors in ChPT

- In the infinite volume limit

$$\nu^{\rm nlo}(\Lambda,m) = \frac{2\Lambda\Sigma V}{\pi}\left\{1+\frac{\Sigma\, m}{(4\pi)^2F^4}\left[3\,\bar{l}_6 - 3\ln\left(\frac{\Sigma\Lambda}{F^2\mu^2}\right) - \ln(2) - \frac{\pi}{2}\frac{m}{\Lambda} + O\left(\frac{m^2}{\Lambda^2}\right)\right]\right\}$$

  is clearly sensitive to the infinite volume chiral condensate

- NLO corrections to leading behaviour (vanishing for $m \to 0$) expected rather small for $\Lambda = 50$–100 MeV and $m \leq 20$ MeV [$\overline{\rm MS}$ @ 2 GeV]. No chiral logs $\propto \ln(m)$

- Finite volume effects: a fraction of a percent at NLO ChPT in the p-regime

- When $\Lambda\Sigma V$ is not very large, threshold effects can be sizeable.
  They can be quantified in ChPT

Spectral Projectors: numerical implementation

- The quantity $\nu(\Lambda, m)$ can be computed as

$$\nu(\Lambda, m) = \langle \mathcal{O} \rangle \ , \qquad \mathcal{O}[U] = \mathrm{Tr} P_M$$

$$P_M = \theta(M^2 - D_m^\dagger D_m) \ , \qquad M^2 = \Lambda^2 + m^2$$

- To avoid the (unnecessary) computation of $O(V)$ eigenvalues

$$\nu(\Lambda, m) = \langle \hat{\mathcal{O}} \rangle \ , \qquad \hat{\mathcal{O}}[U, \eta] = (\eta, P_M \eta)$$

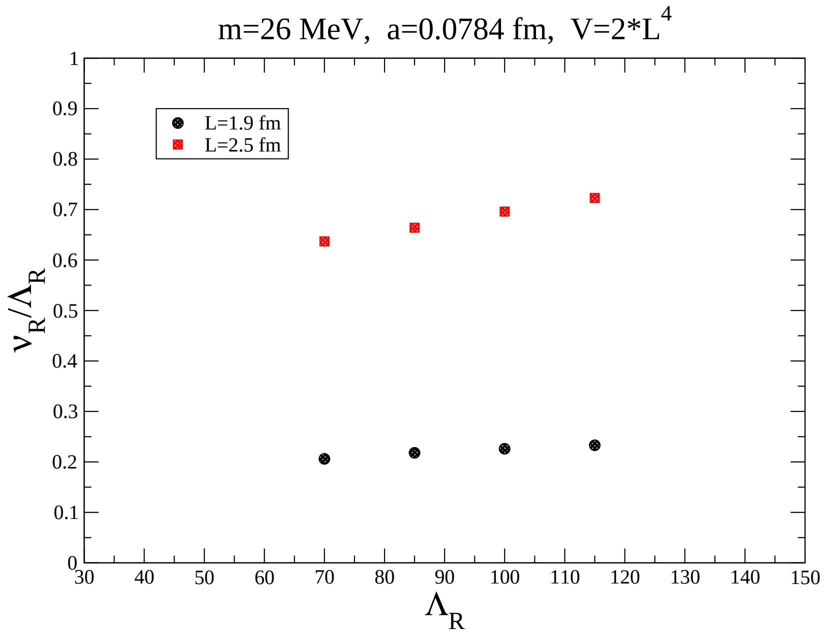
where η are Gaussian random sources

- The cost scales with V rather than V^2, and the variance scales like $V^{-1/2}$.

Infinite volume and continuum limit extrapolation feasible

Lattice details:

- $*$ Action: $O(a)$-improved Wilson
- $*$ $a = 0.0784$ fm
- $*$ $V = 2L \times L^3$, $L = 1.9, 2.5$ fm
- $*$ $m_R^{\overline{\rm MS}}(2\,{\rm GeV}) = 12.8, 26.5, 45.8$ MeV
- $*$ $\Lambda_{\rm R} = 70, 85, 100, 115$ MeV



Numerical computation: parameters and scaling with the volume

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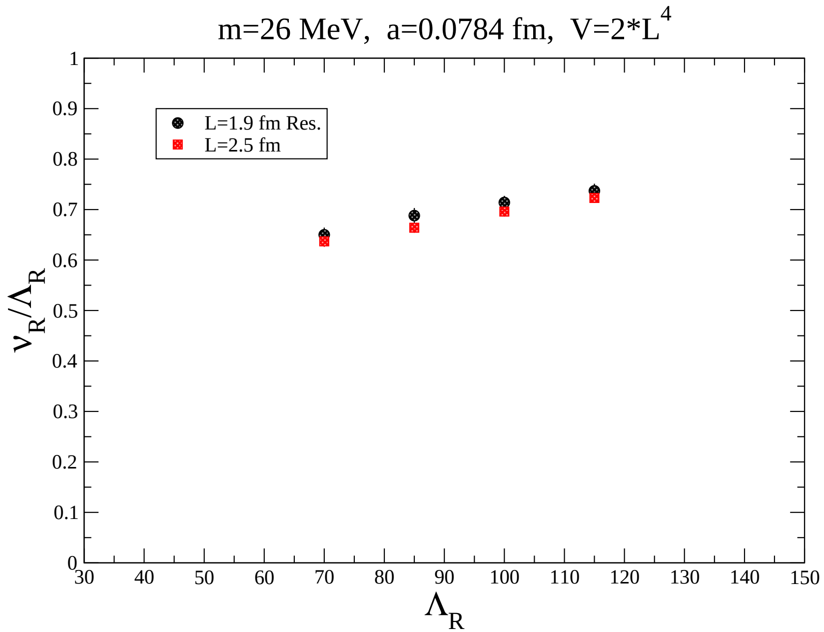

- Scaling is $\propto V$ as expected in presence of SSB.
  Sub-leading effects smaller than statistical errors of $\sim 1.5\%$
- NLO ChPT predicts deviations to be

$$1 - \frac{\nu_{\rm R}^V}{\nu_{\rm R}^\infty} \propto e^{-M_\Lambda L/2}\ , \qquad M_\Lambda^2 = \Lambda \frac{M_\pi^2}{m}$$

a fraction of a percent for our parameter choice

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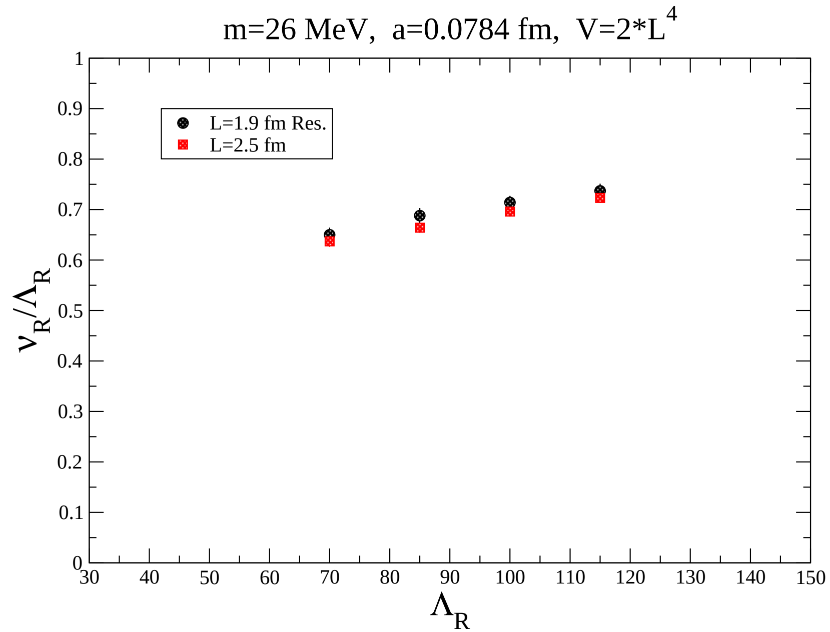
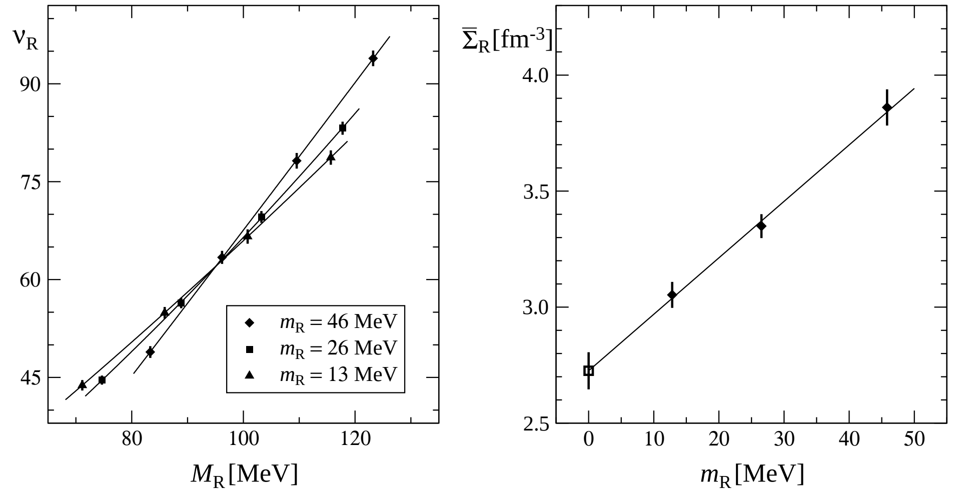


- Scaling is $\propto V$ as expected in presence of SSB.

- Eigenvalues condensate near zero with spacing $\propto 1/V$ even without chiral symmetry!
- One my speculate that SSB is an effect of the condensation

Numerical computation: quark mass dependence



- An effective condensate can defined as

$$\overline{\Sigma}_{\rm R} = \frac{\pi}{2V}\sqrt{1-\left(\frac{m_{\rm R}}{M_{\rm R}}\right)^2}\,\frac{\partial}{\partial M_{\rm R}}\nu_{\rm R}(M_{\rm R}, m_{\rm R})$$

with the prefactor chosen so that $\overline{\Sigma}_{\rm R}$ coincides with Σ to LO

- Extrapolation to the chiral limit mild. At NLO no chiral logs $\propto \ln(m)$

Numerical computation: summary of the results for $L = 2.5$ fm and $M_{\rm R} = 95$ MeV

- The simulation results for $\overline{\Sigma}_{\rm R}$ at $M_{\rm R} = 95$ MeV are

$m_{\rm R}$[MeV]	$\overline{\Sigma}_{\rm R}^{1/3}$[MeV]
$12.8(2)(3)$	$286(2)(4)$
$26.5(2)(6)$	$295(2)(4)$
$45.8(3)(11)$	$310(2)(4)$

and a linear extrapolation to the chiral limit yields the estimate

$$\Sigma_{\rm eff}^{1/3} = 276(3)(4)(5)\text{MeV}$$

- To be compared, for instance, with the value extracted from the GMOR by the ETMC

$$\Sigma_{\rm eff}^{1/3} = 270(7) \qquad \text{[R. Baron et al. ETM Coll. 09]}$$

or from fixed topology simulations by JLQCD

$$\Sigma_{\rm eff}^{1/3} = 243(4)(0)\ \text{MeV} \qquad \text{[Fukaya et al. 09]}$$

Conclusions and outlook

- Simulations with dynamical fermions are feasible in interesting ranges of parameters

$$a = 0.04 - 0.08 \text{ fm} , \qquad M_\pi = 200 - 500 \text{ MeV}$$

  thanks mainly to a breakthrough in algorithms and also to the Moore's law

- Spectral projectors open a new perspective to study the chiral regime of QCD:
  - $*$ Chiral condensate without power divergences
  - $*$ Topological susceptibility without power divergences
  - $*$ Ward Identities . . .

- The moderate computational cost allows for the infinite volume and continuum limits
- More simulations needed to estimate the systematics on Σ with confidence

- Application to finite temperature or other QCD-like theories straightforward

Scaling with the volume BCK1