Implications of Poincaré symmetry for thermal field theories

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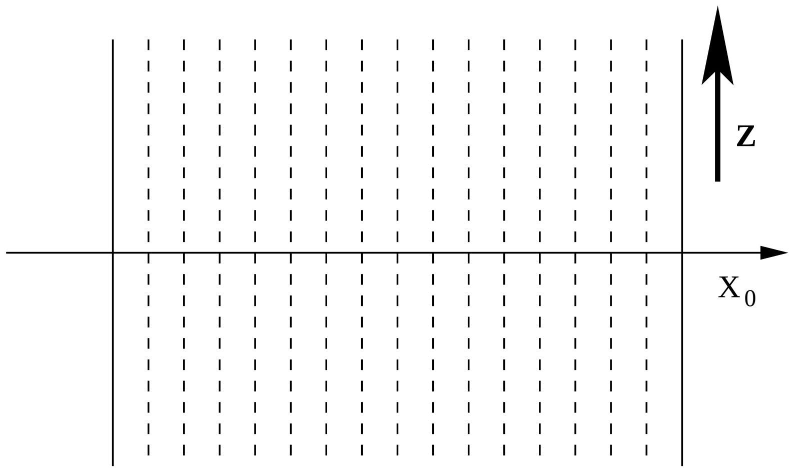
Based on: Della Morte and L. G. 2011; L. G. and H. B. Meyer 2011-2013; L. G. and M. Pepe 2014-2015

Outline

- Introduction
- Free-energy density with "shifted" boundary conditions

$$f\left(\sqrt{L_0^2+\boldsymbol{z}^2}\right) = -\lim_{V\to\infty}\frac{1}{L_0 V}\ln Z(L_0,\boldsymbol{z})$$

$$\phi(L_0,\boldsymbol{x}) = \phi(0,\mathbf{x}-\boldsymbol{z})$$



- Ward Identities in infinite and finite volume
- Definition of $T_{\mu\nu}$ on the lattice
- Non-perturbative measure of the entropy density
- Conclusions and outlook

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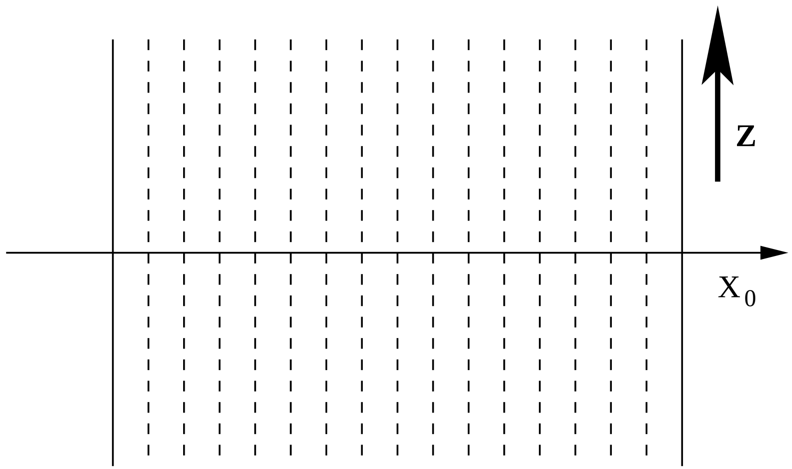

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Thermal field theories in the Euclidean path integral formalism

- From textbooks

$$\phi(x) = \phi(x + V_{\rm pbc} m) \quad m \in \mathbb{Z}^4$$

$$Z(L_0) = \mathrm{Tr}\left\{ e^{-L_0 \widehat{H}} \right\}$$

$$V_{\rm pbc} = \begin{pmatrix} L_0 & 0 & 0 & 0 \\ 0 & L_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

where the temperature is $T = 1/L_0$

- The basic thermodynamic quantities are defined as

$$f = -\frac{1}{L_0 V} \ln Z(L_0) \qquad e = -\frac{1}{V} \frac{\partial}{\partial L_0} \ln Z(L_0) \qquad s = -\frac{L_0^2}{V} \frac{\partial}{\partial L_0} \left\{ \frac{1}{L_0} \ln Z(L_0) \right\}$$

which in the thermodynamic limit lead to

$$p = -f \qquad s = L_0(e + p) \qquad c_v = -L_0 \frac{\partial}{\partial L_0} s$$

Path integrals with shifted boundary conditions: infinite-volume limit (I)

- We are interested in the partition function

$$\phi(x) = \phi(x+V_{\rm sbc}m) \quad m \in \mathbb{Z}^4$$

$$Z(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\left\{e^{-L_0(\widehat{H}-i\boldsymbol{\xi}\cdot\widehat{\boldsymbol{P}})}\right\}$$

$$V_{\rm sbc} = \begin{pmatrix} L_0 & 0 & 0 & 0 \\ L_0\xi_1 & L_1 & 0 & 0 \\ L_0\xi_2 & 0 & L_2 & 0 \\ L_0\xi_3 & 0 & 0 & L_3 \end{pmatrix}$$

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where we have chosen $\boldsymbol{\xi} = \{\xi_1, 0, 0\}$

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- By making an Euclidean "boost" rotation

$$\gamma_1 = \frac{1}{\sqrt{1+\xi_1^2}}$$

$$\Lambda = \begin{pmatrix} \gamma_1 & \gamma_1\xi_1 & 0 & 0 \\ -\gamma_1\xi_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Lorentz [SO(4)] invariance implies

$$Z(L_0,\boldsymbol{\xi}) = \mathrm{Tr}\left\{e^{-L_1\gamma_1(\widetilde{H}+i\xi_1\widetilde{P}_0)}\right\}$$

$$V'_{\rm sbc} = \Lambda\, V_{\rm sbc} = \begin{pmatrix} L_0/\gamma_1 & L_1\gamma_1\xi_1 & 0 & 0 \\ 0 & L_1\gamma_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

Path integrals with shifted boundary conditions: infinite-volume limit (II)

- Assuming that $\widetilde{H}$ has a translationally-invariant vacuum and a mass gap [$\boldsymbol{\xi} = \{\xi_1, 0, 0\}$]

$$Z(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\left\{ e^{-L_1\gamma_1(\widetilde{H} + i\xi_1 \widetilde{P}_0)} \right\}$$

$$V'_{\rm sbc} = \Lambda V_{\rm sbc} = \begin{pmatrix} L_0/\gamma_1 & L_1\gamma_1\xi_1 & 0 & 0 \\ 0 & L_1\gamma_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

the right hand side becomes insensitive to the phase in the limit $L_1 \to \infty$ at fixed ξ_1

$$f\left(L_0\sqrt{1+\xi_1^2}\right) = -\lim_{V\to\infty} \frac{1}{L_0 V} \ln Z(L_0, \boldsymbol{\xi})$$

$$V''_{\rm sbc} = \begin{pmatrix} L_0/\gamma_1 & 0 & 0 & 0 \\ 0 & L_1\gamma_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

- Thanks to cubic symmetry (infinite volume)

$$f\left(L_0, \boldsymbol{\xi}\right) = f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}, \mathbf{0}\right)$$

$$\phi(x_0, \boldsymbol{x}) = \phi(x_0 + L_0, \boldsymbol{x} + L_0\boldsymbol{\xi})$$

for a generic shift $\boldsymbol{\xi}$

Thermal field theory in a moving frame

- If $\widehat{H}$ and $\widehat{\boldsymbol{P}}$ are the Hamiltonian and the total momentum operator expressed in a moving frame, the standard partition function is

$$\mathcal{Z}(L_0, \boldsymbol{v}) \equiv \mathrm{Tr}\left\{e^{-L_0\,(\widehat{H}-\boldsymbol{v}\cdot\widehat{\boldsymbol{P}})}\right\}$$

- If we continue $\mathcal{Z}$ to imaginary velocities $\boldsymbol{v} = i\boldsymbol{\xi}$

$$Z(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\,\{e^{-L_0(\widehat{H}-i\boldsymbol{\xi}\cdot\widehat{\boldsymbol{P}})}\}$$

- The functional dependence $f(L_0\sqrt{1+\boldsymbol{\xi}^2})$ is consistent with modern thermodynamic arguments on the Lorentz transformation of the temperature and the free-energy [Ott 63; Arzelies 65; see Przanowski 11 for a recent discussion]

- In the zero-temperature limit the invariance of the theory (and of its vacuum) under the Poincaré group forces its free energy to be independent of the shift $\boldsymbol{\xi}$

- At non-zero temperature the finite length L_0 breaks SO(4) softly, and the free energy depends on the shift (velocity) but only through the combination $\beta = L_0\sqrt{1+\boldsymbol{\xi}^2}$

Euclidean Ward identities for correlators of $\overline{T}_{\mu\nu}$

- At $\boldsymbol{\xi} = 0$ the dependence of f from the combination $L_0\sqrt{1+\boldsymbol{\xi}^2}$ in turn implies

$$L_0\, \langle \overline{T}_{01}\, T_{01}\rangle_c = \langle T_{00}\rangle - \langle T_{11}\rangle$$

$$L_0^3\, \langle \overline{T}_{01}\, \overline{T}_{01}\, \overline{T}_{01}\, T_{01}\rangle_c = 9\,\langle T_{11}\rangle - 9\,\langle T_{00}\rangle + 3\, L_0\, \langle \overline{T}_{00} T_{00}\rangle_c$$

$$\dots$$

where $\langle T_{00}\rangle = -e$, $\langle T_{11}\rangle = p$, $\widehat{P}_1 \to -i\overline{T}_{01}$ and

$$\overline{T}_{\mu\nu}(x_0) = \int d^3x\, T_{\mu\nu}(x)$$

- Note that:
  - $*$ All operators at non-zero distance
  - $*$ Number of EMT on the two sides different
  - $*$ On the lattice they can be imposed to fix the renormalization of $T_{\mu\nu}$

Ward identities at non-zero shift

- When $\boldsymbol{\xi} \neq 0$ odd derivatives in the ξ_k do not vanish anymore, and new interesting WIs hold. The first non-trivial one is

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{\xi_k}{1-\xi_k^2} \left\{ \langle T_{00} \rangle_{\boldsymbol{\xi}} - \langle T_{kk} \rangle_{\boldsymbol{\xi}} \right\}$$

- By deriving twice with respect to the ξ_k

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{L_0 \xi_k}{2} \sum_{ij} \left\langle \overline{T}_{0i}\, T_{0j} \right\rangle_{\boldsymbol{\xi},\, c} \left[\delta_{ij} - \frac{\xi_i\, \xi_j}{\boldsymbol{\xi}^2} \right]$$

- Note that also in this case:

  $*$ All operators at non-zero distance

  $*$ Number or components of EMT on the two sides different

  $*$ On the lattice they can be imposed to fix the renormalization of $T_{\mu\nu}$

Entropy density from the response to the shift

- The Entropy density can be computed as

$$s = -\frac{L_0\,(1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k}\,\langle T_{0k}\rangle_{\boldsymbol{\xi}}$$

or as

$$s = -\frac{(1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k}\,\lim_{V\to\infty}\frac{1}{V}\,\frac{\partial}{\partial\xi_k}\ln Z(L_0,\boldsymbol{\xi})$$

- With respect to the standard technique:

  $*$ No ultraviolet power divergent subtraction (zero temperature subtraction)

  $*$ On the lattice finite multiplicative renormalization constant fixed non-perturbatively by WIs

Path integrals with shifted boundary conditions: finite-size effects

- The leading finite-size contributions to the free energy are

$$f(V_{\rm sbc}) - f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \cdots$$

where for $L_k = L$

$$\mathcal{I}_i = \frac{\gamma\nu}{2\pi L_0 L^3}\frac{1}{r}\frac{d}{dr}\left[\frac{e^{-MLr}}{r}\right]\Big|_{r=r_i}\,, \quad r_i = \frac{\gamma}{\bar{\gamma}_i}\,, \quad \bar{\gamma}_i = 1/\sqrt{1+\sum_{k\neq i}\xi_k^2}$$

with M and ν being the mass and the multiplicity of the lightest screening state

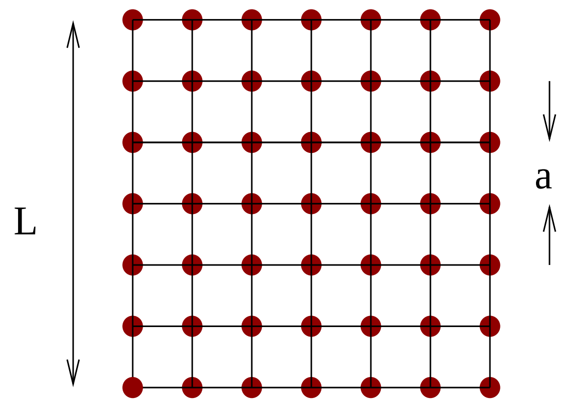
- Analogous formula for the entropy by noticing that

$$\langle T_{0k}\rangle_{V_{\rm sbc}} - \langle T_{0k}\rangle_{\boldsymbol{\xi}} = -\frac{\partial}{\partial\xi_k}\sum_{i=1}^{3}\mathcal{I}_i + \ldots$$

- WIs can be derived analogously in finite volume. They are modified by terms which vanish exponentially in the thermodynamic limit

Lattice gauge theory [Wilson 74]

- A Yang-Mills theory can be defined on a discretized space-time so that gauge invariance is preserved



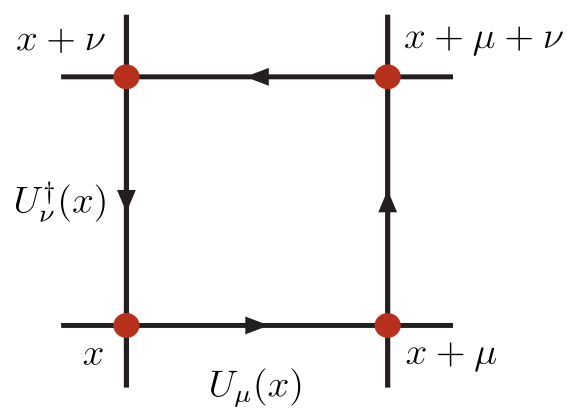
- The the gauge field $U_\mu \in \mathrm{SU}(3)$ resides on links

- The Wilson action is

$$S_G[U] = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{3}\mathrm{ReTr}\left\{U_{\mu\nu}(x)\right\}\right]$$

where $\beta = 6/g_0^2$ and the plaquette is

$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x)$$



- Discrete shifts in the boundary conditions can be implemented straightforwardly

Non-perturbative renormalization of $T_{\mu\nu}$

- On the lattice the Poincaré group is broken down to a discrete group and standard discretizations of $T_{\mu\nu}$ acquire finite ultraviolet renormalizations

- We focus on the SU(3) Yang–Mills. The analysis applies to other theories as well

$$T^{\rm R}_{\mu\nu} = Z_T \Big\{ T^{[1]}_{\mu\nu} + z_T\, T^{[3]}_{\mu\nu} + z_S \left[T^{[2]}_{\mu\nu} - \langle T^{[2]}_{\mu\nu} \rangle_0 \right] \Big\} \, .$$

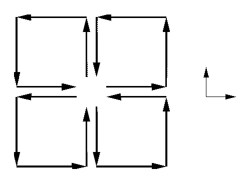
$$T^{[1]}_{\mu\nu} = (1-\delta_{\mu\nu}) \frac{1}{g_0^2} \Big\{ F^a_{\mu\alpha} F^a_{\nu\alpha} \Big\}$$

$$T^{[2]}_{\mu\nu} = \delta_{\mu\nu} \, \frac{1}{4 g_0^2} \, F^a_{\alpha\beta} F^a_{\alpha\beta}$$

$$T^{[3]}_{\mu\nu} = \delta_{\mu\nu} \frac{1}{g_0^2} \Big\{ F^a_{\mu\alpha} F^a_{\mu\alpha} - \frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} \Big\}$$

where

$$F^a_{\mu\nu}(x) = -\frac{i}{4a^2} {\rm Tr} \Big\{ \Big[Q_{\mu\nu}(x) - Q_{\nu\mu}(x) \Big] T^a \Big\} \, , \qquad Q_{\mu\nu}(x) = \sum$$



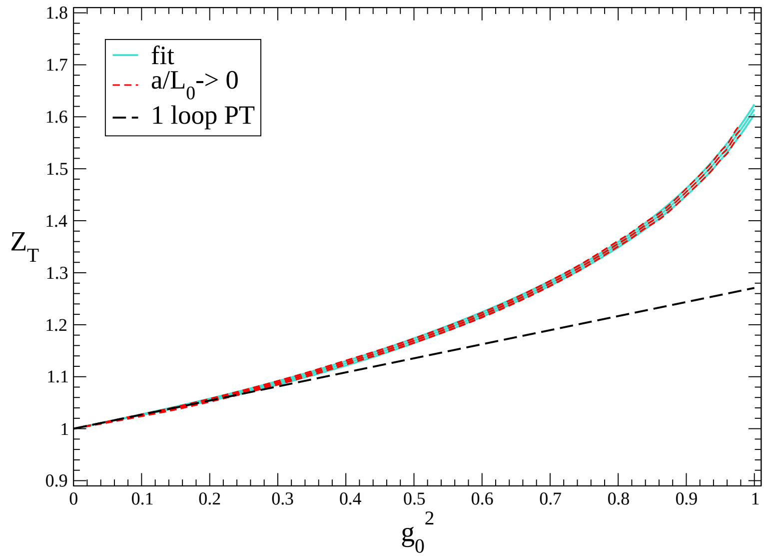
The sextet renormalization constant Z_T

- The continuum relation

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{1}{L_0} \lim_{V\to\infty} \frac{1}{V} \frac{\partial}{\partial \xi_k} \ln Z(L_0, \boldsymbol{\xi})$$

can be imposed on the lattice to fix Z_T

$$Z_T(g_0^2) = -\frac{\Delta f}{\Delta \xi_k} \frac{1}{\langle T^{[1]}_{0k} \rangle_{\boldsymbol{\xi}}}$$



where the derivative in the shift is discretized by the symmetric finite difference

$$\frac{\Delta f}{\Delta \xi_k} = \frac{1}{2aV} \ln \Big[\frac{Z(L_0, \boldsymbol{\xi} - a\hat{k}/L_0)}{Z(L_0, \boldsymbol{\xi} + a\hat{k}/L_0)} \Big]$$

- The final results for $Z_T(g_0^2)$ are well represented by

$$Z_T(g_0^2) = \frac{1 - 0.4457\, g_0^2}{1 - 0.7165\, g_0^2} - 0.2543\, g_0^4 + 0.4357\, g_0^6 - 0.5221\, g_0^8$$

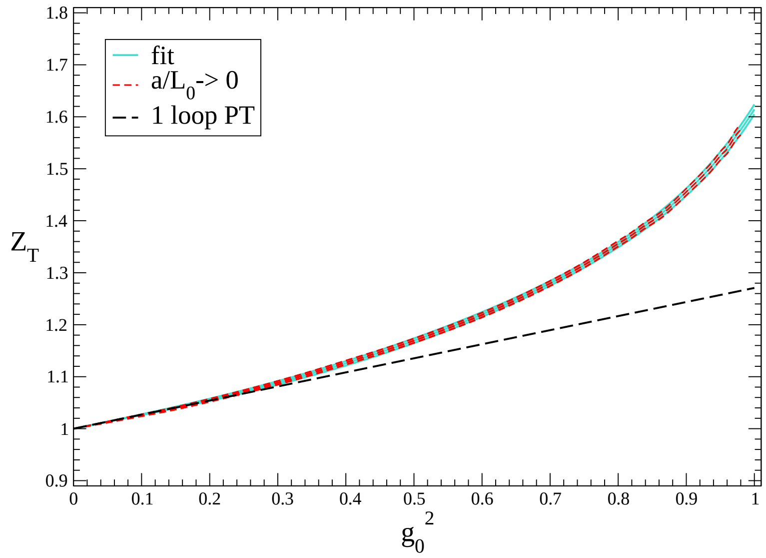
with the error that varies from 0.4% up 0.7% in the range $0 \le g_0^2 \le 1$

- The continuum relation

$$\langle T_{0k}\rangle_{\boldsymbol{\xi}} = \frac{1}{L_0} \lim_{V\to\infty} \frac{1}{V} \frac{\partial}{\partial \xi_k} \ln Z(L_0, \boldsymbol{\xi})$$

can be imposed on the lattice to fix Z_T

$$Z_T(g_0^2) = -\frac{\Delta f}{\Delta \xi_k} \frac{1}{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}$$



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$$\frac{\Delta f}{\Delta \xi_k} = \frac{1}{2aV} \ln \Big[\frac{Z(L_0, \boldsymbol{\xi} - a\hat{k}/L_0)}{Z(L_0, \boldsymbol{\xi} + a\hat{k}/L_0)}\Big]$$

- Within statistical errors, the non-perturbative determination starts to deviate significantly from the one-loop result [Caracciolo et al. 88, 90]

$$Z_T(g_0^2) = 1 + 0.27076\, g_0^2$$

already at $g_0^2 \sim 0.25$

The triplet renormalization constant z_T

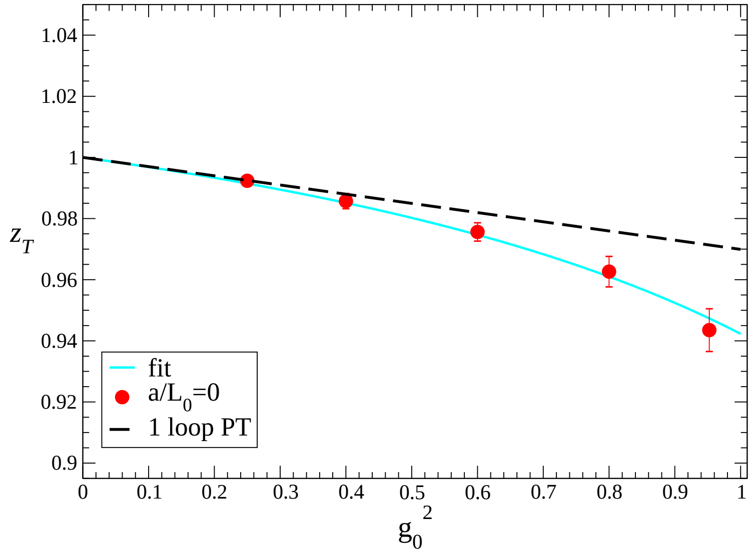
- The continuum relation

$$\langle T_{0k}\rangle_{\boldsymbol{\xi}} = \frac{\xi_k}{1-\xi_k^2}\left\{\langle T_{00}\rangle_{\boldsymbol{\xi}} - \langle T_{kk}\rangle_{\boldsymbol{\xi}}\right\}$$

is enforced on the lattice to determine z_T

$$z_T(g_0^2) = \frac{1-\xi_k^2}{\xi_k}\frac{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}{\langle T_{00}^{[3]}\rangle_{\boldsymbol{\xi}} - \langle T_{kk}^{[3]}\rangle_{\boldsymbol{\xi}}}$$

with the condition $\frac{L\,\xi_k}{L_0(1+\xi_k^2)} = q \in \mathbb{Z}$



- The results for $z_T(g_0^2)$ are well represented by

$$z_T(g_0^2) = \frac{1-0.5090\,g_0^2}{1-0.4789\,g_0^2}$$

where the error grows linearly from 0.15% to 0.75% in the interval $0 \le g_0^2 \le 1$

The triplet renormalization constant z_T

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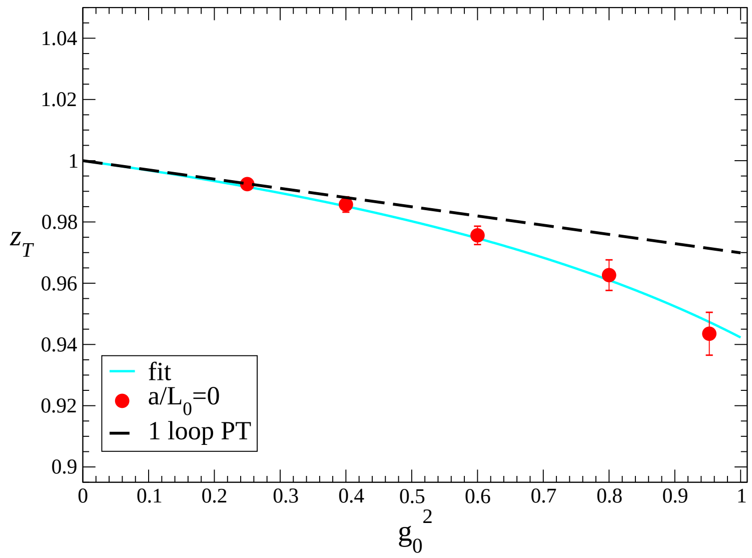


- Within statistical errors, the non-perturbative determination starts to deviate significantly from the one-loop result [Caracciolo et al. 88, 90]

$$z_T(g_0^2) = 1 - 0.03008\; g_0^2$$

already at $g_0^2 \sim 0.4$

Computation of the entropy density

- Once Z_T has been determined, the entropy density can be computed as ($\xi_k \neq 0$)

$$s = -\frac{Z_T L_0 (1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k} \langle T_{0k}^{[1]} \rangle_{\boldsymbol{\xi}}$$

  thanks to the misalignment of the lattice axes with respect to the periodic directions

- A step-scaling function for $s(T)/T^3$ can be defined as

$$\Sigma_s(T,r) \equiv \frac{1}{r^3}\frac{s(rT)}{s(T)} = \frac{(1+\boldsymbol{\xi}^2)^3}{(1+\boldsymbol{\zeta}^2)^3}\frac{\zeta_k}{\xi_k}\frac{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\zeta}}}$$

  where the step is given by $r = \sqrt{1+\boldsymbol{\zeta}^2}/\sqrt{1+\boldsymbol{\xi}^2}$

- The entropy density at a given T can be obtained by solving the recursive relation

$$v_0 = \frac{s(T_0)}{T_0^3}\,, \qquad v_{k+1} = \Sigma_s(T_k, r) v_k\,, \qquad T_k = r^k T_0$$

- The step-scaling function does not require any ultraviolet renormalization factor, and it has a universal continuum limit

Computation of the entropy density

- Once Z_T has been determined, the entropy density can be computed as ($\xi_k \neq 0$)

$$s = -\frac{Z_T L_0 (1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k} \langle T^{[1]}_{0k} \rangle_{\boldsymbol{\xi}}$$

thanks to the misalignment of the lattice axes with respect to the periodic directions

- In practice we consider the step-scaling function

$$\Sigma_s(T, \sqrt{2}) = \frac{1}{8} \frac{\langle T^{[1]}_{01} \rangle_{(1,0,0)}}{\langle T^{[1]}_{01} \rangle_{(1,1,1)}}$$

- Being T the only relevant scale in the problem (no zero-temperature subtraction needed), various orders of magnitude in T can be spanned this way in the spirit of the multi-step matching technique [Lüscher et al. 93-94]

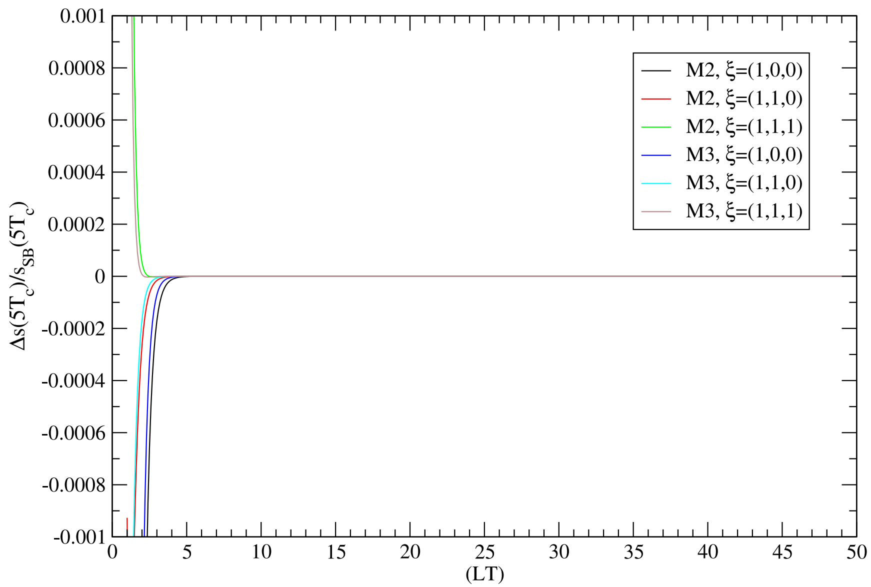
Finite-size effects in the entropy density

- The leading finite-size corrections are given by

$$\frac{\Delta s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\frac{L_0^4}{\gamma^6\xi_k}\left\{\langle T_{0k}\rangle_{V_{\rm sbc}} - \langle T_{0k}\rangle_{\boldsymbol{\xi}}\right\}$$

- The perturbative expression for the lightest screening mass is [Laine, Vepsalainen 09]

$$M_2 = 2\,Tg + \ldots\ , \qquad M_3 = 30\,T\,\frac{g^2}{4\pi} + \ldots$$



- It is realistic to consider boxes with $(LT) > 10$ where finite-size effects are negligible.

Thanks to the locality of the observable, the cost of the simulation is volume independent at fixed statistical error

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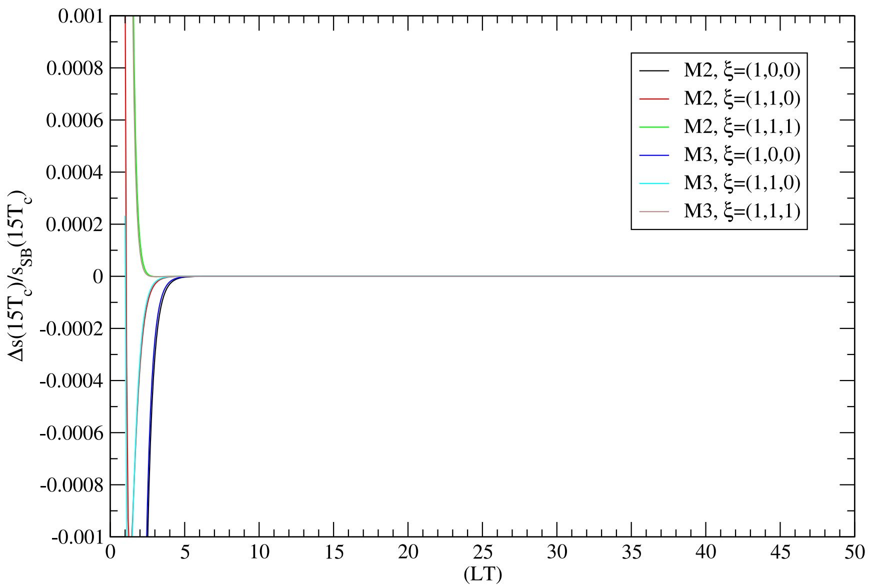

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Numerical computation of the step-scaling function (I)

- We have considered 10 different values of the temperature in the range T_c–$20\,Tc$

$$\frac{T_0}{2}\,,\ T_0\,,\ \ldots\,,\ 8T_0\,, 8\sqrt{2}\,T_0$$

where $T_0 = 1/L_{\rm max} \simeq 1.802 T_c$ [Capitani et al. 98]

- For each temperature, 4 values of the lattice spacing have been simulated to extrapolate the step-scaling function to the continuum limit

- We have chosen aspect ratios $(LT) > 10$. Finite-size effects checked explicitly with dedicated runs. They are negligible within statistical errors

id	L/a	L_0/a	β	TL
A_1	80	3	5.8506	13.3
A_2	128	4	6.0056	16.0
A_3	128	5	6.1429	12.8
A_4	128	6	6.2670	10.7
B_1	80	3	6.0403	13.3
B_2	128	4	6.2257	16.0
B_3	128	5	6.3875	12.8
B_4	128	6	6.5282	10.7
C_1	80	3	6.2670	13.3
C_2	128	4	6.4822	16.0
C_3	128	5	6.6575	12.8
C_4	128	6	6.7981	10.7
D_1	80	3	6.5282	13.3
D_2	128	4	6.7533	16.0
D_3	128	5	6.9183	12.8
D_4	128	6	7.0750	10.7
...	...	...	...	...
...	...	...	...	...
...	...	...	...	...
H_1	80	3	7.7039	13.3
H_2	128	4	7.9489	16.0
H_3	128	5	8.1405	12.8
H_4	128	6	8.2982	10.7
I_1	80	3	8.0060	13.3
I_2	128	4	8.2458	16.0
I_3	128	5	8.4346	12.8
I_4	128	6	8.5908	10.7

Numerical computation of the step-scaling function (II)

- For the C lattices we fix $T = T_0 = 1/L_{\rm max}$

- In the first four steps the bare coupling constant is fixed by requiring that [Necco, Sommer 01]

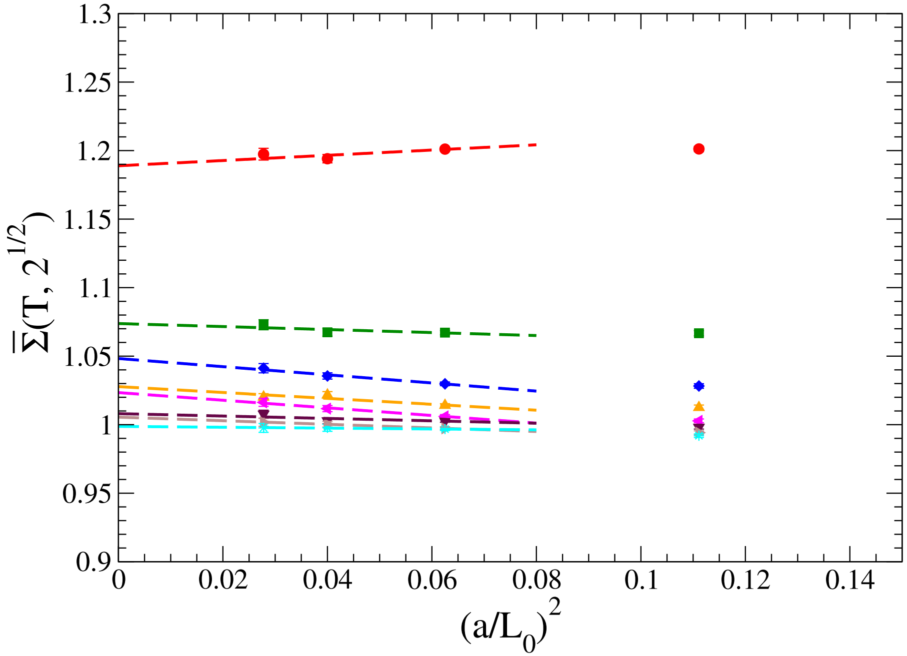
$$L_{\rm max}/r_0 = 0.738(16)$$

- From the 4^{th} step, we interpolate quadratically in $\ln(L/a)$ each set of data at constant $\bar{g}^2(L)$ and we choose [Capitani et al. 98]

$$\frac{1}{aT_k} = \frac{L_k}{a} = 2^{-k/2}\frac{L_{\rm max}}{a}$$

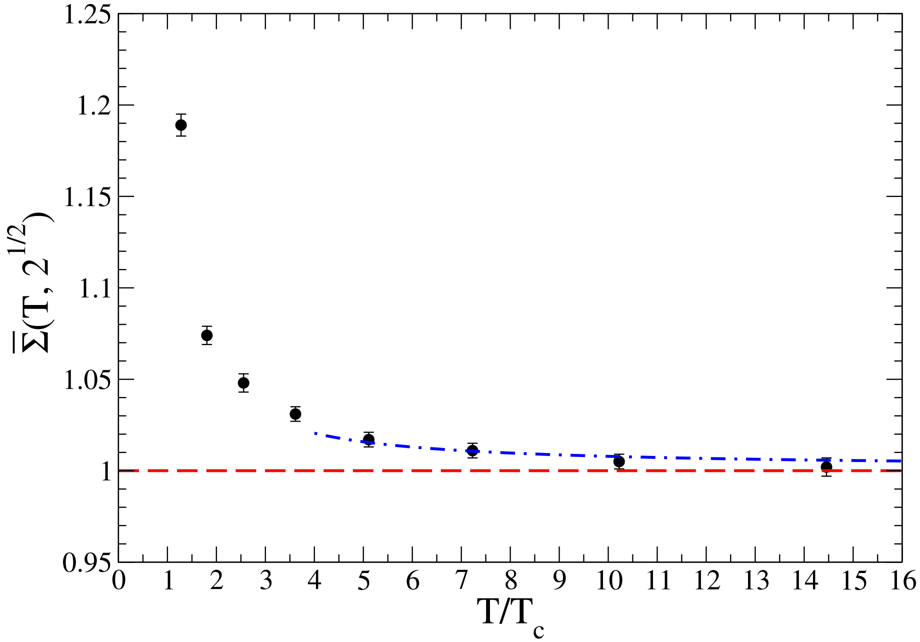
id	L/a	L_0/a	β	TL
A_1	80	3	5.8506	13.3
A_2	128	4	6.0056	16.0
A_3	128	5	6.1429	12.8
A_4	128	6	6.2670	10.7
B_1	80	3	6.0403	13.3
B_2	128	4	6.2257	16.0
B_3	128	5	6.3875	12.8
B_4	128	6	6.5282	10.7
C_1	80	3	6.2670	13.3
C_2	128	4	6.4822	16.0
C_3	128	5	6.6575	12.8
C_4	128	6	6.7981	10.7
D_1	80	3	6.5282	13.3
D_2	128	4	6.7533	16.0
D_3	128	5	6.9183	12.8
D_4	128	6	7.0750	10.7
...	...	...	...	...
...	...	...	...	...
...	...	...	...	...
H_1	80	3	7.7039	13.3
H_2	128	4	7.9489	16.0
H_3	128	5	8.1405	12.8
H_4	128	6	8.2982	10.7
I_1	80	3	8.0060	13.3
I_2	128	4	8.2458	16.0
I_3	128	5	8.4346	12.8
I_4	128	6	8.5908	10.7

Extrapolation to the continuum limit of Σ_s



- Discretization effects and statistical errors are at the level of per mille
- A precision of half a percent in the continuum limit is reached
- At fixed statistical error, the cost of the simulation is volume independent thanks to the locality of the observable

Step-scaling function in the continuum



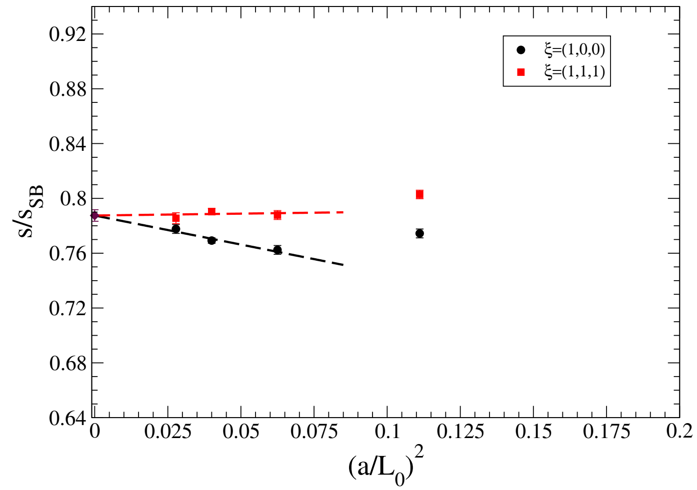
- Precision at the level of 0.5%
- At $T \sim 15\,T_c$, Σ_s already compatible with the high temperature limit within errors
- Contribution from various orders (blue line) in the perturbative series is oscillating (see below)

Entropy density in the continuum

- At the reference temperature T_0 the entropy density is obtained by extrapolating

$$\frac{s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\frac{(1+\xi^2)}{\xi_k}\,\frac{Z_T\,\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}{T^4}$$

to the continuum limit

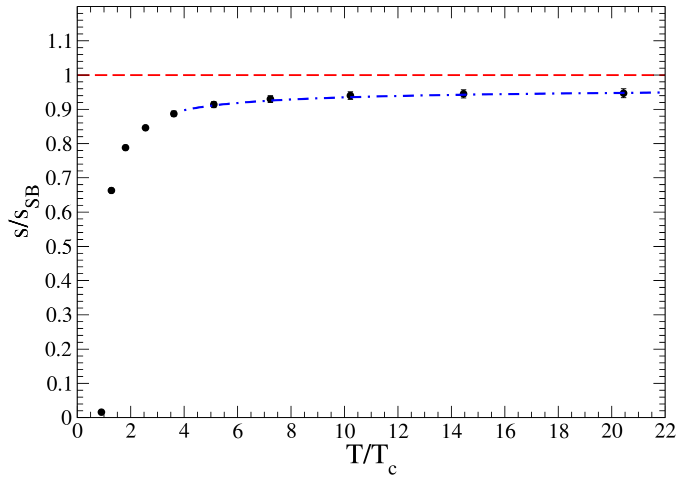


Entropy density in the continuum

- At the reference temperature T_0 the entropy density is obtained by extrapolating

$$\frac{s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\,\frac{(1+\xi^2)}{\xi_k}\,\frac{Z_T\,\langle T^{[1]}_{0k}\rangle_{\boldsymbol{\xi}}}{T^4}$$

to the continuum limit



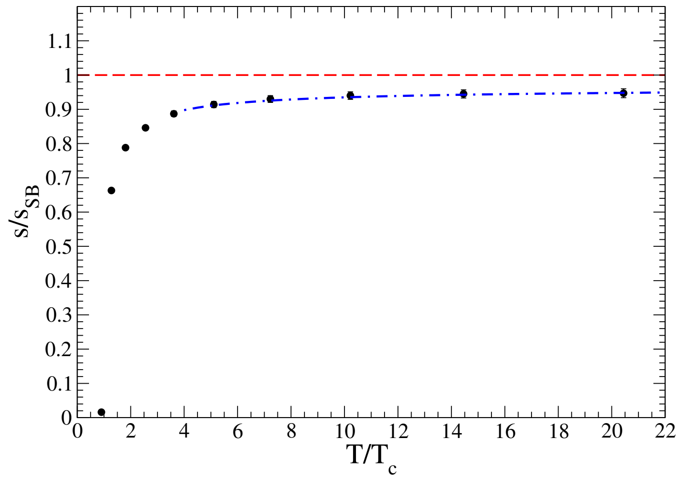
- Precision between 0.5%–1.5%. Will be reduced to $\sim 0.5\%$ in the next few months
- At $T \sim 20\,T_c$ the entropy still differs from the Stefan-Boltzmann value by roughly 5%
- When matching with perturbation theory (blue line), the series has oscillating coeffs. At $T \sim 20\,T_c$, the $O(g^6)$ is roughly 40% of total correction with respect to SB

Entropy density in the continuum

- At the reference temperature T_0 the entropy density is obtained by extrapolating

$$\frac{s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\frac{(1+\xi^2)}{\xi_k}\frac{Z_T\,\langle T^{[1]}_{0k}\rangle_{\boldsymbol{\xi}}}{T^4}$$

to the continuum limit



- Results for $T \leq 4\,T_c$ agree with [Boyd et al 96, Meyer 09]

- For $T \geq 2\,T_c$ agree with [Borsanyi et al 13] within errors. We observe a tension with these data, however, for $T \leq 2\,T_c$

- The computation at more temperature values in the region $T \leq 2\,T_c$ and at $T > 20\,T_c$ is in progress

Conclusions and outlook (I)

- Lorentz invariance implies a great degree of redundancy in defining a relativistic thermal theory in the Euclidean path-integral formalism

- In the thermodynamic limit, the orientation of the compact periodic direction with respect to the coordinate axes can be chosen at will and only its length is relevant

$$f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right) = -\lim_{V\to\infty}\frac{1}{L_0 V}\ln Z(L_0,\boldsymbol{\xi})$$

- The redundancy in the description implies that the total energy and momentum distributions in the canonical ensemble are related

- For a finite-size system, the lengths of the box dimensions break this invariance. Being a soft breaking, however, interesting exact Ward Identities survive

- As in the standard case, if the lightest screening mass $M \neq 0$, leading finite-size corrections are exponentially small in (ML)

Conclusions and outlook (II)

- When the theory is regularized on a lattice, the overall orientation of the periodic directions with respect to the lattice coordinate system affects renormalized observables at the level of lattice artifacts

- As the cutoff is removed, the artifacts are suppressed by a power of the spacing

- The flexibility in the lattice formulation added by the introduction of a triplet $\boldsymbol{\xi}$ of (renormalized) parameters has interesting consequences:

  $*$ WIs to renormalize non-perturbatively $T_{\mu\nu}$

  $*$ Simpler ways to compute thermodynamic potentials

$$s = -\frac{Z_T L_0 (1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k} \langle T_{0k} \rangle_{V_{\rm sbc}}$$

  $* \ldots$

- In the Yang–Mills theory we defined non-perturbatively $T_{\mu\nu}$, and we computed the entropy density over several orders of magnitude in T. Discretization and statistical errors are at the level of a few per mille in both cases