Lattice QCD: a theoretical femtoscope for non-perturbative strong dynamics

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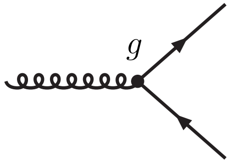
Outline

- Introduction to (lattice) QCD:
  - ∗ Asymptotic freedom and dimensional transmutation
  - ∗ Quantum chromodynamics on a lattice
- Spontaneous symmetry breaking:
  - ∗ Banks–Casher relation
  - ∗ Renormalization of the spectral density
  - ∗ Exploratory numerical study
- Witten–Veneziano solution to the $\mathrm{U}(1)_\mathrm{A}$ problem:
  - ∗ Definition of the topological susceptibility
  - ∗ Non-perturbative computation
- Conclusions

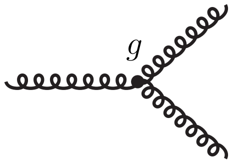
Quantum chromodynamics (QCD)

- QCD is assumed to be the quantum field theory of strong interactions in Nature. Its action
  [Fritzsch, Gell-Mann, Leutwyler 73; Gross, Wilczek 73; Weinberg 73]

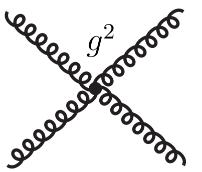
$$S[A, \bar{\psi}_i, \psi_i; g, m_i, \theta]$$



is fixed by few simple principles:



  - $*$ SU(3)$_c$ gauge (local) invariance
  - $*$ Quarks in fundamental representation $\psi_i = u, d, s, c, b, t$



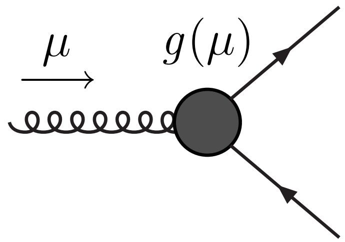
  - $*$ Renormalizability

- Present experimental results compatible with $\theta = 0$

- It is fascinating that such a simple action and few parameters $[g, m_i]$ can account for the variety and richness of strong-interaction physics phenomena

- Quantization breaks scale invariance at $m_i = 0$

- The renormalized coupling constant is scale dependent



$$\mu \frac{d}{d\mu} g = \beta(g)$$

and QCD is asymptotically free [$b_0 > 0$]
[Gross, Wilczek 73; Politzer 73]

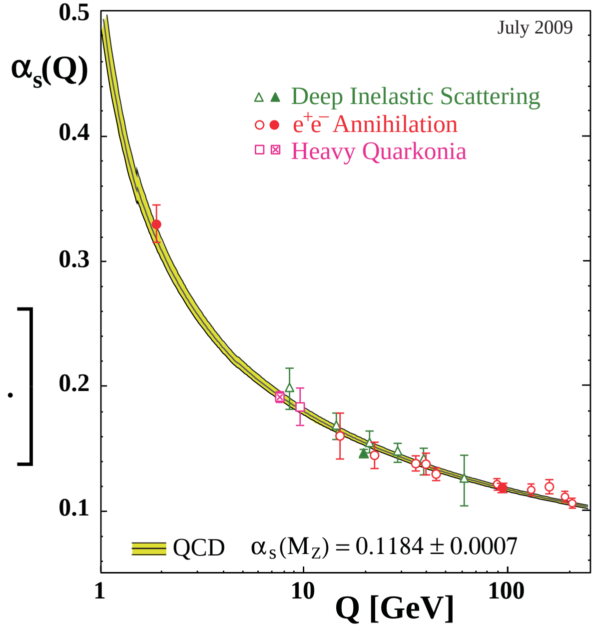
$$\beta(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

- The theory develops a fundamental scale

$$\Lambda = \mu \left[b_0 g^2(\mu)\right]^{-b_1/2b_0^2} e^{-\frac{1}{2b_0 g^2(\mu)}} e^{-\int_0^{g(\mu)} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]}$$

which is a non-analytic function of the coupling constant at $g^2 = 0$

[Bethke 09]

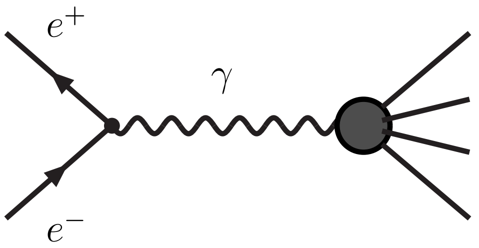


- Processes where the relevant energy scale is $\mu \gg \Lambda$ can be studied by perturbative expansion

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} = \frac{1}{4\pi b_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \left[1 - \frac{b_1}{b_0^2} \frac{\ln(\ln(\frac{\mu^2}{\Lambda^2}))}{\ln(\frac{\mu^2}{\Lambda^2})} + \ldots \right]$$

- An example is given by

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_i Q_i^2 \cdot \left[1 + \frac{\alpha_s(\mu)}{\pi} + C_2 \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 + \cdots \right]$$

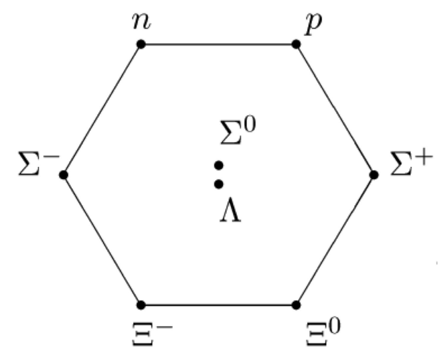


- Experimental results significantly prove the logarithmic dependence in μ/Λ predicted by perturbative QCD

Scale of the strong interactions

- By comparing these measurements to theory

$$\Lambda \sim 0.2 \text{ GeV} \qquad 1/\Lambda \sim 1 \text{ fm} = 10^{-15} \text{ m}$$

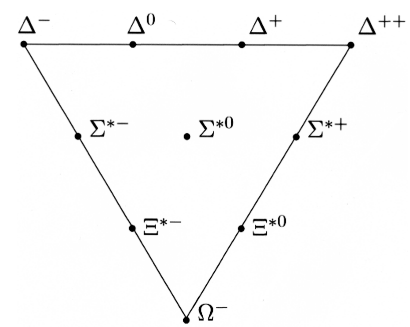


- At these distances the dynamics of QCD is non-perturbative

- A rich spectrum of hadrons is observed at these energies. Their properties such as the mass

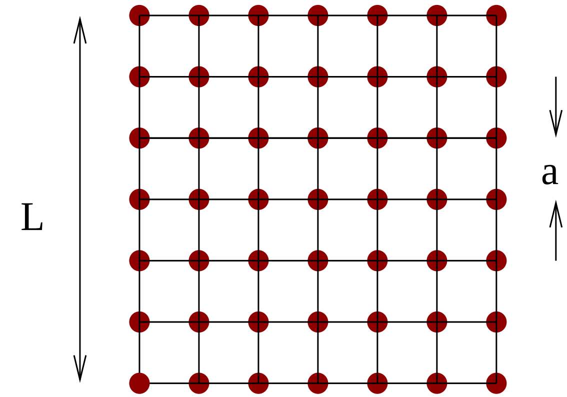
$$M_n = b_n \, \Lambda$$

need to be computed non-perturbatively



- The theory is highly predictive: in the (interesting) limit $m_{u,d,s} = 0$ and $m_{c,b,t} \to \infty$, for instance, dimensionless quantities are parameter-free numbers

- QCD can be defined on a discretized space-time so that gauge invariance is preserved



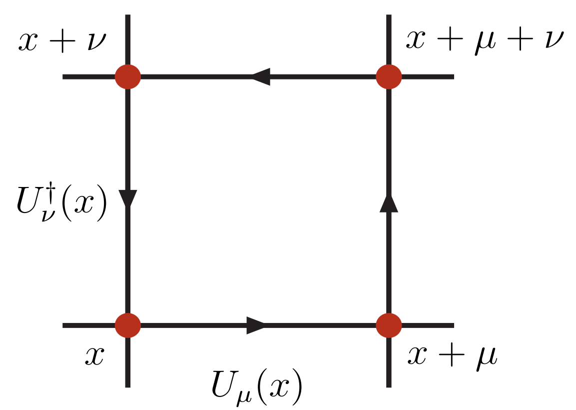
- Quark fields reside on a four-dimensional lattice, the gauge field $U_\mu \in \mathrm{SU}(3)$ resides on links

- The Wilson action for the gauge field is

$$S_G[U] = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{3}\mathrm{ReTr}\left\{U_{\mu\nu}(x)\right\}\right]$$

where $\beta = 6/g^2$ and the plaquette is defined as

$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x)$$



- Popular discretizations of fermion action: Wilson, Domain-Wall-Neuberger, perfect actions, tmQCD

Lattice QCD: path integral [Wilson 74]

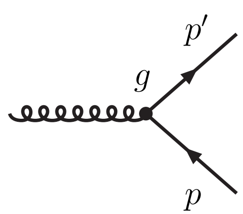
- The lattice provides a non-perturbative definition of QCD. The path integral at finite spacing and volume is mathematically well defined (Euclidean time)

$$Z = \int DU D\bar{\psi}_i D\psi_i \; e^{-S[U,\bar{\psi}_i,\psi_i;g,m_i]}$$

- Nucleon mass, for instance, can be extracted from the behaviour of a suitable two-point correlation function at large time-distance

$$\langle O_N(x)\bar{O}_N(y)\rangle = \frac{1}{Z}\int DU D\bar{\psi}_i D\psi_i \; e^{-S}\, O_N(x)\bar{O}_N(y) \longrightarrow R_N \; e^{-M_N\,|x_0-y_0|}$$

- For small gauge fields, the perturb. expansion differs from the usual one for terms of $O(a)$



$$= -igT^a\left\{\gamma_\mu - \tfrac{i}{2}(p_\mu + p'_\mu)a + O(a^2)\right\}$$

Consistency of lattice QCD with the standard perturbative approach is thus guaranteed

Lattice QCD: universality

- Continuum and infinite-volume limit of LQCD is the $non\text{-}perturbative\ definition$ of QCD

- Details of the discretization become irrelevant in the continuum limit, and any reasonable lattice formulation tends to the same continuum theory

$$M_N(a) = M_N + c_N a + \ldots$$

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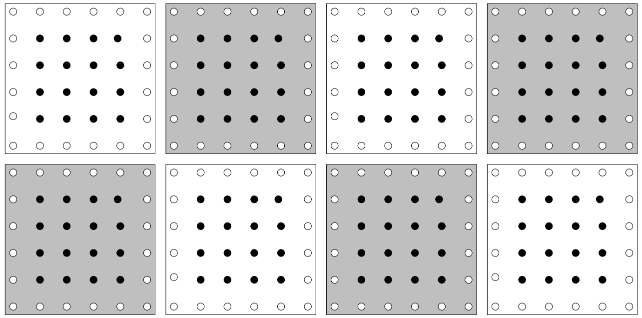
- Details of the discretization become irrelevant in the continuum limit, and any reasonable lattice formulation tends to the same continuum theory

$$M_N(a) = M_N + d_N a^2 + \dots$$

- By a proper tuning of the action and operators, convergence to continuum can be accelerated without introducing extra free-parameters [Symanzik 83; Sheikholeslami Wohlert 85; Lüscher et al. 96]

- Finite-volume effects are proportional to $\exp(-M_\pi L)$ at asymptotically large volumes

Numerical lattice QCD: machines

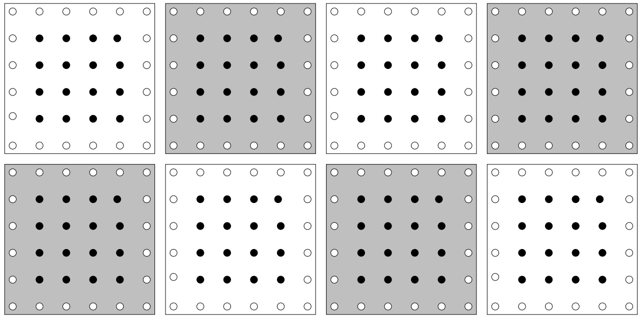


Correlation functions at $finite\ volume$ and $finite\ lattice\ spacing$ can be computed by Monte Carlo techniques $exactly$ up to statistical errors

Numerical lattice QCD: machines

- Typical lattice parameters:

$$a = 0.05 \text{ fm} \qquad\qquad (a\Lambda)^2 \sim 0.25\%$$
$$L = 3.2 \text{ fm} \qquad \Longrightarrow \qquad M_\pi L \geq 4, \quad M_\pi \geq 0.25 \text{ GeV}$$
$$V = 2L \times L^3 \qquad\qquad \#\text{points} = 2^{25} \sim 3.4 \cdot 10^7$$



- Monte Carlo algorithms integrate over 10^7–10^9 SU(3) link variables

- A typical cluster of PCs:
  - ∗ Standard CPUs [AMD, Intel]
  - ∗ Fast connection [40Gbit/s]

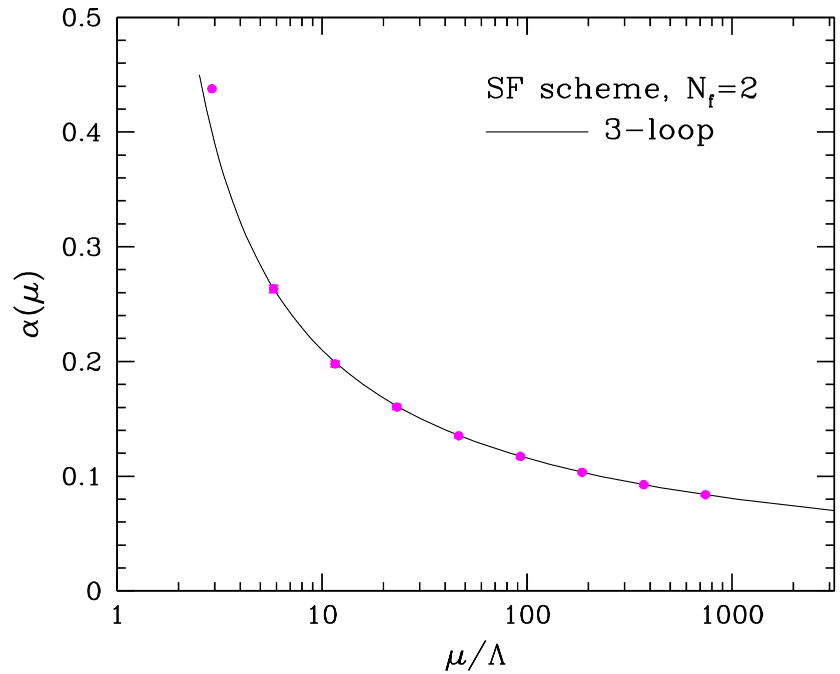
- Lattice partitioned in blocks which are distributed over the nodes (128 a good example)

- Data exchange among nodes minimized thanks to the locality of the action

Numerical lattice QCD: algorithms

- Extraordinary algorithmic progress over the last 30 years, keywords:
  - ∗ Hybrid Monte Carlo (HMC)
    Duane et al. 87
  - ∗ Multiple time-step integration
    Sexton, Weingarten 92
  - ∗ Frequency splitting of determinant
    Hasenbusch 01
  - ∗ Domain Decomposition
    Lüscher 04
  - ∗ Mass preconditioning and rational HMC
    Urbach et al 05; Clark, Kennedy 06
  - ∗ Deflation of the low quark modes
    Lüscher 07

[Della Morte et al. 05]



- Light dynamical quarks can be simulated (continuum limit still problematic). Chiral regime of QCD is becoming accessible
- Algorithms are designed to produce exact results up to statistical errors

Lattice QCD: a theoretical femtoscope

- Lattice QCD is the femtoscope for studying strong dynamics. Its lenses are made of quantum field theory, numerical techniques and computers

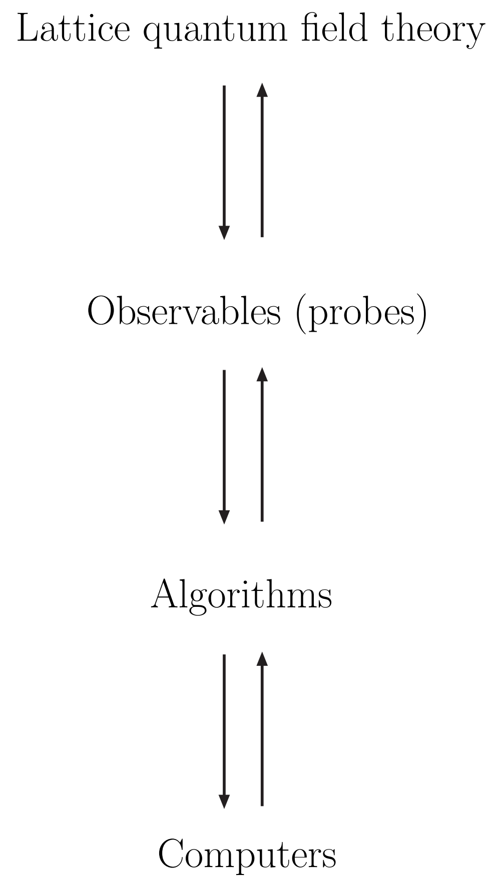
- It allows us to look also at quantities not accessible to experiments which may help understanding the underlying mechanisms

- Femtoscope still rather crude. Often we compute what we can and not what would like to

- An example: the signal-to-noise ratio of the nucleon two-point correlation function

$$\frac{\langle O_N \bar{O}_N \rangle^2}{\Delta^2} \propto n\, e^{-(2M_N - 3M_\pi)|x_0 - y_0|}$$

  decreases exp. with time-distance of sources. At physical point $2M_N – 3M_\pi \simeq$ 7 fm^{-1}



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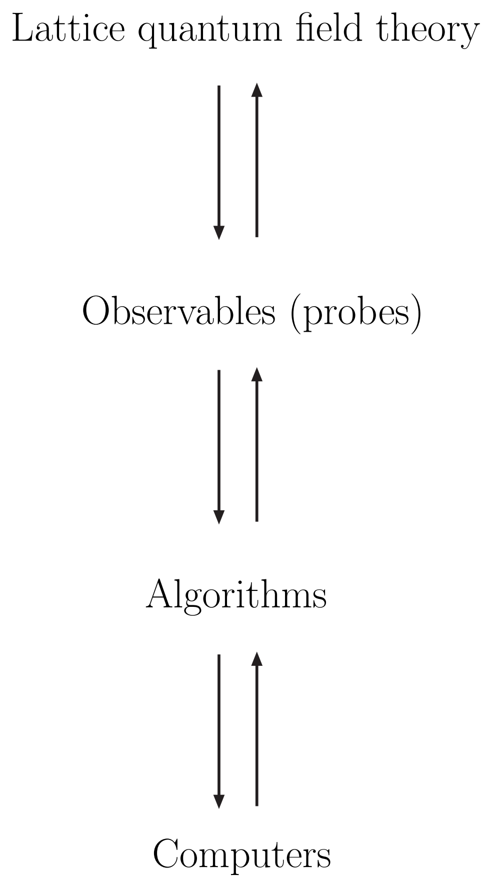
- Analogous problem for glueballs in Yang–Mills theory solved by decomposing the path integral and by enforcing the global symmetries of the theory into the Monte Carlo [Della Morte, LG 08-10]



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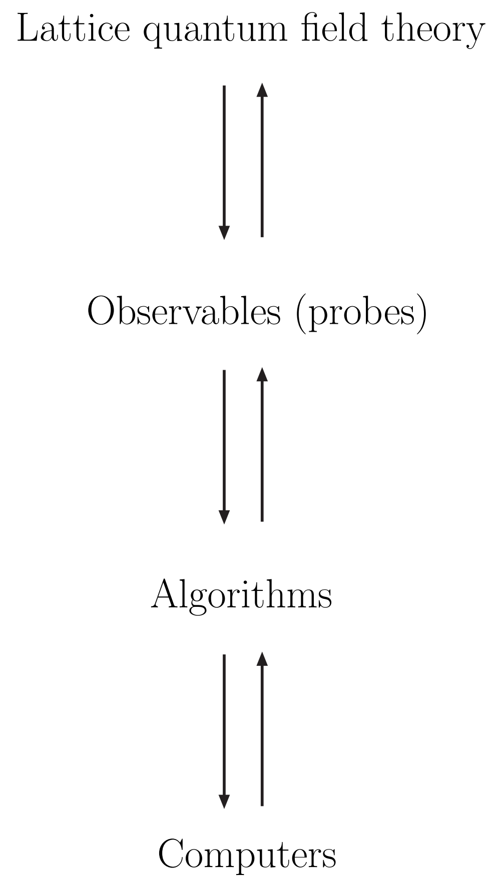
- Femtoscope still rather crude. Often we compute what we can and not what would like to
- A rather general strategy is emerging: design special purpose algorithms which exploit known math. and phys. properties of the theory to be faster
- Results from first-principles when all syst. uncertainties quantified. This achieved without introducing extra free parameters or dynamical assumptions but just by improving the femtoscope



QCD action and its (broken) symmetries

- QCD action for $N_f = 3$, $M = \text{diag}(m_u, m_d, m_s)$

$$S = S_G + \int d^4x \left\{ \bar\psi D\psi + \bar\psi M\psi \right\}, \qquad D = \gamma_\mu(\partial_\mu + iA_\mu)$$

- For $M = 0$ chiral symmetry

$$\psi_{\rm R,L} \to V_{\rm R,L}\psi_{\rm R,L} \quad \psi_{\rm R,L} = \left(\frac{1\pm\gamma_5}{2}\right)\psi$$

Chiral anomaly: measure not invariant

SSB: vacuum not symmetric

- Gauge symmetry

$$\psi(x) \to G(x)\psi(x)$$

Confinement: no isolated coloured charge

$$\mathrm{SU}(3)_c \times \mathrm{SU}(3)_{\rm L} \times \mathrm{SU}(3)_{\rm R} \times \mathrm{U}(1)_{\rm L} \times \mathrm{U}(1)_{\rm R} \times \mathcal{R}_{\rm scale}$$

$\downarrow$ (dim. transm., chiral anomaly)

$$\mathrm{SU}(3)_c \times \mathrm{SU}(3)_{\rm L} \times \mathrm{SU}(3)_{\rm R} \times \mathrm{U}(1)_{\rm B=L+R}$$

$\downarrow$ (Spont. Sym. Break.)

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↓ (dim. transm., **chiral anomaly**)

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↓ **(Spont. Sym. Break.)**

$$\mathrm{SU}(3)_c \times \mathrm{SU}(3)_{\mathrm{L+R}} \times \mathrm{U}(1)_{\mathrm{B}}$$

↓ (Confinement)

$$\mathrm{SU}(3)_{\mathrm{L+R}} \times \mathrm{U}(1)_{\mathrm{B}}$$

- Confinement and SSB due to non-perturbative dynamics
- The mechanisms are still not know
- Today focus on SSB and chiral anomaly

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- An axial Ward identity of the chiral group is [for simplicity $M =$ diag(m, m, m)]

$$\langle \bar\psi_1 \psi_1 \rangle = m \int d^4x \, \langle P_{12}(x) P_{21}(0) \rangle \, , \qquad P_{ij} = \bar\psi_i \gamma_5 \psi_j$$

- In the limit $m \to 0$

$$\Sigma = - \lim_{m \to 0} \langle \bar\psi_1 \psi_1 \rangle \neq 0 \qquad \Longrightarrow \qquad M^2 = \frac{2m\Sigma}{F^2}$$ [Gell-Mann, Oakes, Renner 68]

where the decay constant is defined as

$$|\langle 0 | \hat{A}_{12,\mu} | \pi^-, p \rangle| = \sqrt{2}\, F_\pi \, p_\mu \, , \qquad F = \lim_{m \to 0} F_\pi$$

- Octet compatible with SSB pattern

$$\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \rightarrow \mathrm{SU}(3)_{\mathrm{L+R}}$$

and soft explicit symmetry breaking

$$m_u, m_d \ll m_s < \Lambda$$

- $m_u, m_d \ll m_s \Longrightarrow m_\pi \ll m_{\mathrm{K}}$

- A 9^{th} pseudoscalar with $m_{\eta'} \sim \mathcal{O}(\Lambda)$

I	I_3	S	Meson	Quark Content	Mass (GeV)
1	1	0	π^+	$u\bar{d}$	0.140
1	-1	0	π^-	$d\bar{u}$	0.140
1	0	0	π^0	$(d\bar{d} - u\bar{u})/\sqrt{2}$	0.135
$\frac{1}{2}$	$\frac{1}{2}$	+1	K^+	$u\bar{s}$	0.494
$\frac{1}{2}$	$-\frac{1}{2}$	+1	K^0	$d\bar{s}$	0.498
$\frac{1}{2}$	$-\frac{1}{2}$	-1	K^-	$s\bar{u}$	0.494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\overline{\mathrm{K}}^0$	$s\bar{d}$	0.498
0	0	0	η	$\cos\vartheta\eta_8 - \sin\vartheta\eta_0$	0.548
0	0	0	η'	$\sin\vartheta\eta_8 + \cos\vartheta\eta_0$	0.958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\vartheta \sim -10^\circ$$

- Chiral effective theory for pions

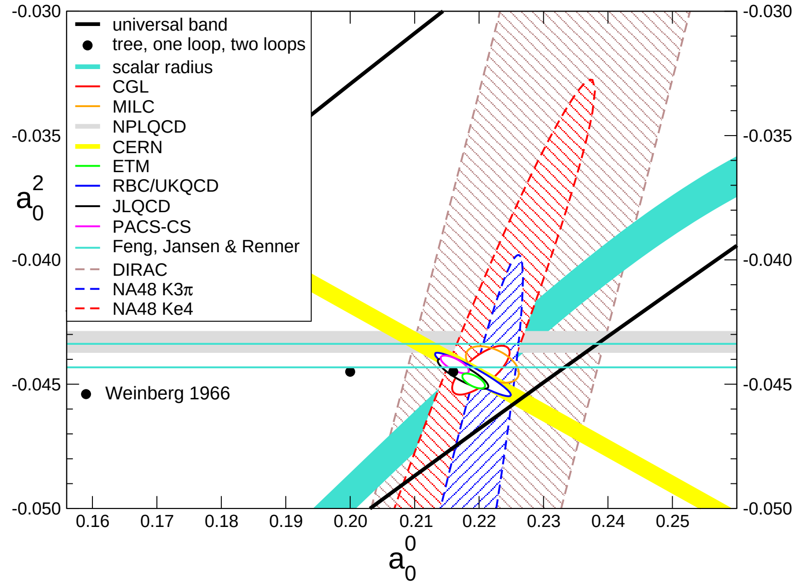
$$S_{\rm eff} = S^2_{\rm eff}(U; m, F, \Sigma) + S^4_{\rm eff}(U; m, F, \Sigma, \Lambda_i) + \cdots$$

encodes spontaneous symmetry breaking

- For $m = 0$ pions can interact only if they carry momentum. Expansion in p and m

- Chiral dynamics parameterized by effective low-energy coupling constants

[Colangelo, Gasser, Leutwyler 01; Leutwyler 09]



- For instance the pion mass and decay constant at $\mathcal{O}(p^4)$ are given by

$$M_\pi^2 = M^2 \left\{ 1 + \frac{M^2}{32\pi^2 F^2} \ln\left(\frac{M^2}{\Lambda_3^2}\right) \right\}, \qquad F_\pi = F \left\{ 1 - \frac{M^2}{16\pi^2 F^2} \ln\left(\frac{M^2}{\Lambda_4^2}\right) \right\}$$

Analogous expressions for other quantities such as S-wave $\pi\pi$ scattering lengths a_0^0 and a_0^2

Lattice QCD with two light dynamical quarks [Del Debbio, LG, Lüscher, Petronzio, Tantalo 06-08]

- Chiral regime is becoming accessible to lattice QCD simulations

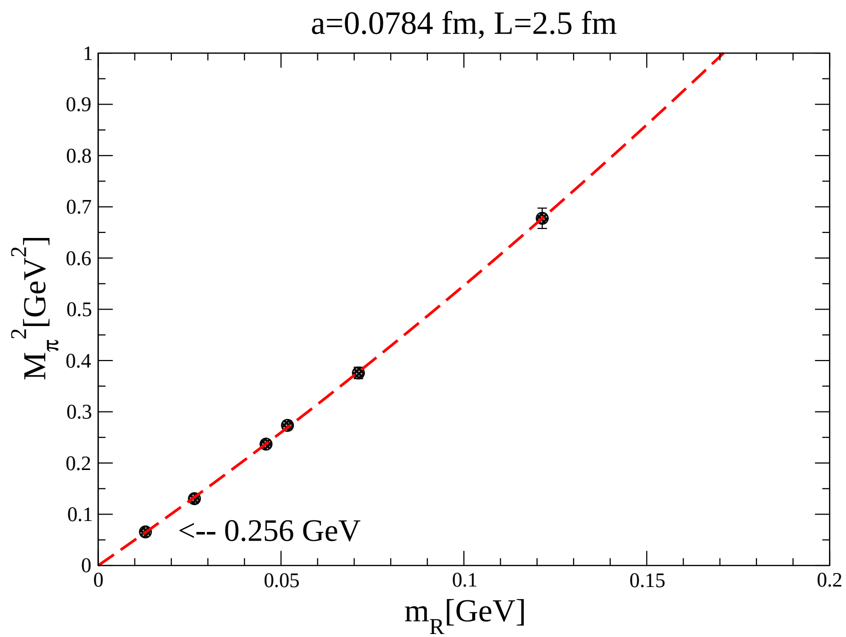
- The pion mass squared is found to be a nearly linear function of quark mass up to $(0.5\ \text{GeV})^2$. At smallest masses non-linear correction is 1 - 3%

- Non-Abelian chiral symmetry spontaneously broken as expected

- Compatible with the fact that the bulk of the mass is given by the leading term in standard ChPT

- Relations dictated by SSB can be verified quantitatively. GMOR is maybe the simplest to start with

- Low-energy constants will finally be determined



Banks–Casher relation [Banks, Casher 80]

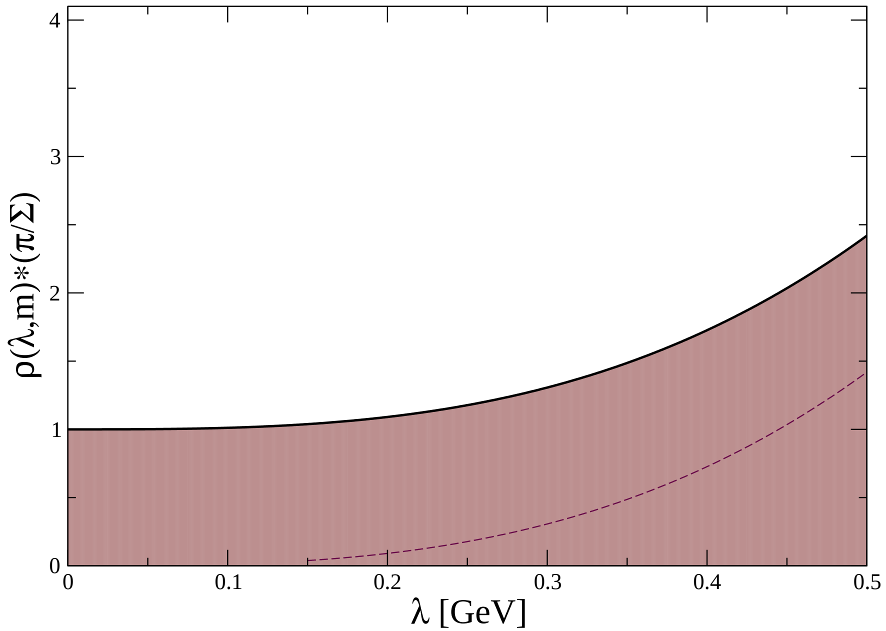
- For each gauge configuration

$$D_m \chi_k = (m + i\lambda_k)\chi_k$$

- The spectral density of D is

$$\rho(\lambda, m) = \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle$$

where $\langle \ldots \rangle$ indicates path-integral average



- The Banks–Casher relation

$$\lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m) = \frac{\Sigma}{\pi}$$

provides a link between the condensate and the (non-zero) spectral density at the origin.
To be compared, for instance, with the free case $\rho(\lambda) \propto |\lambda^3|$

- For each gauge configuration

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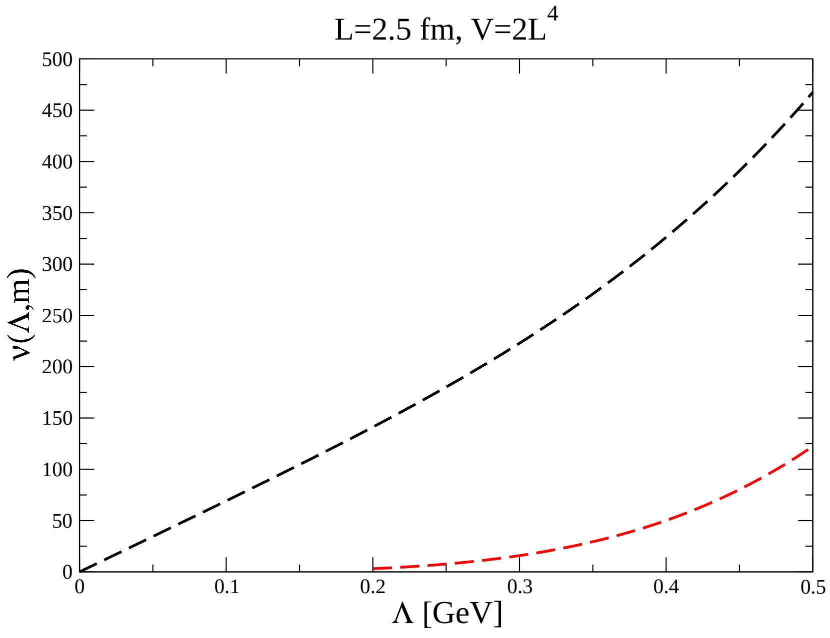

- The number of modes in a given energy interval

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda\, \rho(\lambda, m) \qquad \nu(\Lambda, m) = \frac{2}{\pi}\Lambda\Sigma V + \ldots$$

grows linearly with Λ, and they condense near the origin with values $\propto 1/V$

In the free case $\nu(\Lambda, m) \propto V\Lambda^4$

Renormalization and continuum limit [LG, Lüscher 09]

- Instead of the spectral density, consider the spectral sum

$$\sigma_k(m_v, m) = V \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m_v^2)^k}$$

$$= -a^{8k} \sum_{x_1 \dots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \dots P_{2k1}(x_{2k}) \rangle$$

$*$ Integral converges if $k \geq 3$

$*$ The relation between $\sigma_k(m_v, m)$ and $\rho(\lambda, m)$ invertible for every k

- Renormalization properties of $\rho(\lambda, m)$ can thus be inferred from those of σ_k

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- Corr. functions of pseudoscalar densities at physical distance renormalized by $(1/Z_m)^{2k}$

- At short distance the flavour structure implies

$$P_{12}(x_1) P_{23}(x_2) \sim C(x_1 - x_2) S_{13}(x_1) \qquad S_{13} = \bar{\psi}_1 \psi_3$$

where $C(x)$ diverges like $|x|^{-3}$ and it is therefore integrable. Analogous argument for all other short-distance singularities. No extra contact terms needed to renormalize σ_k

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$$\begin{aligned}\sigma_k(m_v,m) &= V\int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda,m)}{(\lambda^2+m_v^2)^k} \\ &= -a^{8k}\sum_{x_1\ldots x_{2k}} \langle P_{12}(x_1)P_{23}(x_2)\ldots P_{2k1}(x_{2k})\rangle\end{aligned}$$

- Once the gauge coupling and the mass(es) are renormalized, the spectral sum

$$\sigma_{k,\mathrm{R}}(m_{v\mathrm{R}},m_{\mathrm{R}}) = Z_m^{-2k}\sigma_k\left(\frac{m_{v\mathrm{R}}}{Z_m},\frac{m_{\mathrm{R}}}{Z_m}\right)$$

is ultraviolet finite. Continuum limit universal (if same renormalization conditions are used)

Renormalization and continuum limit [LG, Lüscher 09]

- Instead of the spectral density, consider the spectral sum

$$
\begin{aligned}
\sigma_k(m_v, m) &= V \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m_v^2)^k} \\
&= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle
\end{aligned}
$$

- The spectral density thus renormalizes as

$$
\rho_{\mathrm{R}}(\lambda_{\mathrm{R}}, m_{\mathrm{R}}) = Z_m^{-1} \rho \left(\frac{\lambda_{\mathrm{R}}}{Z_m}, \frac{m_{\mathrm{R}}}{Z_m} \right)
$$

- For Wilson fermions similar derivation but twisted-mass valence quarks

Renormalization and continuum limit [LG, Lüscher 09]

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$$\sigma_k(m_v, m) = V \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m_v^2)^k}$$

$$= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle$$

- It follows that the mode number is a renormalization-group invariant

$$\nu_{\rm R}(\Lambda_{\rm R}, m_{\rm R}) = \nu(\Lambda, m)$$

and its continuum limit is universal for any value of Λ and m

Numerical computation (I) [LG, Lüscher 09]

- Lattice details:

  ∗ $N_f = 2$ degenerate quarks

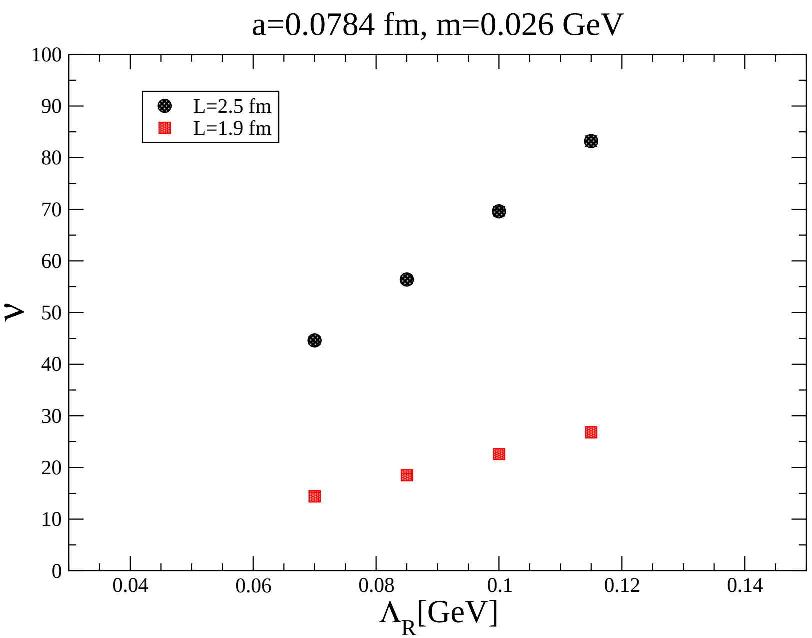
  ∗ Action: $O(a)$-improved Wilson

  ∗ $a = 0.0784$ fm

  ∗ $V = 2L \times L^3$, $L = 1.9, 2.5$ fm

  ∗ $m_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.013, 0.026, 0.046$ GeV

  ∗ $\Lambda_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.07, 0.085, 0.1, 0.115$ GeV



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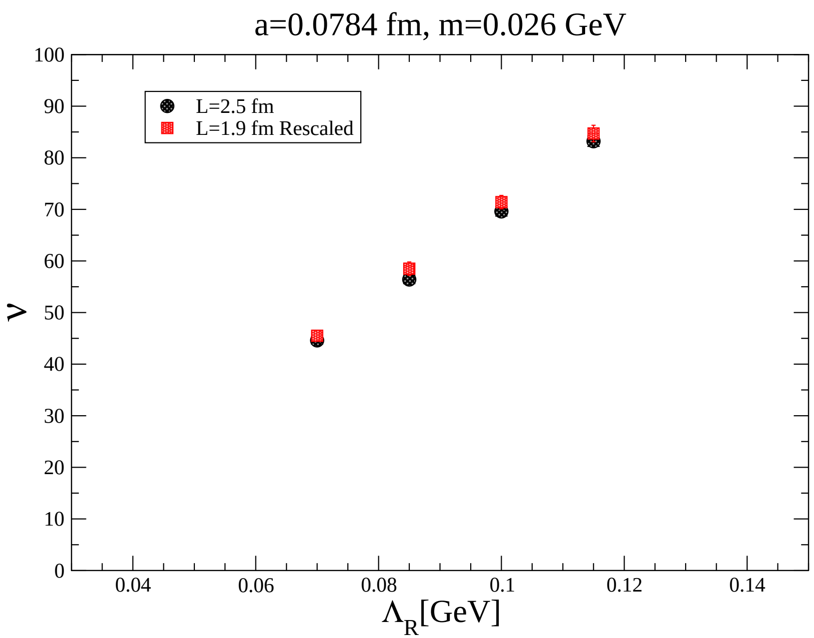

- Finite volume effects below stat. errors (1.5%). ChPT suggests a fraction of a percent

- A nearly linear function up to 0.1 Gev. Qualitative in line with ChPT, but the fact that the linear behaviour extends to such large values of $\Lambda_{\rm R}$ is rather striking and unexpected

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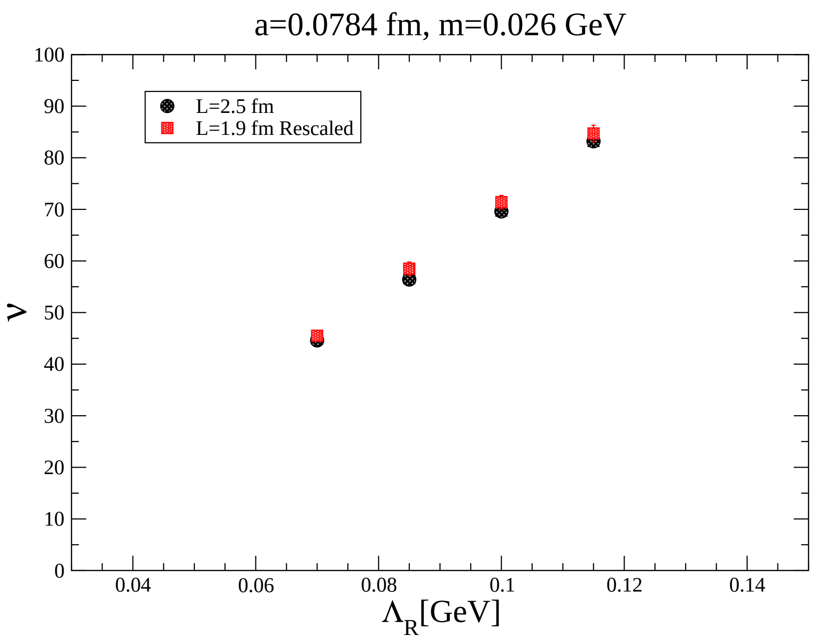


- Finite volume effects below stat. errors (1.5%).
  ChPT suggests a fraction of a percent

- In the effective theory at NLO

$$\nu^{\rm nlo}(\Lambda, m) = \frac{2\Lambda\Sigma V}{\pi}\left\{1 - \frac{m\,\Sigma}{(4\pi)^2 F^4}\left[3\ln\left(\frac{\Lambda\Sigma}{F^2\Lambda_6^2}\right) + \ln(2) + \frac{\pi}{2}\frac{m}{\Lambda} + O\left(\frac{m^2}{\Lambda^2}\right)\right]\right\}$$

corrections of $\mathcal{O}(10\%)$ for $\Lambda = 0.05$–0.1 GeV and $m \leq 0.02$ GeV. No chiral logs $\propto m\ln(m)$

Numerical computation (II) [LG, Lüscher 09]

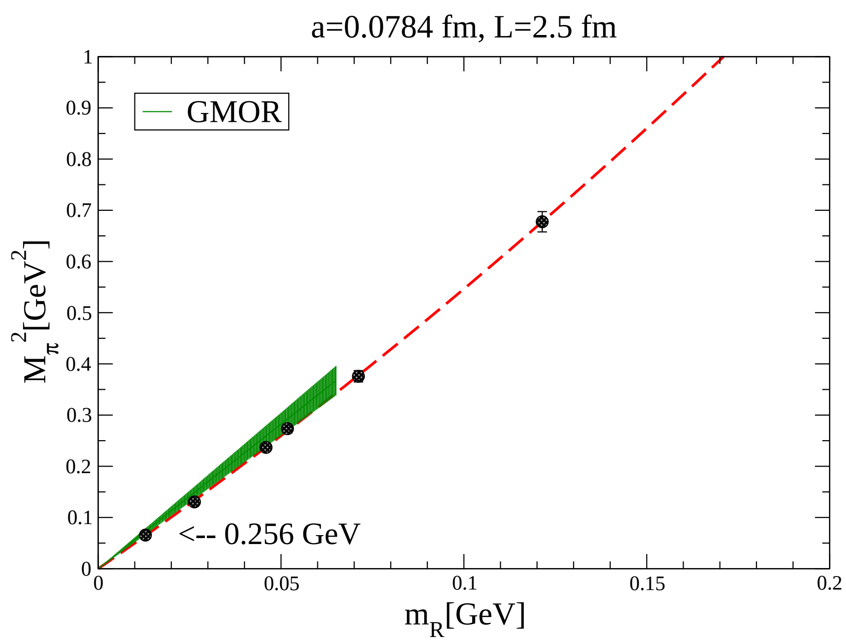
- An effective condensate can be defined as

$$\overline{\Sigma}_{\rm R} = \frac{\pi}{2V}\frac{\partial}{\partial \Lambda_{\rm R}}\,\nu_{\rm R}(\Lambda_{\rm R}, m_{\rm R})$$

prefactor so that $\overline{\Sigma}_{\rm R}$ coincides with Σ at LO

- A linear extrapolation to the chiral limit yields

$$\left[\Sigma_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV})\right]^{1/3} = 0.276(3)(4)(5)\ \text{GeV}$$

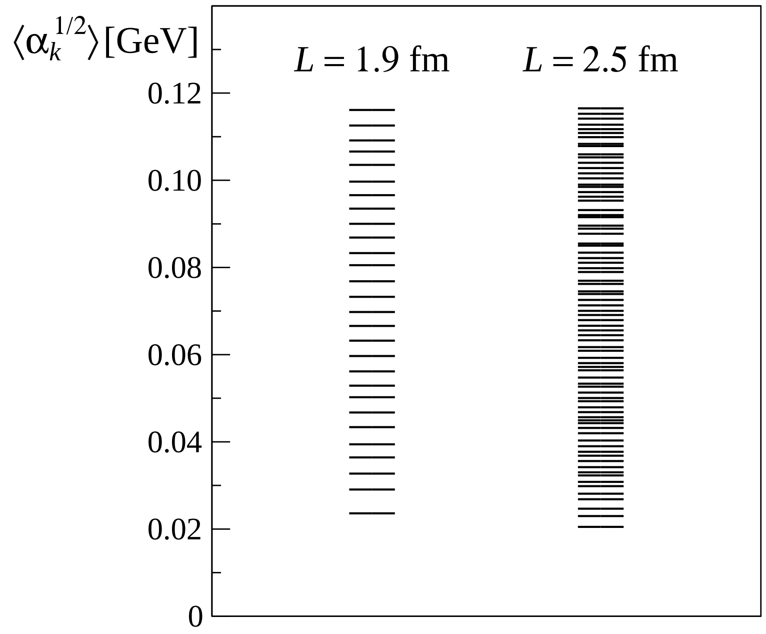


- A clear and consistent picture is emerging. For $m_R \leq 0.05$ GeV the GMOR formula accounts for the bulk of the pion mass. But discretization errors not quantified yet

Why is the symmetry spontaneously broken?

- Dynamical process not yet known. Studies of low modes can provide important clues

- The Banks–Casher mechanism is:
  - ∗ insensitive to lattice details (universality)
  - ∗ largely insensitive to dynamical quark effects
  - ∗ present also in quenched QCD



Eigenvalues of $D_m^\dagger D_m$

- It is tempting to read the relation in the other direction, i.e. chiral symmetry is broken because the low-modes of the Dirac operator condense

$$\frac{\Sigma}{\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty} \rho(\lambda, m)$$

Outline

- Introduction to (lattice) QCD:
  - ∗ Asymptotic freedom and dimensional transmutation
  - ∗ Quantum chromodynamics on a lattice
- Spontaneous symmetry breaking:
  - ∗ Banks–Casher relation
  - ∗ Renormalization of the spectral density
  - ∗ Exploratory numerical study
- Witten–Veneziano solution to the $\mathrm{U}(1)_\mathrm{A}$ problem:
  - ∗ Definition of the topological susceptibility
  - ∗ Non-perturbative computation
- Conclusions

Light pseudoscalar meson spectrum

- Octet compatible with SSB pattern

$$\text{SU}(3)_{\rm L} \times \text{SU}(3)_{\rm R} \rightarrow \text{SU}(3)_{\rm L+R}$$

and soft explicit symmetry breaking

$$m_u, m_d \ll m_s < \Lambda$$

- $m_u, m_d \ll m_s \Longrightarrow m_\pi \ll m_{\rm K}$

- A $9^{\rm th}$ pseudoscalar with $m_{\eta'} \sim \mathcal{O}(\Lambda)$

I	I_3	S	Meson	Quark Content	Mass (GeV)
1	1	0	π^+	$u\bar{d}$	0.140
1	-1	0	π^-	$d\bar{u}$	0.140
1	0	0	π^0	$(d\bar{d} - u\bar{u})/\sqrt{2}$	0.135
$\frac{1}{2}$	$\frac{1}{2}$	+1	K^+	$u\bar{s}$	0.494
$\frac{1}{2}$	$-\frac{1}{2}$	+1	K^0	$d\bar{s}$	0.498
$\frac{1}{2}$	$-\frac{1}{2}$	-1	K^-	$s\bar{u}$	0.494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\overline{\mathrm{K}}^0$	$s\bar{d}$	0.498
0	0	0	η	$\cos\vartheta\eta_8 - \sin\vartheta\eta_0$	0.548
0	0	0	η'	$\sin\vartheta\eta_8 + \cos\vartheta\eta_0$	0.958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\vartheta \sim -10^\circ$$

- An axial Ward identity of the chiral group is

$$\int d^4x \,\langle Q(x) Q(0)\rangle = m_1 m_2 \int d^4x\,\langle P_{11}(x) P_{22}(0)\rangle\,, \quad Q(x) = -\frac{1}{32\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\Big[F_{\mu\nu}(x) F_{\rho\sigma}(x)\Big]$$

- In the limit $N_c \to \infty$

$$\chi_\infty = \lim_{N_c\to\infty} \int d^4x\,\langle Q(x) Q(0)\rangle \neq 0 \qquad \Longrightarrow \qquad \lim_{N_c\to\infty}\lim_{m_i\to 0} \frac{F^2 M^2_{\eta'}}{2N_f} = \chi_\infty$$

- Note that for $N_c \to \infty$:

  $*$ $\mathrm{U}(1)_\mathrm{A}$ is restored

  $*$ η' becomes a Nambu–Goldstone boson $\Longrightarrow M_{\eta'} = 0$

  $*$ At first order in $1/N_c$, $M^2_{\eta'} = \mathcal{O}(N_f/N_c)$

The Witten–Veneziano mechanism [Witten 79; Veneziano 79]

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- An unambiguous definition of the topological susceptibility is required. Naive definition would diverge as

$$\chi = \int d^4x \, \langle Q(x) Q(0) \rangle \propto \frac{1}{a^4}$$

The Witten–Veneziano mechanism with Ginsparg–Wilson fermions

- With Ginsparg–Wilson fermions the lattice Ward identity is

[Neuberger 97; Hasenfratz, Laliena, Niedermayer 98; Lüscher 98; LG, Rossi, Testa, Veneziano 01]

$$\sum_x a^4 \langle Q(x)Q(0)\rangle = m_1 m_2 \sum_x a^4 \langle P_{11}(x)P_{22}(0)\rangle \,, \qquad Q(x) = -\frac{1}{2a^3}\mathrm{Tr}\Big[\gamma_5 D(x,x)\Big]$$

- In the limit $N_c \to \infty$

$$\chi_\infty = \lim_{N_c\to\infty} \sum_x a^4 \langle Q(x)Q(0)\rangle \neq 0 \qquad \Longrightarrow \qquad \lim_{N_c\to\infty}\lim_{m_i\to 0} \frac{F^2 M^2_{\eta'}}{2N_f} = \chi_\infty$$

- Need to demonstrate that the topological susceptibility suggested by GW fermions

$$\chi = \sum_x a^4 \langle Q(x)Q(0)\rangle$$

is ultraviolet finite and unambiguously defined

Renormalization and continuum limit [LG, Rossi, Testa 04; Lüscher 04]

- A chain of Ward identities holds

$$
\begin{array}{ll}
N_f = 2 & \chi = m_1 m_2 \sum_{x_1} a^4 \langle P_{11}(x_1) P_{22}(0) \rangle \\
\dots & \dots\dots \\
\dots & \dots\dots \\
N_f = 5 & \chi = m_1 \dots m_5 \sum_{x_1 \dots x_4} a^{16} \langle P_{31}(x_1) S_{12}(x_2) S_{23}(x_3) P_{54}(x_4) S_{45}(0) \rangle
\end{array}
$$

- It follows that the topological susceptibility is finite, it is renormalization-group invariant and its continuum limit is universal for any value of m

- A definition of χ even if the regularization breaks chiral symmetry

- The limit $N_c \to \infty$ is given by

$$
\lim_{N_c \to \infty} \chi = \lim_{N_c \to \infty} \chi^{\rm YM}
$$

and finiteness in Yang–Mills theory is proven analogously by introducing pseudofermions

Algorithm for zero-mode counting [LG, Hoelbling, Lüscher and Wittig 02]

- A Monte Carlo computation of

$$\chi^{\rm YM} = \frac{1}{V}\Big\langle (n_+ - n_-)^2 \Big\rangle^{\rm YM}$$

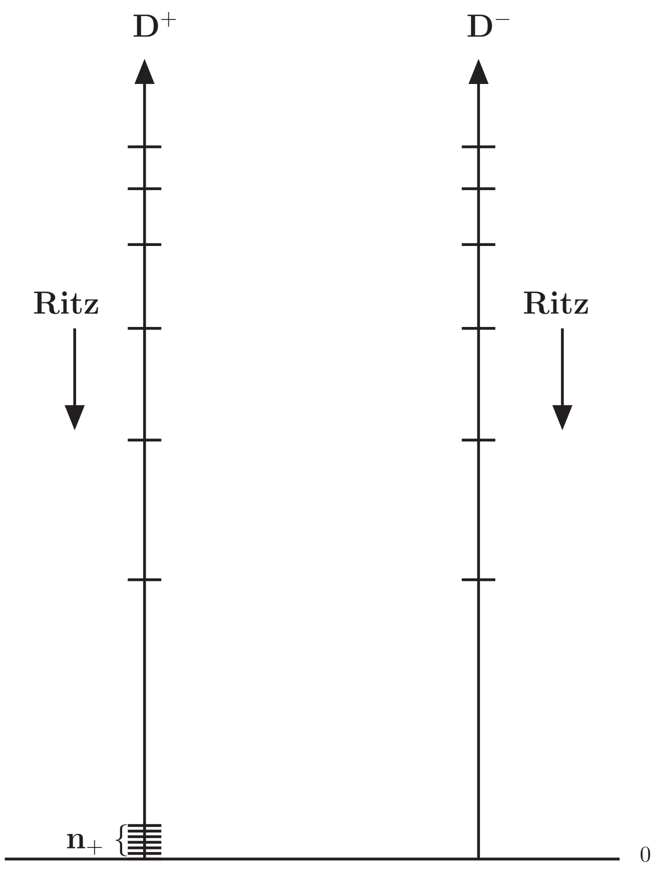
is challenging for several reasons

- $L \sim 2$ fm and $a \sim 0.08$ fm $\Longrightarrow \mathrm{dim}[D] \sim 4.5 \cdot 10^6$
- In finite V null probability for $n_+ \neq 0$ and $n_- \neq 0$
- Simultaneous minimization of Ritz functionals for

$$D^{\pm} = P_{\pm} D P_{\pm} \qquad P_{\pm} = \frac{1 \pm \gamma_5}{2}$$

to find the gap in one of the sectors and to count the zero modes in the other

- No contamination from quasi-zero modes



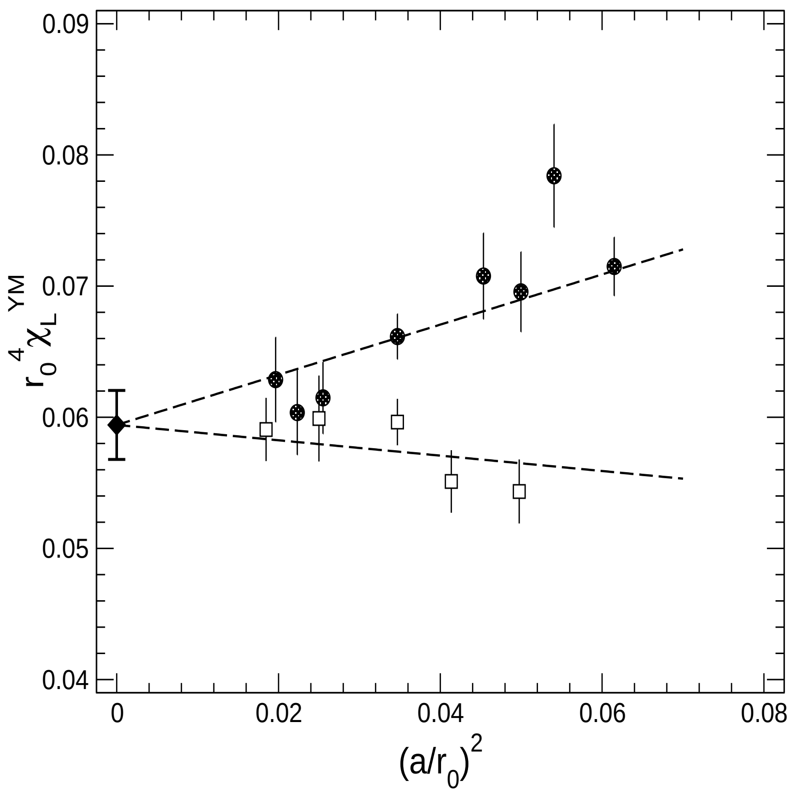
Non-perturbative computation for $N_c = 3$ [Del Debbio, LG, Pica 04]

- Combined fit of the form [$\chi^2_{\rm dof} = 0.73$]

$$r_0^4 \chi^{\rm YM}(a,s) = r_0^4 \chi^{\rm YM} + c_1(s)\frac{a^2}{r_0^2}$$

gives

$$r_0^4 \chi^{\rm YM} = 0.059 \pm 0.003$$



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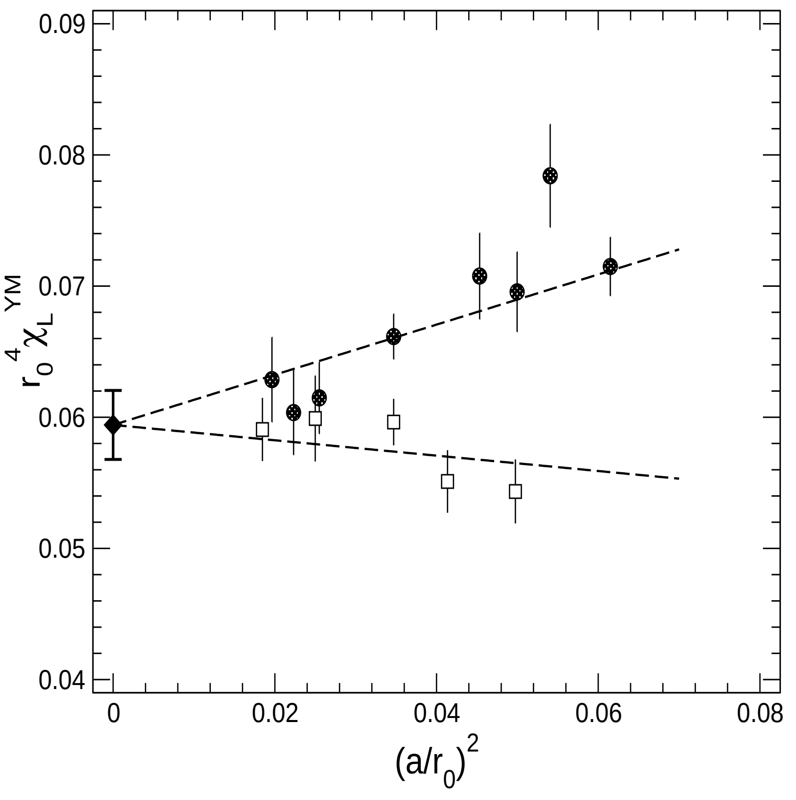
- By setting the scale $F_{\rm K} = 0.113(1)$ GeV

$$\chi^{\rm YM} = (0.191 \pm 0.005 \text{ GeV})^4$$



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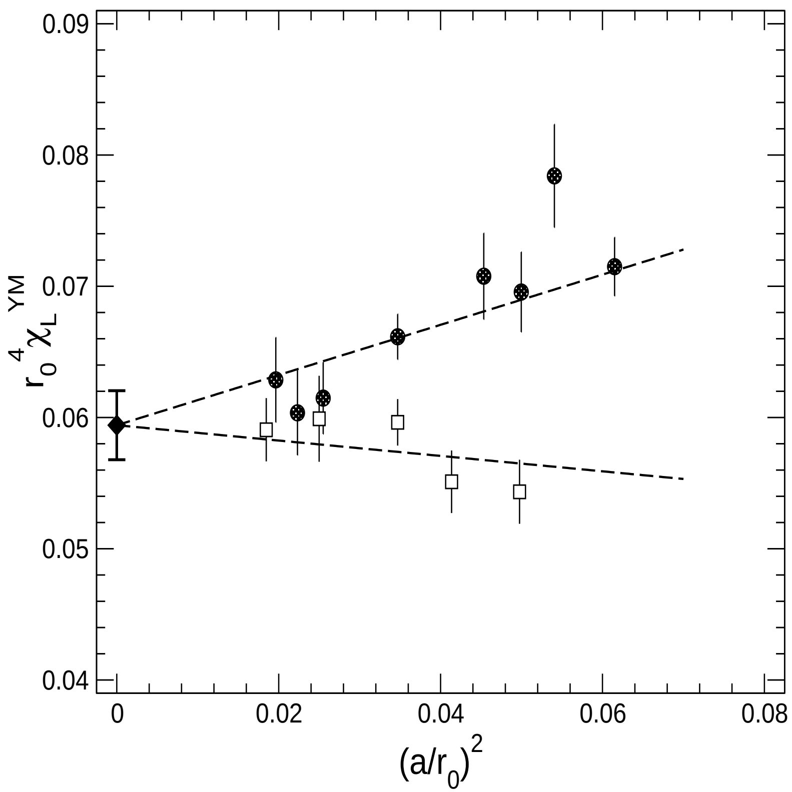
$$\boxed{\chi^{\rm YM} = (0.191 \pm 0.005\ {\rm GeV})^4}$$

to be compared with

$$\frac{F^2}{2N_f}(M_\eta^2 + M_{\eta'}^2 - 2M_K^2) \underset{\rm exp}{\approx} (0.175\ {\rm GeV})^4$$



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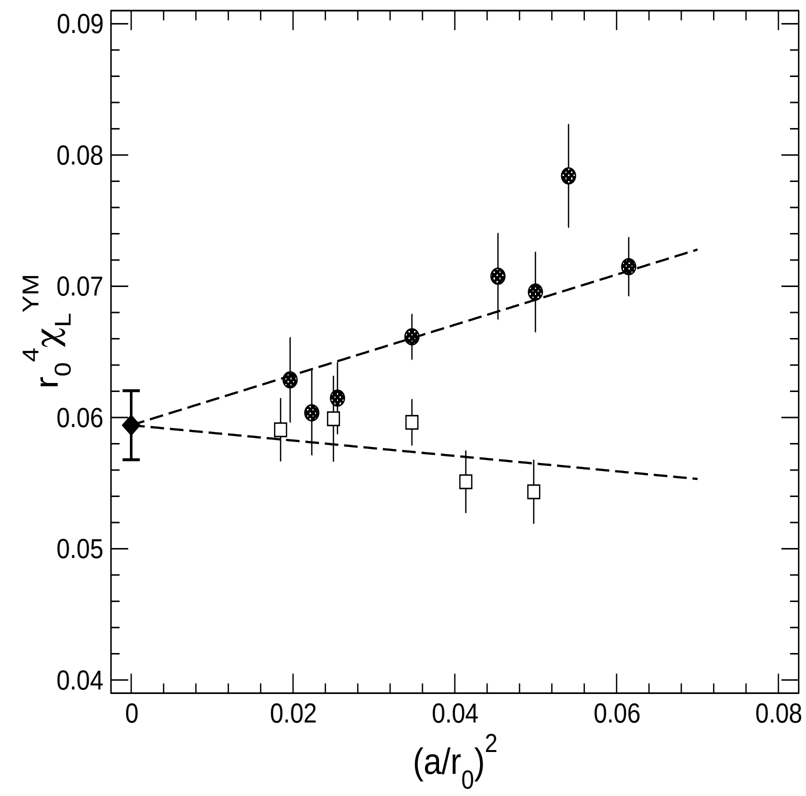
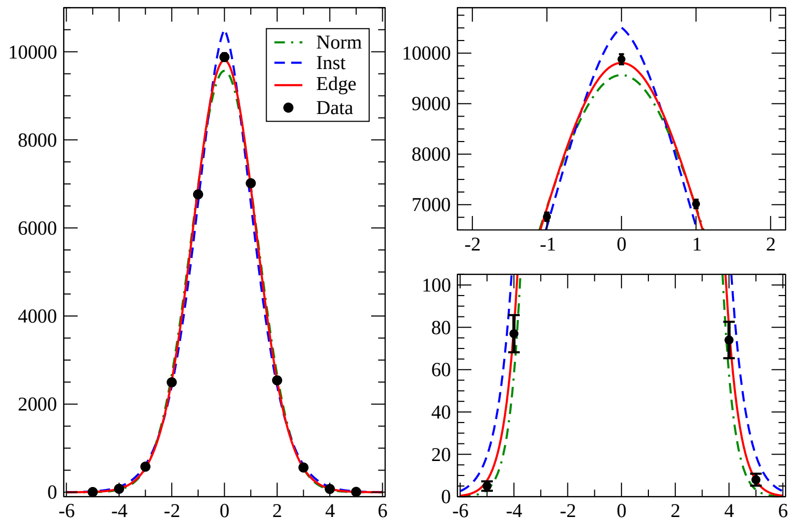


- The (leading) QCD anomalous contribution to $M^2_{\eta'}$, supports the Witten–Veneziano explanation for its large experimental value

- Vacuum energy and charge distribution are

$$e^{-F(\theta)} = \langle e^{i\theta Q}\rangle\ , \qquad P_Q = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} e^{-F(\theta)}$$

Their behaviour is a distinctive feature of the configurations that dominate the path integral



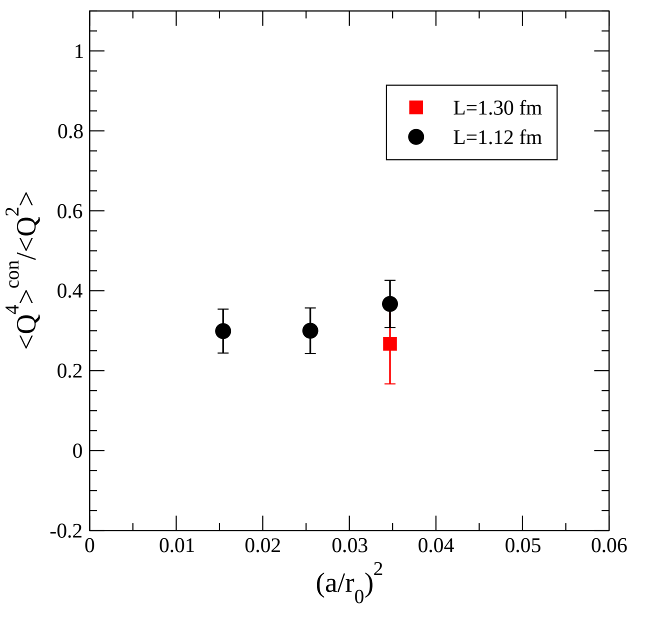
- Large N_c expansion predicts

$$\frac{\langle Q^{2n}\rangle^{\rm con}}{\langle Q^2\rangle} \propto \frac{1}{N_c^{2n-2}}$$

- Various conjectures. For example, dilute-gas instanton model gives ['t Hooft 74; Callan et al. 76; ...]

$$F^{\rm Inst}(\theta) = -VA\{\cos(\theta) - 1\}$$

$$\frac{\langle Q^{2n}\rangle^{\rm con}}{\langle Q^2\rangle} = 1$$



Do the instantons play a rôle ? [LG, Petrarca, Taglienti 07]

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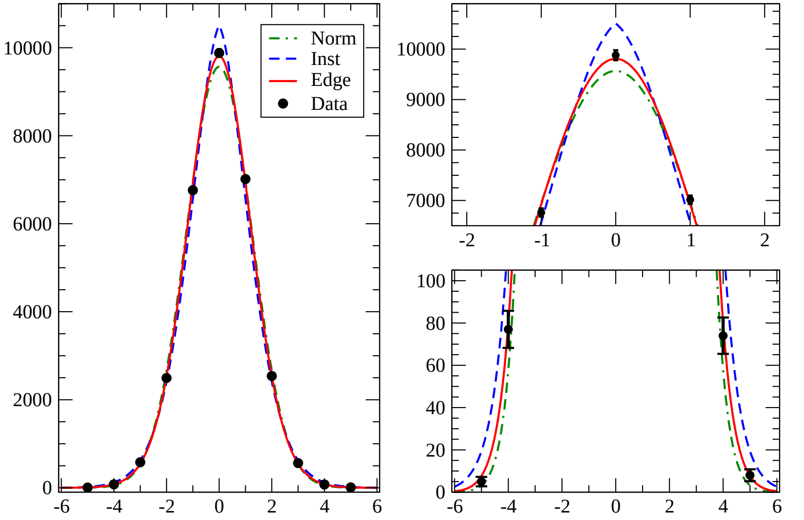
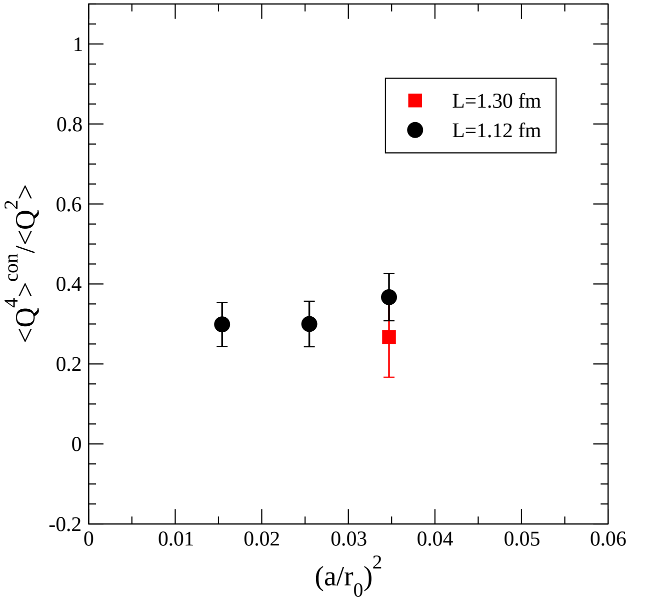


- A lattice computation gives

$$\frac{\langle Q^4 \rangle^{\rm con}}{\langle Q^2 \rangle} = 0.30 \pm 0.11$$



- Witten–Veneziano mechanism: the anomaly gives a mass to the η' boson thanks to the non-perturbative quantum fluctuations of the topological charge

Conclusions

- Lattice QCD is a phenomenal theoretical femtoscope to explore strong dynamics. Its lenses are made of quantum field theory, numerical techniques and computers

- It allows us to look at quantities not accessible to experiments that may unveil the the underlying mechanisms of non-perturbative strong dynamics

- A large variety of physics applications: QCD, flavour physics, beyond Standard Model physics, etc.

- Thanks to the recent extraordinary conceptual, technical and algorithmic advances the chiral regime of the theory is becoming accessible

- Today two particularly interesting applications:

  $*$ Banks–Casher relation

  $*$ Witten–Veneziano mechanism

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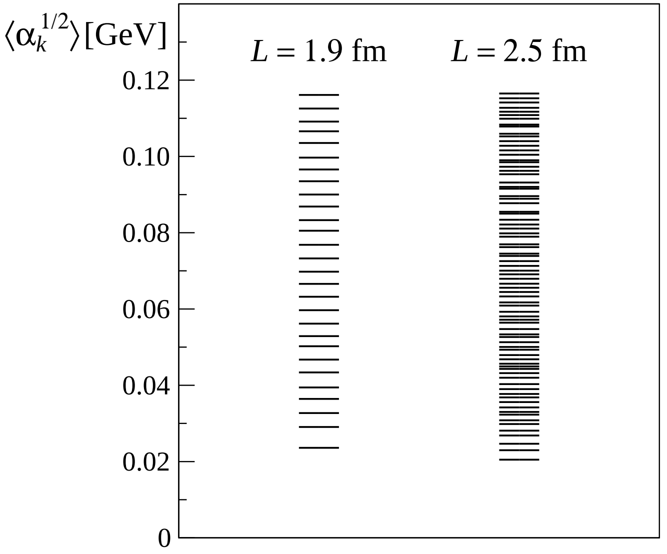
- Condensation of low-modes of the Dirac operator most direct piece of theoretical evidence for SSB

- The rate of condensation explains the bulk of the pion mass up to 0.5 GeV



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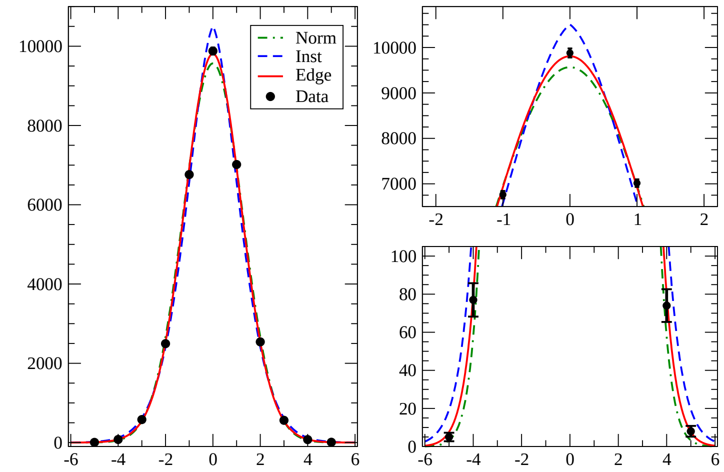
- Quantum fluctuations of the topological charge in Yang–Mills theory generate a non-zero value of χ^{YM}

- Its value supports the Witten–Veneziano explanation for the large mass of the η'



Conclusions

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- It allows us to look at quantities not accessible to experiments that may unveil the the underlying mechanisms of non-perturbative strong dynamics

- A large variety of physics applications: QCD, flavour physics, beyond Standard Model physics, etc.

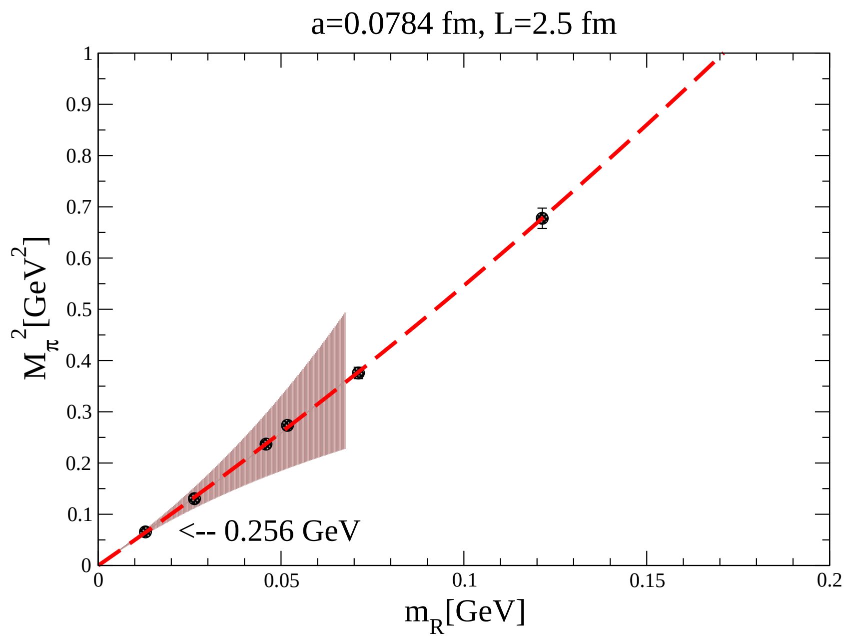
- The femtoscope, however, is still rather crude. There is continuous conceptual and technical progress to empower it

- LQCD will lead us to a precise quantitative understanding of QCD in the low-energy regime, and to validate the theory to be the one of the strong interactions in Nature

Lattice QCD with two light dynamical quarks [Del Debbio, LG, Lüscher, Petronzio, Tantalo 06-08]

- Chiral regime is becoming accessible to lattice QCD simulations

- The pion mass squared is found to be a nearly linear function of quark mass up to $(0.5\ \text{GeV})^2$. At smallest masses non-linear correction is 1 - 3%



- Non-Abelian chiral symmetry spontaneously broken as expected

- Compatible with the fact that the bulk of the mass is given by the leading term in standard ChPT

- Relations dictated by SSB can be verified quantitatively. GMOR is maybe the simplest to start with

- An example of the potentiality. From a fit to the curve

$0.47 \leq \Lambda_3 \leq 0.86$ GeV to be compared with $0.2 \leq \Lambda_3 \leq 2$ GeV [Gasser, Leutwyler 84]

Numerical computation (II) [LG, Lüscher 09]

- An effective condensate can be defined as

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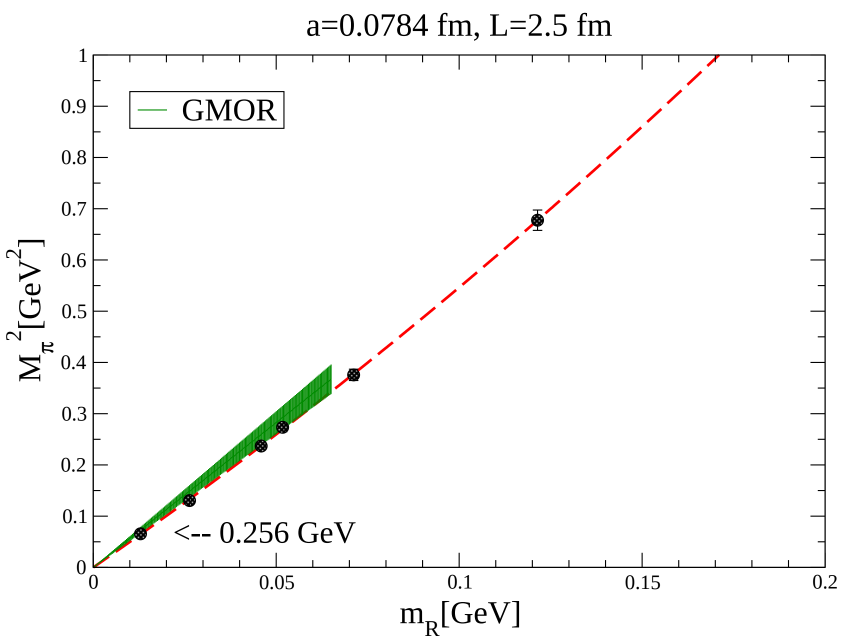

- The ETM collaboration from an overall fit of the pion mass and decay constant

$$\left[\Sigma_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV})\right]^{1/3}_{\rm GMOR} = 0.270(7)\ \text{GeV} \qquad \text{[ETM Coll. 09]}$$

- A clear and consistent picture is emerging. For $m_R \leq 0.05$ GeV the GMOR formula accounts for the bulk of the pion mass. But discretization errors not quantified yet