Recent progress on chiral symmetry breaking in QCD

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QCD action and its (broken) symmetries

- QCD action for $N_\mathrm{f} = 2$, $M = \mathrm{diag}(m, m)$

$$S = S_G + \int d^4x \left\{ \bar\psi D \psi + \bar\psi M \psi \right\}\, , \qquad D = \gamma_\mu(\partial_\mu - i A_\mu)$$

- For $M = 0$ chiral symmetry

$$\psi_{\mathrm{R,L}} \to V_{\mathrm{R,L}}\, \psi_{\mathrm{R,L}} \qquad \psi_{\mathrm{R,L}} = \left(\frac{1 \pm \gamma_5}{2}\right) \psi$$

Chiral anomaly: measure not invariant

SSB: vacuum not symmetric

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R} \times \mathcal{R}_\mathrm{scale}$$

$\downarrow$ (dim. transm., **chiral anomaly**)

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_{\mathrm{B=L+R}}$$

$\downarrow$ **(Spont. Sym. Break.)**

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L+R}} \times \mathrm{U}(1)_\mathrm{B}$$

$\downarrow$ (Confinement)

$$\mathrm{SU}(2)_{\mathrm{L+R}} \times \mathrm{U}(1)_\mathrm{B}$$

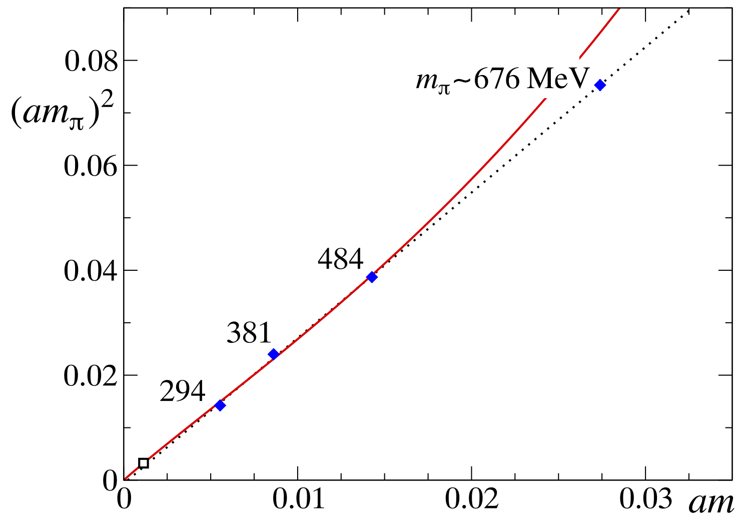
- Breaking due to non-perturbative dynamics. Precise quantitative computations are being made on the lattice

Overwhelming evidence of Spontaneous Symmetry Breaking in QCD

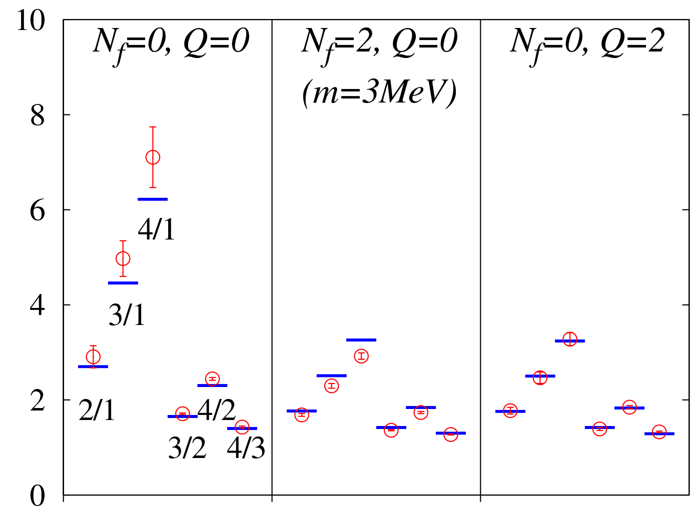
- Since the very first simulations with light dynamical quarks, the GMOR relation was beautifully observed on the lattice

- In the ϵ-regime ratios of low-lying eigenvalues of the Dirac operator are parameter-free predictions of ChPT/RMT up to NLO corrections. Lattice results turned out to be in spectacular agreement with expectations

[Luscher Lattice 05]



[Fukaya et al. 07]



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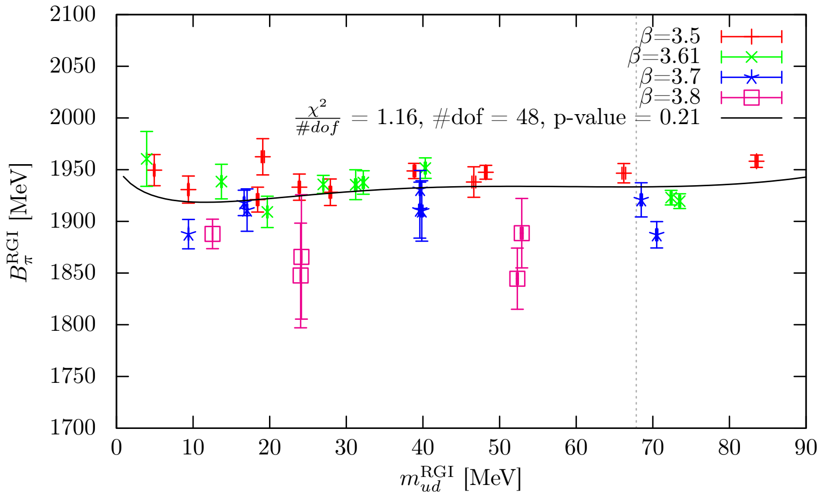
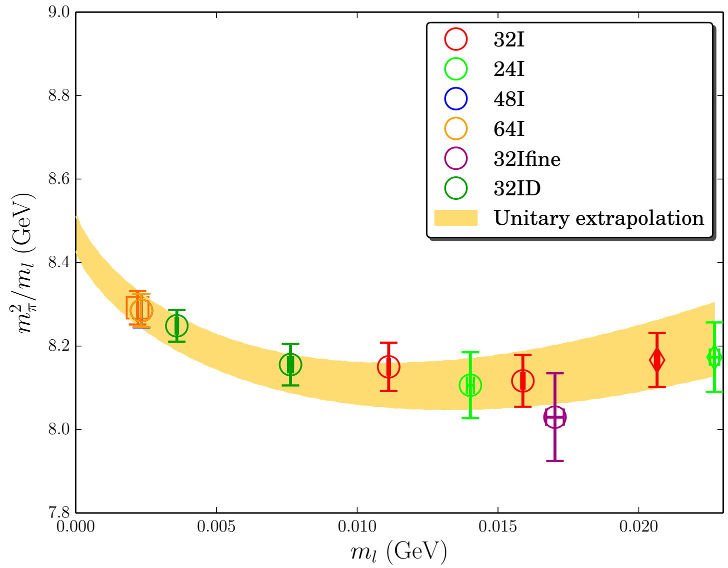
- After only a decade we have many results with $N_{\rm f}=2$, $2+1$ and $2+1+1$ flavours with light quarks down to the physical point

- The linear term explains the bulk of the pion mass. Generalized ChPT can be dismissed.

[Dürr et al. 13]



[Blum et al. 14; Murphy and Mawhinney on Thu 16th]



Review of lattice results by the FLAG Working Group [Aoki et al. 14]

- Chiral effective theory for pions

$$S_{\rm eff} = S^2_{\rm eff}(F,\Sigma) + S^4_{\rm eff}(F,\Sigma,\Lambda_i) + \cdots$$

encodes spontaneous symmetry breaking

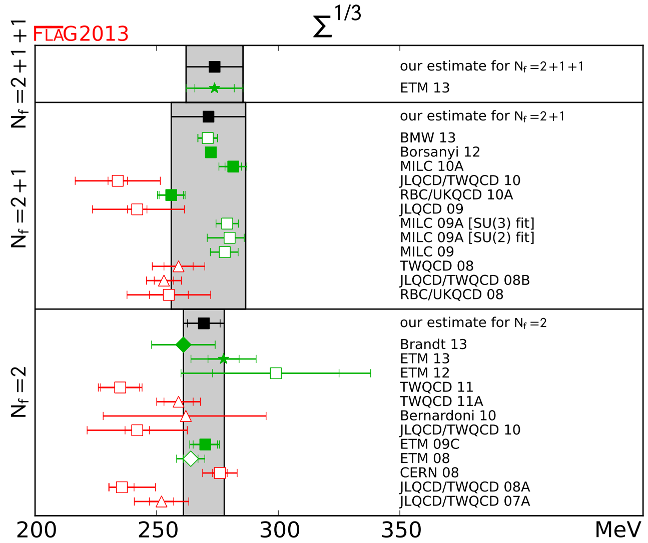
- For $m=0$ pions can interact only if they carry momentum. Expansion in p and m

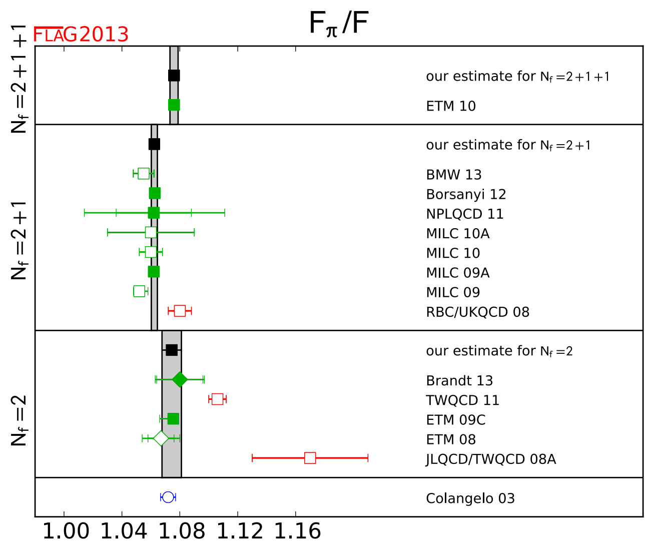
- Chiral dynamics parameterized by effective low-energy coupling constants

- In the chiral limit pion mass and decay constant satisfy

$$\frac{M^2_\pi F^2_\pi}{2m}\Big|_{m=0} = \Sigma\,, \qquad F_\pi\Big|_{m=0} = F$$

Analogous expressions for other quantities such a^0_0 and a^2_0, etc.





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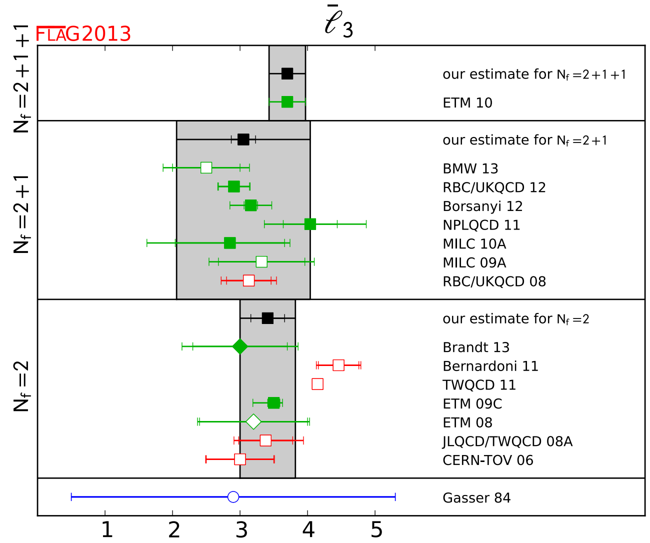
- Chiral dynamics parameterized by effective low-energy coupling constants

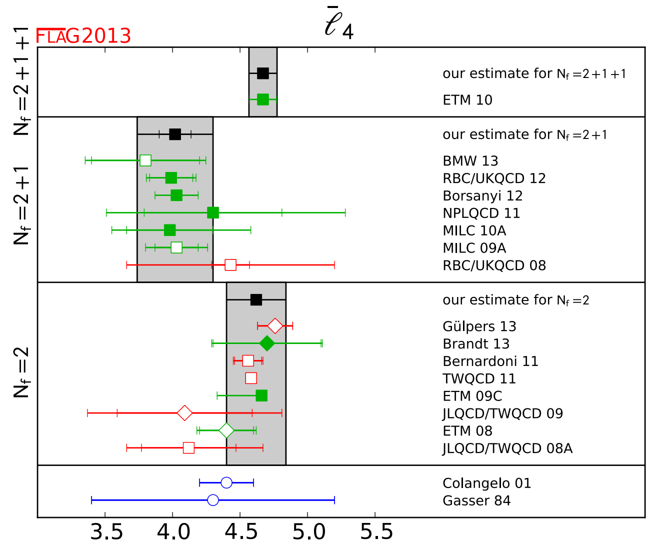
- At $\mathcal{O}(p^4)$ pion mass and decay constant are

$$M_\pi^2 = M^2 \left\{ 1 + \frac{M^2}{32\pi^2 F^2} \ln\left(\frac{M^2}{\Lambda_3^2}\right)\right\}$$

$$F_\pi = F \left\{ 1 - \frac{M^2}{16\pi^2 F^2} \ln\left(\frac{M^2}{\Lambda_4^2}\right)\right\}$$

where $M^2 = 2\Sigma m/F^2$. Analogous expressions for other quantities such as a_0^0 and a_0^2, etc.





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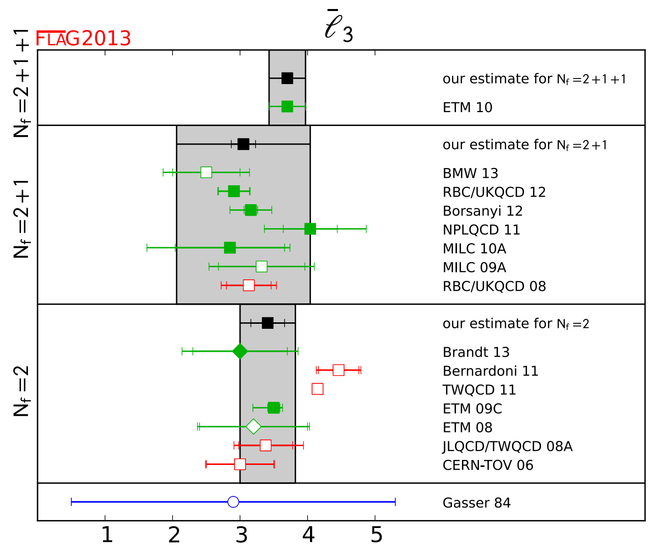
- For $m = 0$ pions can interact only if they carry momentum. Expansion in p and m

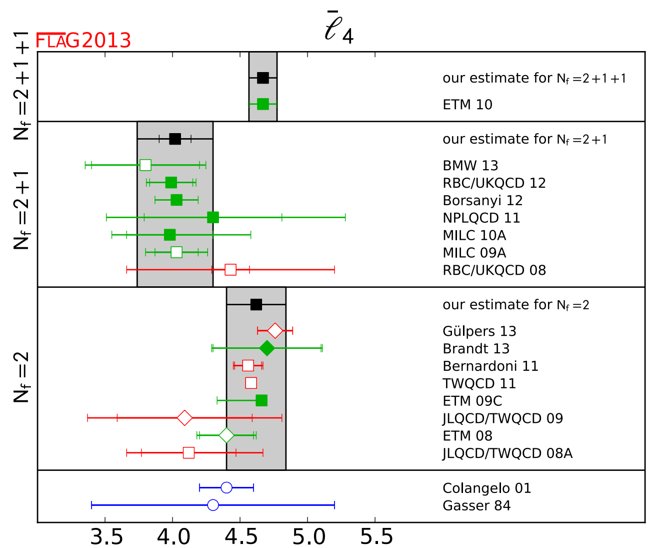
- A comprehensive review of lattice results is provided by the FLAG WG ($n = 3, 4$)

$$\bar{l}_n = \ln\left(\frac{\Lambda_n^2}{M^2}\right)\Bigg|_{M=139.6{\rm MeV}}$$

- Significant reduction in the errors quoted since last FLAG update for $\Sigma, \bar{l}_3, \bar{l}_4$ for $N_{\rm f}=2+1$

[Dürr et al. 13; Blum et al. 14; Murphy and Mawhinney on Thu 16th]





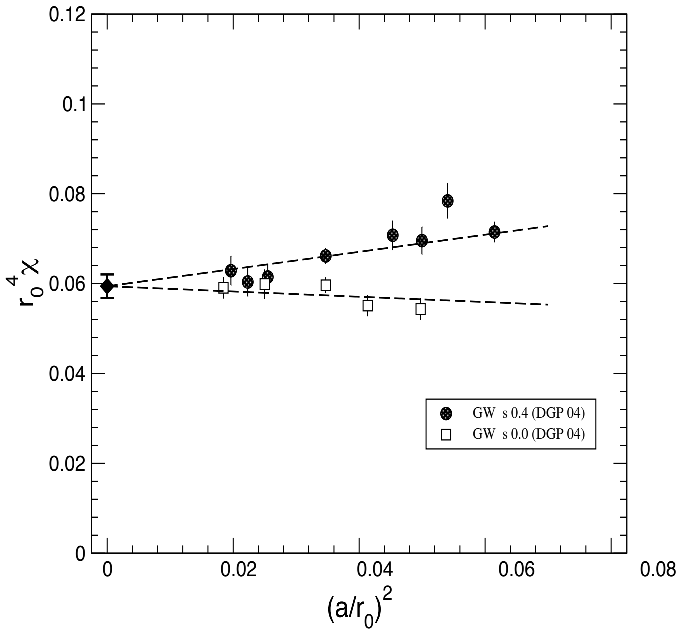
Growing evidence for the Witten–Veneziano mechanism [Witten 79; Veneziano 79]

- Analogously to the GMOR relation, if we expand also in $1/N_c$

$$\left.\frac{F_{\eta'}^2 M_{\eta'}^2}{2N_f}\right|_{\frac{1}{N_c}=0}^{m=0} = \left.\chi^{\rm YM}\right|_{\frac{1}{N_c}=0}$$

where χ has to be defined so that in presence of fermions it satisfies the anomalous AWI

[Del Debbio et al. 04]



- Exploratory studies with cooling techniques well summarized in [Teper 00; Vicari, Panagopoulos 08]

- With Neuberger's definition of the topological charge by setting the scale with F_K

$$\chi^{\rm YM} = (193 \pm 4\ {\rm MeV})^4 \qquad (N_c = 3)$$

to be compared with

$$\frac{F_\pi^2}{6}(M_\eta^2 + M_{\eta'}^2 - 2M_K^2) \underset{\rm exp}{\approx} (180\ {\rm MeV})^4$$

- The value of the (leading) QCD anomalous contribution to $M_{\eta'}^2$ supports the Witten–Veneziano explanation for its large experimental value. A lot of work still needed.

Parallel talks closely related

- Hadron Spectroscopy and Interactions [Tue 14th]

  Bulava, Donald, Fahy, Fukaya, Hörz, Janowski, Soeldner, Ukita

- Chiral Symmetry [Thu 16th]

  Alexandru, Murphy, Mawhinney, Nishigaki, Ottnad, Pak, Verbaarschot, Zafeiropoulos

- Vacuum structure and confinement [Fri 17th]

  Cè , Doi, Glozman , Hasegawa, Horvath

- Poster session [Fri 17th]

  Jeong, Hashimoto, Holland

Outline

- Banks–Casher mechanism:
  - $*$ Spectral density and the mode number
  - $*$ The density in QCD Lite
  - $*$ First results with $N_{\rm f}=2+1+1$ flavours
- Witten–Veneziano mechanism:
  - $*$ Definition of the topological susceptibility from the gradient flow
  - $*$ Recent numerical results in the Yang–Mills theory
  - $*$ Universality tests
- Topological susceptibility in QCD
  - $*$ Numerical results
- Conclusions and outlook

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Banks–Casher relation [Banks, Casher 80]

- For each gauge configuration

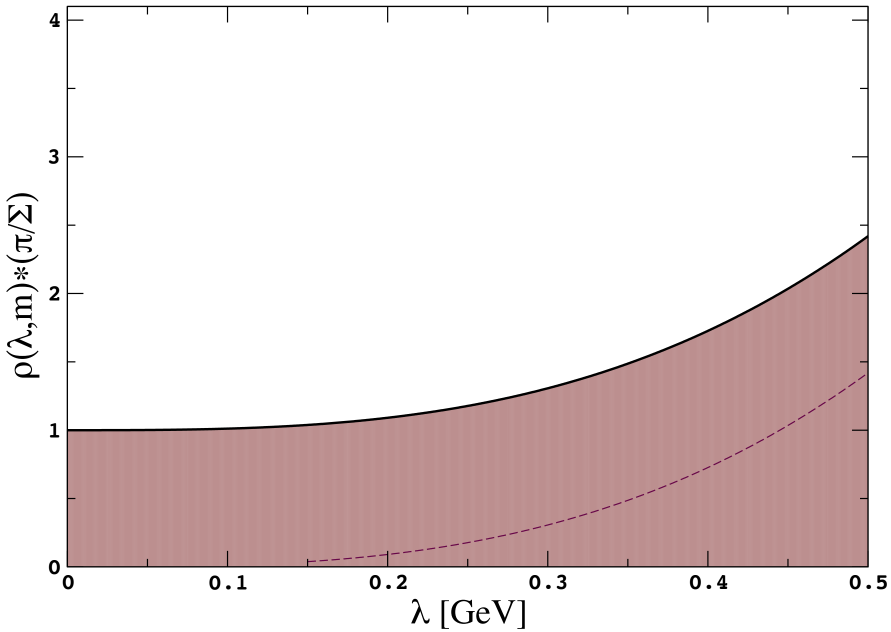
$$D_m \chi_k = (m + i\lambda_k)\chi_k$$

- The spectral density of D is

[Banks, Casher 80; Leutwyler, Smilga 92; Shuryak, Verbaarschot 93]

$$\rho(\lambda, m) = \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle$$

where $\langle \ldots \rangle$ indicates path-integral average



- The Banks–Casher relation

$$\lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m) = \frac{\Sigma}{\pi}$$

can be read in both directions: a non-zero spectral density implies that the symmetry is broken with a non-vanishing Σ and vice versa.

To be compared, for instance, with the free case $\rho(\lambda) \propto |\lambda^3|$

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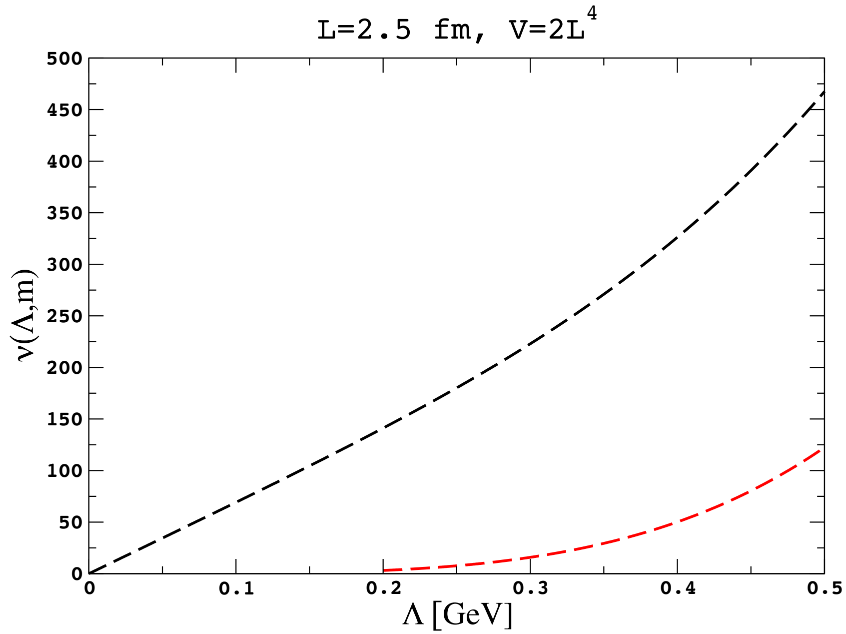

- The number of modes in a given energy interval

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda\, \rho(\lambda, m) \qquad \nu(\Lambda, m) = \frac{2}{\pi}\Lambda\Sigma V + \dots$$

grows linearly with Λ, and they condense near the origin with values $\propto 1/V$

In the free case $\nu(\Lambda, m) \propto V\Lambda^4$

Renormalizability and universality of the spectral density [LG, Lüscher 09]

- Instead of the spectral density, consider the spectral sum

$$\sigma_k(m_v, m) = V \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m_v^2)^k}$$

$$= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle$$

$*$ Integral converges if $k \geq 3$

$*$ The relation between $\sigma_k(m_v, m)$ and $\rho(\lambda, m)$ invertible for every k

- Renormalization properties of $\rho(\lambda, m)$ can thus be inferred from those of σ_k

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- With exact chiral symmetry, correlation functions of pseudoscalar densities at physical distance are renormalized by $(1/Z_m)^{2k}$

- At short distance the flavour structure implies

$$P_{12}(x_1) P_{23}(x_2) \sim C(x_1 - x_2) S_{13}(x_1) \qquad S_{13} = \bar{\psi}_1 \psi_3$$

where $C(x)$ diverges like $|x|^{-3}$ and it is therefore integrable. Analogous argument for all other short-distance singularities. No extra contact terms needed to renormalize σ_k

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- Once the gauge coupling and the mass(es) are renormalized, the spectral sum

$$\sigma_{k,\mathrm{R}}(m_{v\mathrm{R}}, m_\mathrm{R}) = Z_m^{-2k} \sigma_k \left(\frac{m_{v\mathrm{R}}}{Z_m}, \frac{m_\mathrm{R}}{Z_m} \right)$$

is ultraviolet finite. Continuum limit universal (if same renormalization cond. are used)

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- The spectral density thus renormalizes as

$$\rho_{\rm R}(\lambda_{\rm R}, m_{\rm R}) = Z_m^{-1} \rho\left(\frac{\lambda_{\rm R}}{Z_m}, \frac{m_{\rm R}}{Z_m}\right)$$

- For Wilson fermions similar derivation but twisted-mass valence quarks

- The rate of condensation is indeed a renormalizable universal quantity in QCD, and is unambiguously defined once the bare parameters in the action of the theory have been renormalized

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$$= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle$$

- It follows that the mode number is a renormalization-group invariant

$$\nu_{\rm R}(\Lambda_{\rm R}, m_{\rm R}) = \nu(\Lambda, m)$$

and its continuum limit is universal for any value of Λ and m

- When chiral symmetry is spontaneously broken, the spectral density can be computed in ChPT. At the NLO

$$\rho^{\rm nlo}(\lambda, m) = \frac{\Sigma}{\pi}\Big\{1 + \frac{m\Sigma}{(4\pi)^2 F^4}\Big[3\,\bar{l}_6 + 1 - \ln(2) - 3\ln\Big(\frac{\Sigma m}{F^2\bar{\mu}^2}\Big) + g_\nu\Big(\frac{\lambda}{m}\Big)\Big]\Big\}$$

where $g_\nu(x)$ is a parameter-free known function

- The NLO formula has properties which can be confronted against the NP results:

  * at fixed λ no chiral logs are present when $m \to 0$

$$g_\nu(x) \xrightarrow{x\to\infty} -3\ln(x)$$

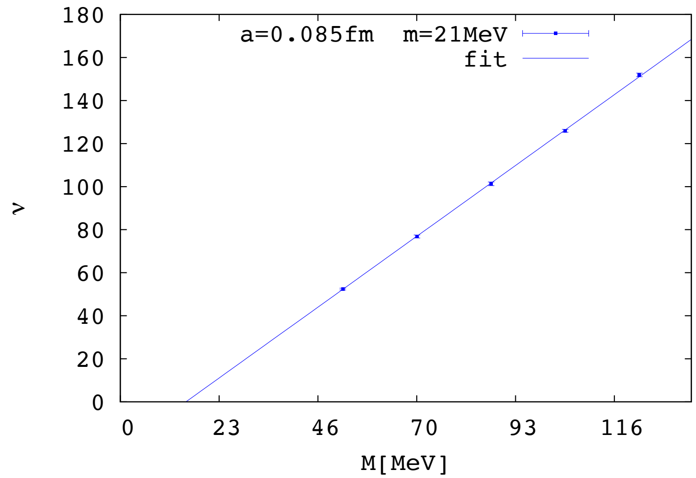
  * in the chiral limit $\rho^{\rm nlo}(\lambda, m)$ becomes independent of λ
    This is an accident of the $N_{\rm f} = 2$ ChPT theory at NLO [Smilga, Stern 93]

  * the λ dependence of $\rho^{\rm nlo}(\lambda, m)$ is a known function (up to overall constant).
    The spectral density is a slowly decreasing function of λ at fixed m

Numerical computation for $N_{\rm f} = 2$ [Cichy et al. 13; Engel et al. 14]

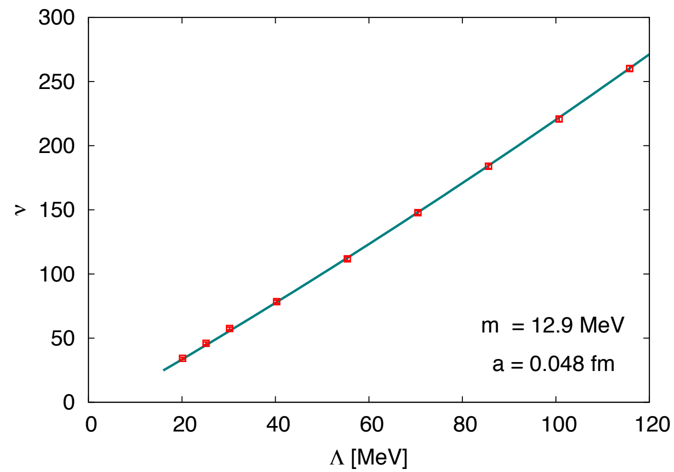
- Twisted-mass QCD [Cichy et al. 13]:

  - $* \; a = 0.054$–0.085 fm
  - $* \; m = 16$–47 MeV
  - $* \; M = 50$–120 MeV
  - $* \; M = \sqrt{\Lambda^2 + m^2}$



- $O(a)$–improved Wilson fermions [Engel et al. 14]:

  - $* \; a = 0.048$–0.075 fm
  - $* \; m = 6$–37 MeV
  - $* \; \Lambda = 20$–500 MeV
  - $* \; \nu = -9.0(13){+}2.07(7)\Lambda{+}0.0022(4)\Lambda^2$



- The mode number is a nearly linear function in Λ up to approximatively 100 MeV. The modes do condense near the origin as predicted by the Banks–Casher mechanism

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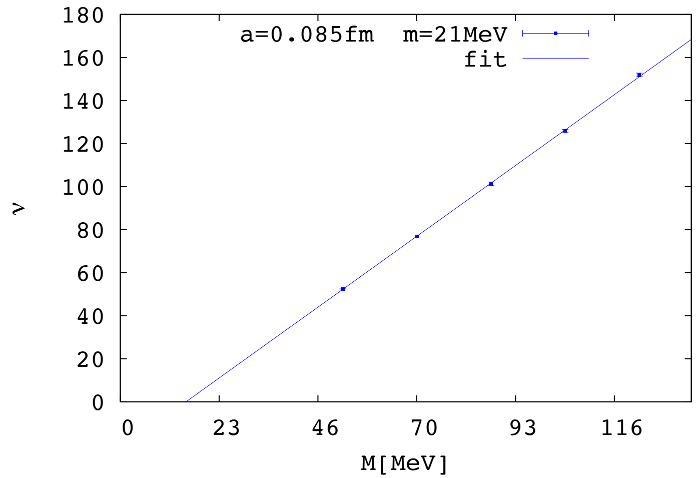

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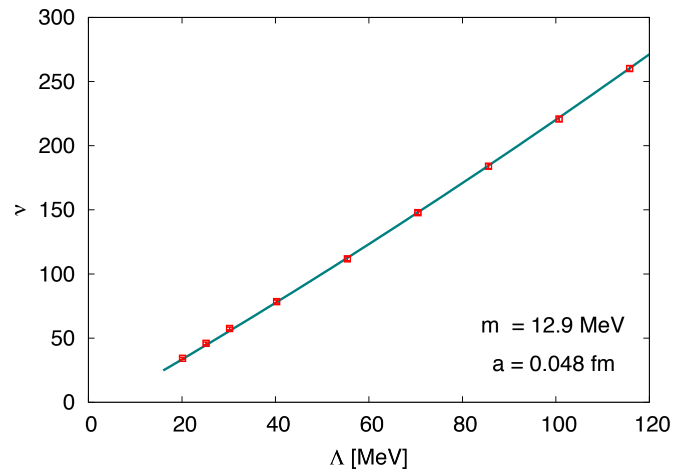


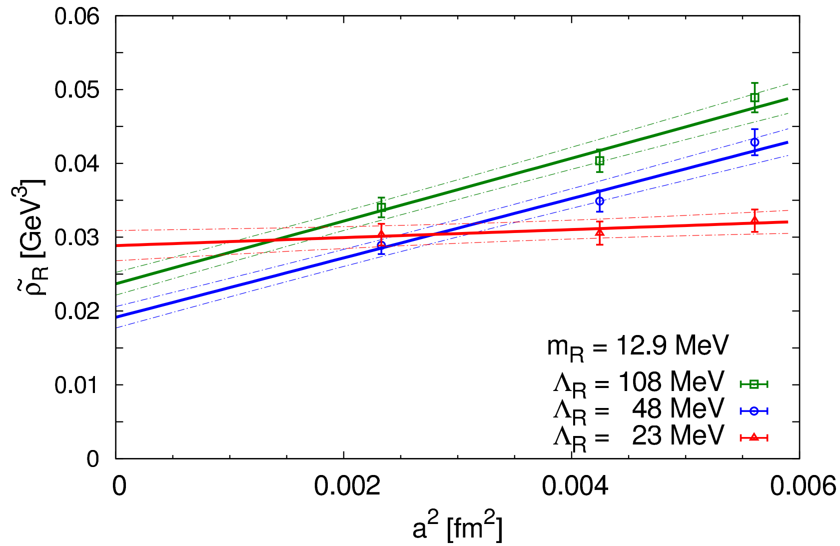
- At fixed lattice spacing and at the percent precision, however, data show statistically significant deviations from the linear behaviour of $O(10\%)$.

- By defining

$$\widetilde{\rho}(\Lambda_1, \Lambda_2, m) = \frac{\pi}{2V} \frac{\nu(\Lambda_2) - \nu(\Lambda_1)}{\Lambda_2 - \Lambda_1}$$

  the continuum limit is taken at fixed m, Λ_1 and Λ_2 $[\Lambda = (\Lambda_1 + \Lambda_2)/2]$

- Data are extrapolated linearly in a^2 as dictated by the Symanzik analysis



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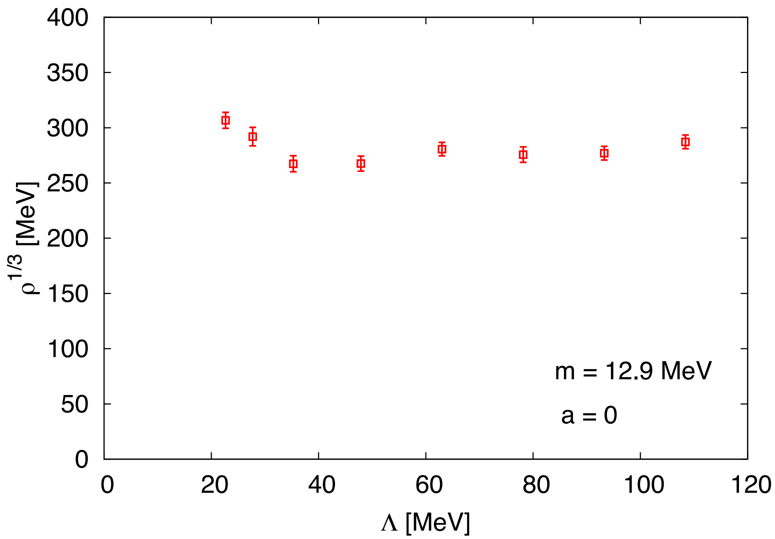

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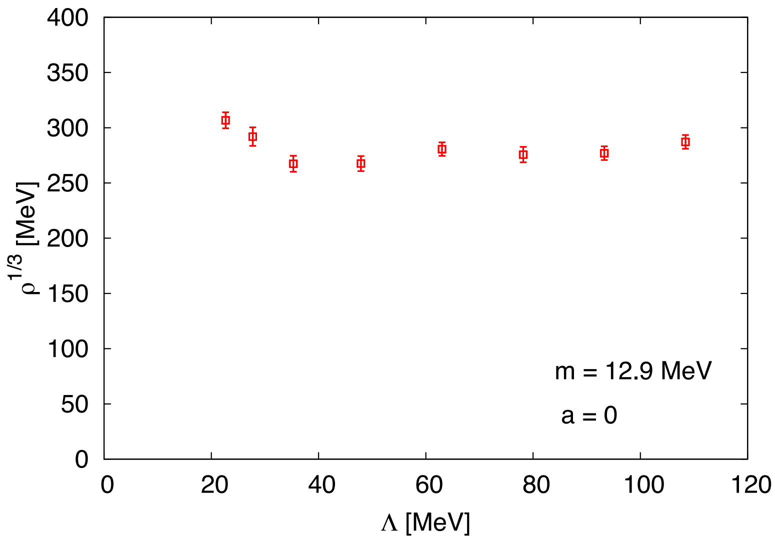

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- It is noteworthy that no assumption on the presence of SSB was needed so far

- The results show that at small quark masses the spectral density is non-zero and (almost) constant in Λ near the origin

- Data are consistent with the expectations from the Banks–Casher mechanism in the presence of SSB. In this case NLO ChPT indeed predicts ($N_\mathrm{f} = 2$)

$$\widetilde{\rho}^{\,\mathrm{nlo}} = \Sigma\Big\{1 + \frac{m\Sigma}{(4\pi)^2 F^4}\Big[3\,\bar{l}_6 + 1 - \ln(2) - 3\ln\Big(\frac{\Sigma m}{F^2\bar{\mu}^2}\Big) + \tilde{g}_\nu\Big(\frac{\Lambda_1}{m}, \frac{\Lambda_2}{m}\Big)\Big]\Big\}$$

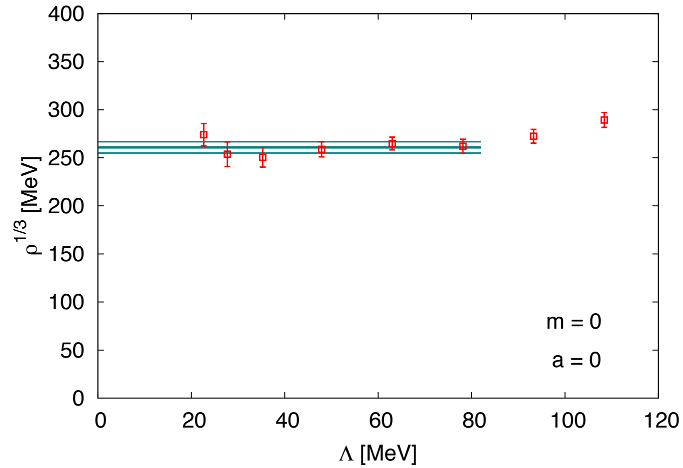
Chiral limit

- In the chiral limit NLO ChPT predicts $\widetilde{\rho}$ to be Λ-independent. By extrapolating to $m = 0$

$$[\widetilde{\rho}^{\,\overline{\rm MS}}]^{1/3} = [\Sigma^{\overline{\rm MS}}_{\rm BK}(2\,\text{GeV})]^{1/3} = 261(6)(8)\,\text{MeV}$$

where the spacing is fixed by introducing a quenched strange quark with $F_K = 109.6$ MeV

[Engel et al. 14]



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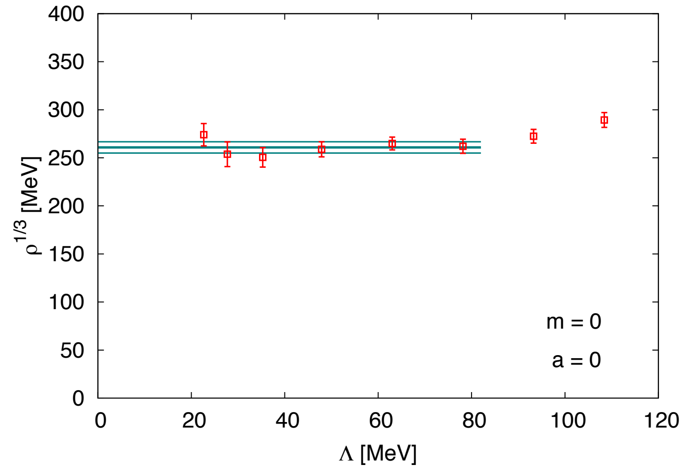
- From the average slope of the mode number with respect to M ($50 \leq M \leq 120$ MeV), and by extrapolating linearly in m

$$r_0[\Sigma^{\overline{\rm MS}}_{\rm BK}(2\,\text{GeV})]^{1/3} = 0.689(16)(29)$$

where the residual M-dependence is accounted for in the systematic error

- No value in physical units given due to the large uncertainty in the determination of the lattice spacing from ETMC

[Engel et al. 14]



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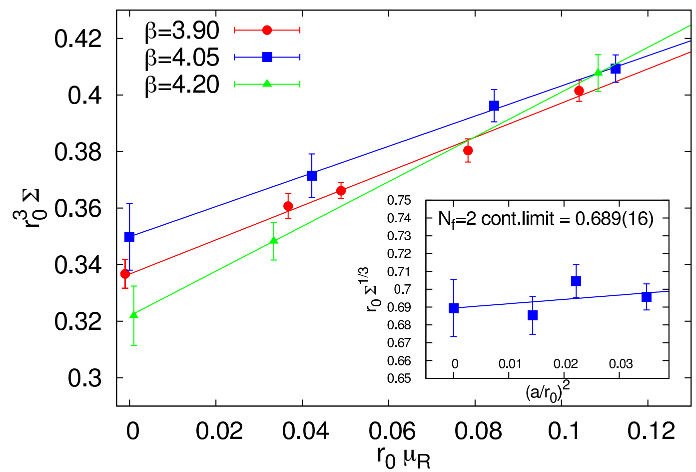

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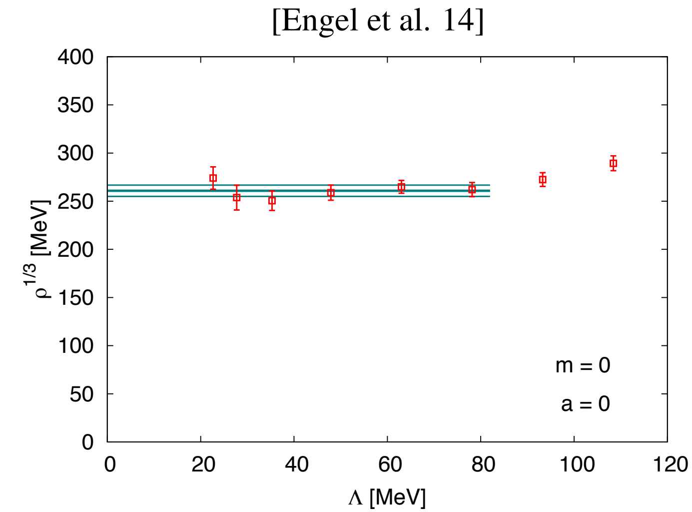
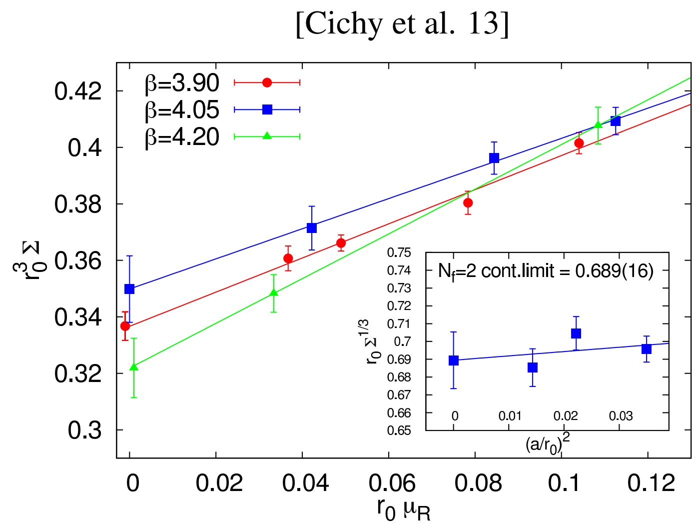


- By using $r_0 F_K = 0.2794(44)$ fm from [Fritzsch et al. 12] I get

$$[\Sigma^{\overline{\rm MS}}_{\rm BK}(2\,\text{GeV})]^{1/3} = 270(6)(11)(4)\,\text{MeV}$$

an exercise that shows that the value is not inconsistent with the result above. More work by ETMC is desirable to clarify this issue

- When extrapolated to the chiral limit by using NLO ChPT (linearly), the spectral density in the continuum limit is definitively non-zero

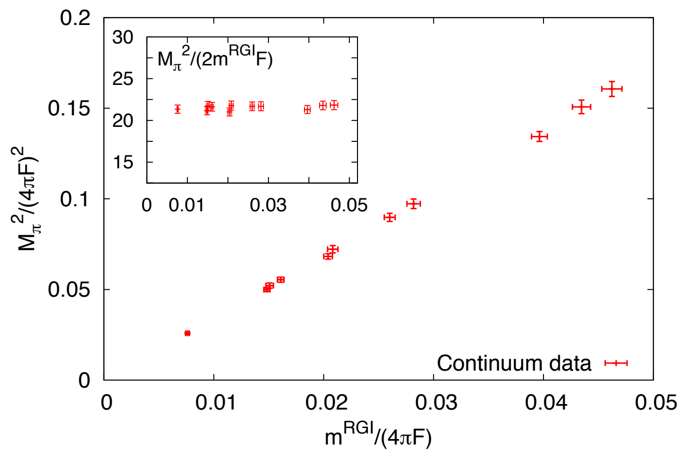


Gell-Mann–Oakes–Renner relation [Engel et al. 14]

- The distinctive signature of SSB is the agreement between $\widetilde{\rho}$ and the slope of $M_\pi^2 F_\pi^2/2$ with respect to m in the chiral limit

- On the same set of configurations by fitting the data with NLO (W)ChPT for $M_\pi < 400$ MeV

$$[\Sigma^{\overline{\rm MS}}_{\rm GMOR}(2\,\text{GeV})]^{1/3} = 263(3)(4)\,\text{MeV}$$



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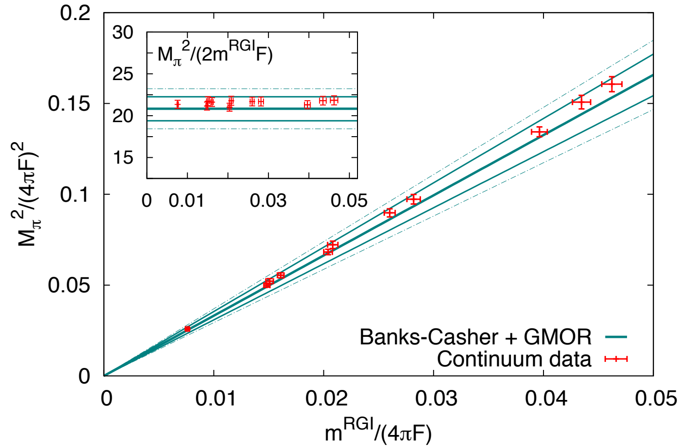
$$[\Sigma^{\overline{\rm MS}}_{\rm GMOR}(2\,\text{GeV})]^{1/3} = 263(3)(4)\,\text{MeV}$$

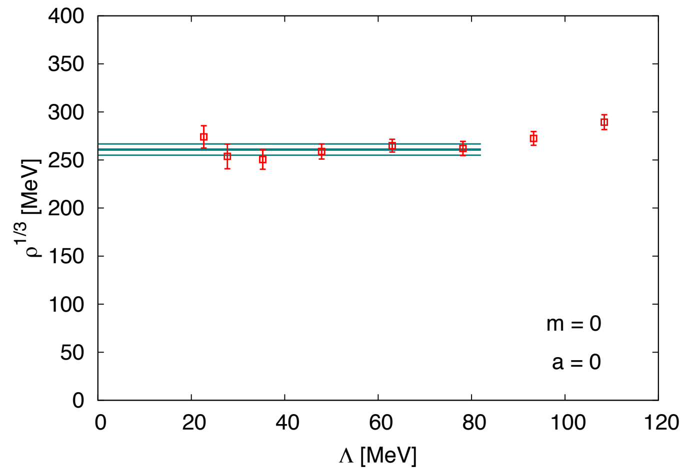
to be compared with the previous result

$$[\widetilde{\rho}^{\,\overline{\rm MS}}]^{1/3} = [\Sigma^{\overline{\rm MS}}_{\rm BK}(2\,\text{GeV})]^{1/3} = 261(6)(8)\,\text{MeV}$$

- The spectral density of the Dirac operator in the continuum is $\neq 0$ at the origin for $m = 0$

- The low-modes of the Dirac operator do condense as expected in the Banks–Casher mechanism





- The rate of condensation agrees with the GMOR relation, and it explains the bulk of the pion mass up to $M_\pi \leq 500$ MeV

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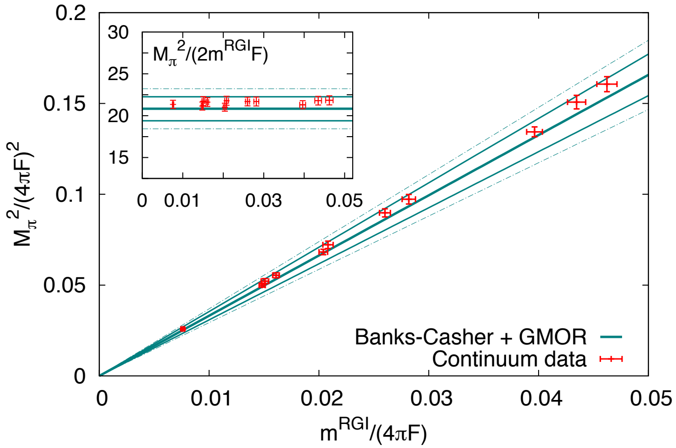

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$$[\Sigma^{\overline{\rm MS}}_{\rm GMOR}(2\,\text{GeV})]^{1/3} = 263(3)(4)\,\text{MeV}$$

- The dimensionless ratios

$$[\Sigma^{\rm RGI}]^{1/3}/F = 2.77(2)(4)\;, \quad \Lambda^{\overline{\rm MS}}/F = 3.6(2)$$

are “geometrical” properties of the theory. They belong to the category of unambiguous quantities in the two flavour theory that should be used for quoting and comparing results rather than those expressed in physical units

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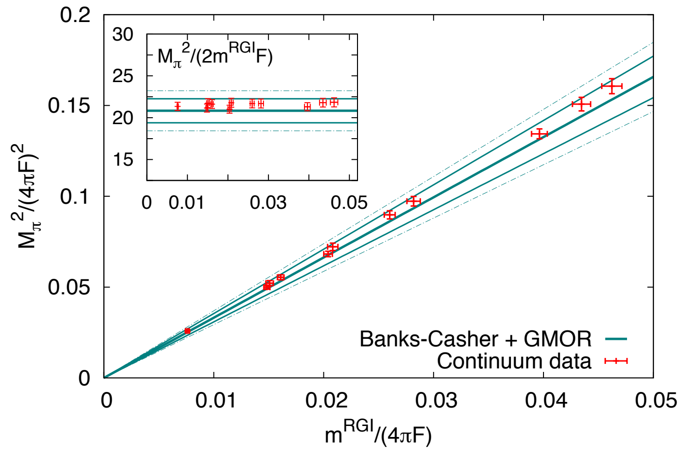

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- The dimensionless ratios

$$[\Sigma^{\rm RGI}]^{1/3}/F = 2.77(2)(4)\;, \quad \Lambda^{\overline{\rm MS}}/F = 3.6(2)$$

are "geometrical" properties of the theory. They can be directly compared with your preferred approximation/model

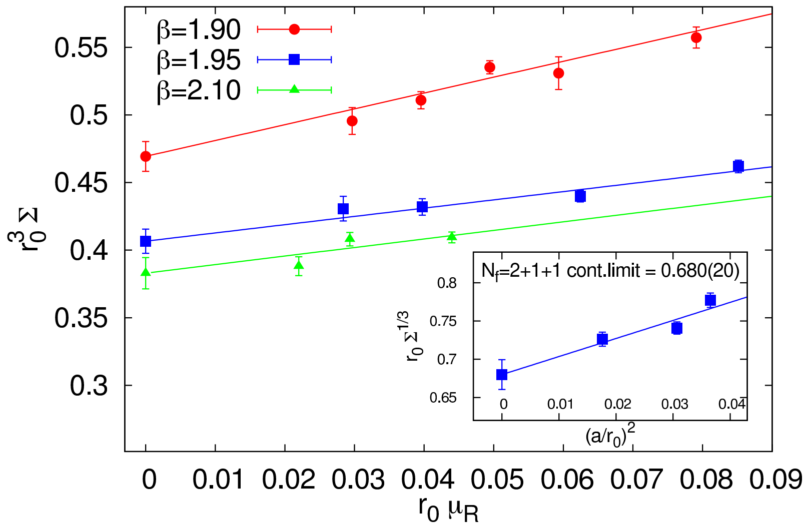
- For instance a large-N_c computation plus model assumptions give [Armoni et al. 06]

$$\frac{[\Sigma^{\rm RGI}]^{1/3}}{\Lambda^{\overline{\rm MS}}} = 1.43\Big[\frac{N_c}{2\pi^2}\tilde{K}\Big]^{1/3} \qquad \tilde{K} = 1 + O(1/N_c)$$

The above lattice measures give $[\Sigma^{\rm RGI}]^{1/3}/\Lambda^{\overline{\rm MS}} = 0.77(4) \implies \tilde{K}^{1/3} = 1.01(5)$

- Twisted-mass QCD

  * $a = 0.061$–0.086 fm
  * $m = 9$–45 MeV
  * $M = 50$–110 MeV
  * $M = \sqrt{\Lambda^2 + m^2}$



- As for the $N_\mathrm{f} = 2$ theory, from the average slope of the mode number with respect to M in the region $50 \leq M \leq 110$ MeV, and by extrapolating linearly in m

$$r_0[\Sigma_{\mathrm{BK}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})]^{1/3} = 0.680(20)(21)$$

  where the residual M-dependence is accounted for in the systematic error

- No significant dependence on the strange and the charm quark mass is seen

Outline

- Banks–Casher mechanism:
  - ∗ Spectral density and the mode number
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Definitions of the topological susceptibility on the lattice

- Three families of definitions of χ ultraviolet finite, unambiguous and satisfying AWIs

- From a Ginsparg–Wilson Dirac operator
  [Neuberger 97; Hasenfratz 98; Lüscher 98; LG, Rossi, Testa, Veneziano 01; LG, Rossi, Testa 04; Lüscher 04]

$$\chi_{\rm N} = a^4 \sum_x \langle q_{\rm N}(x) q_{\rm N}(0)\rangle \qquad q_{\rm N}(x) = -\frac{1}{2a^3}\,{\rm Tr}\Big[\gamma_5 D(x,x)\Big]$$

- From spectral projectors acting on Dirac fields [LG, Lüscher 09]

$$\chi_{\rm P} = \frac{\langle {\rm Tr}\{{\rm P}_M\}\rangle}{V}\,\frac{\langle {\rm Tr}\{\gamma_5 {\rm P}_M\}{\rm Tr}\{\gamma_5 {\rm P}_M\}\rangle}{\langle {\rm Tr}\{\gamma_5 {\rm P}_M \gamma_5 {\rm P}_M\}\rangle}$$

- From the Yang–Mills gradient flow [Lüscher 10]

$$\chi^t_{\rm L} = a^4 \sum_x \langle q^t_{\rm L}(x) q^t_{\rm L}(0)\rangle\;, \quad q^t_{\rm L}(x) = \frac{1}{32\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,{\rm tr}\Big[G_{\mu\nu}(x) G_{\rho\sigma}(x)\Big] \qquad (t>0)$$

- All of them are in the same universality class

Universality of the continuum limit of χ^t_{L} (I) [Cè et al. 15; Cè Fri 17th]

- In the continuum to all orders in perturbation theory [Lüscher 98, 00]

$$\langle q^t(x)\, O(y)\rangle = \langle q^{t=0}(x)\, O(y)\rangle + \partial_\rho \int_0^t dt' \langle w_\rho^{t'}(x)\, O(y)\rangle \qquad (x \neq y)$$

where $w_\rho^{t'}$ is a dimension-5 gauge-invariant pseudovector field. Since there are no local composite fields of dimension $d < 5$ with the same transformation properties

$$\langle q^{t=0}(x)\, O(y)\rangle \equiv \lim_{t\to 0} \langle q^t(x)\, O(y)\rangle \qquad (x \neq y)$$

- The small-t expansion of the charge density is of the form

$$\langle q^t(x)\, O(y)\rangle = \langle q^{t=0}(x)\, O(y)\rangle + \mathcal{O}(t) \qquad (x \neq y)$$

with no divergences when $t \to 0$

- Let us consider in the continuum the correlator

$$\langle q^{t=0}(0) P_{51}(z_1) S_{12}(z_2) \ldots S_{45}(z_5) \rangle$$

in which non-integrable short-distance singularities can arise only when (same for P_{ij})

$$q^{t=0}(x)\, S_{ij}(0) \xrightarrow{x \to 0} c(x)\, P_{ij}(0) + \ldots$$

where $c(x)$ diverges as $|x|^{-4}$ when $|x| \to 0$.

- Being the leading short-distance singularity in this product of fields, $c(x)$ can be computed in perturbation theory. Again to all orders ($y \neq 0$, $y \neq x$)

$$\langle q^{t=0}(x) S_{ij}(0)\, O(y) \rangle = \langle q^{t}(x) S_{ij}(0)\, O(y) \rangle - \partial_\rho \int_0^t dt' \langle w_\rho^{t'}(x) S_{ij}(0)\, O(y) \rangle$$

which implies that the Wilson coefficient is of the form

$$c(x) = \partial_\rho u_\rho(x)$$

and thus it does not contribute to the fully integrated correlation function

Universality of the continuum limit of $\chi^t_{\rm L}$ (III) [Cè et al. 15; Cè Fri 17th]

- Since there are no other $d \leq 4$ gauge-invariant pseudoscalar operators

$$\lim_{a\to 0} Z_q\, a^4 \sum_x \langle q^t_{\rm N}(0)\, q^{t=0}_{\rm N}(x)\rangle = \text{finite} \qquad (t>0)$$

- When the density at $t=0$ is replaced by its density-chain expression

$$a^4 \sum_x \langle q^t_{\rm N}(0)\, q^{t=0}_{\rm N}(x)\rangle = -m^5 a^{20} \sum_{z_1,\dots,z_5} \langle q^t_{\rm N}(0)\, P_{51}(z_1)\, S_{12}(z_2)\dots S_{45}(z_5)\rangle$$

the r.h.s. is finite as it stands when $t>0$

- This in turn implies that $Z_q = 1$, and therefore

$$\lim_{a\to 0} \langle q^{t=0}_{\rm N}(0)\, q^{t=0}_{\rm N}(x)\rangle = \langle q^{t=0}(0)\, q^{t=0}(x)\rangle \qquad (x\neq 0)$$

The field $q^{t=0}(x)$ is the one appearing in the singlet AWIs when fermions are included

Universality of the continuum limit of $\chi^t_{\rm L}$ (IV) [Cè et al. 15; Cè Fri 17th]

- The absence of short-distance singularities on r.h.s. and small-t expansion lead to

$$\lim_{a\to 0} a^{20} \sum_{z_1,\dots,z_5} \langle q^{t=0}_{\rm N}(0)\dots S_{45}(z_5)\rangle = \lim_{t\to 0}\lim_{a\to 0} a^{20} \sum_{z_1,\dots,z_5} \langle q^{t}_{\rm N}(0)\dots S_{45}(z_5)\rangle$$

which can be re-written as

$$\lim_{a\to 0} a^4 \sum_x \langle q^{t=0}_{\rm N}(x)\, q^{t=0}_{\rm N}(0)\rangle = \lim_{t\to 0}\lim_{a\to 0} a^4 \sum_x \langle q^{t}_{\rm N}(x)\, q^{t=0}_{\rm N}(0)\rangle$$

- By using the small-t expansion again, and by remembering that the susceptibility is t-independent for $t > 0$

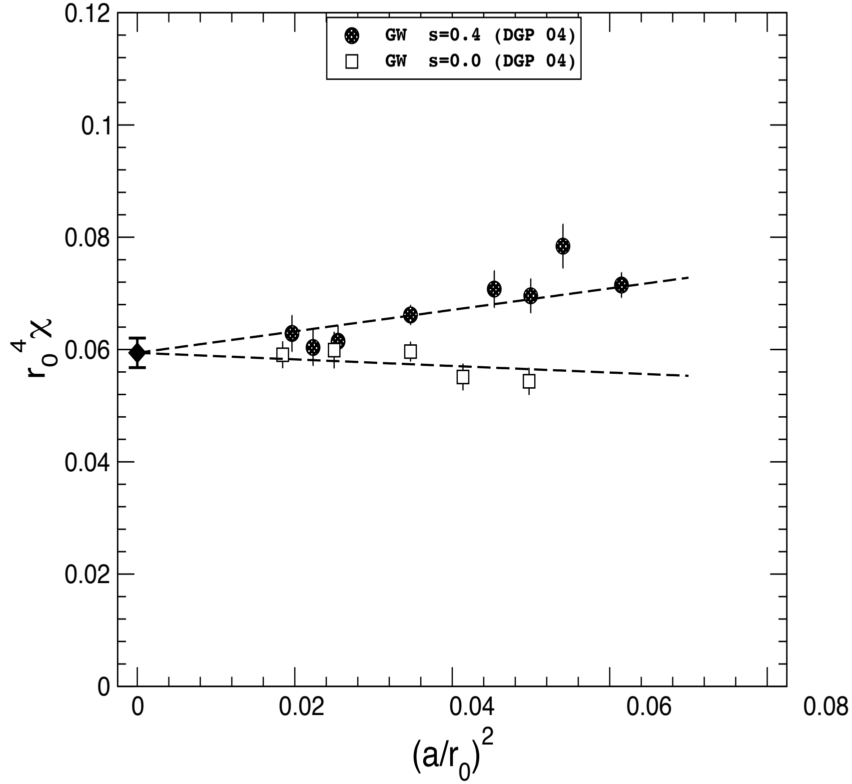
$$\lim_{a\to 0} a^4 \sum_x \langle q^{t}_{\rm N}(x)\, q^{t}_{\rm N}(0)\rangle = \lim_{a\to 0} a^4 \sum_x \langle q^{t=0}_{\rm N}(x)\, q^{t=0}_{\rm N}(0)\rangle$$

- Since $q^t_{\rm N}(x)$ and $q^t_{\rm L}(x)$ have the same asymptotic behaviour in the classical continuum limit, at positive flow-time [Lüscher 10; Lüscher, Weisz 11]

$$\lim_{a\to 0} a^4 \sum_x \langle q^{t}_{\rm L}(x)\, q^{t}_{\rm L}(0)\rangle = \lim_{a\to 0} a^4 \sum_x \langle q^{t=0}_{\rm N}(x)\, q^{t=0}_{\rm N}(0)\rangle \qquad (t > 0)$$

- The GW definition gives

$$r_0^4 \chi^{\mathrm{YM}} = 0.059 \pm 0.003 \quad \text{[Del Debbio et al. 04]}$$



Numerical computations in the Yang–Mills theory (I)

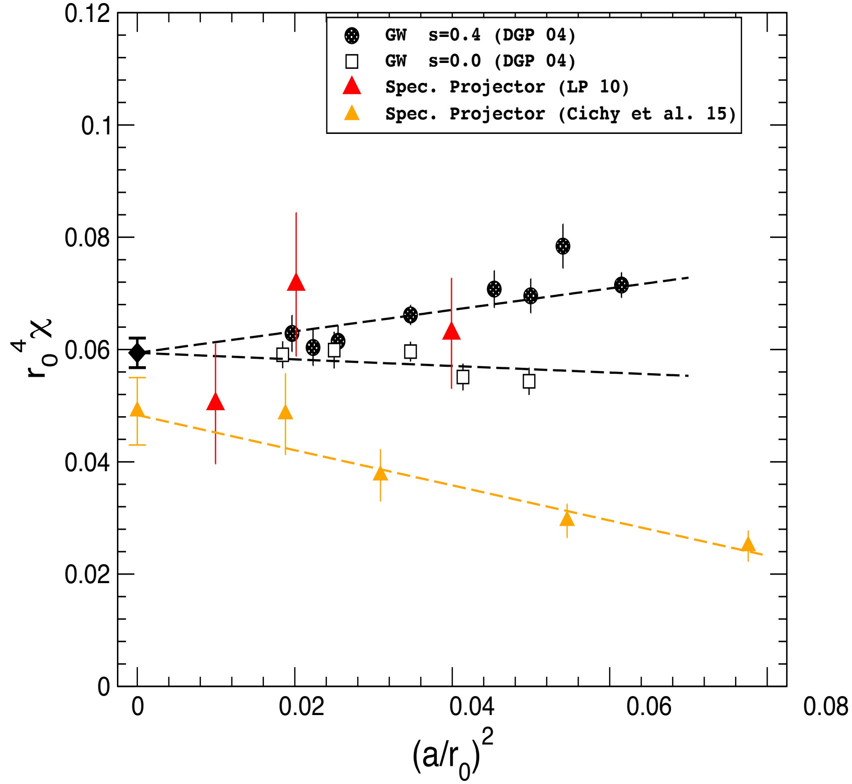
- The GW definition gives

$$r_0^4\chi^{\rm YM} = 0.059 \pm 0.003$$ [Del Debbio et al. 04]

- A (modified) projector definition leads to

$$r_0^4\chi^{\rm YM} = 0.049 \pm 0.006$$

[Cichy et al. 15; Ottnad Thu 16th]



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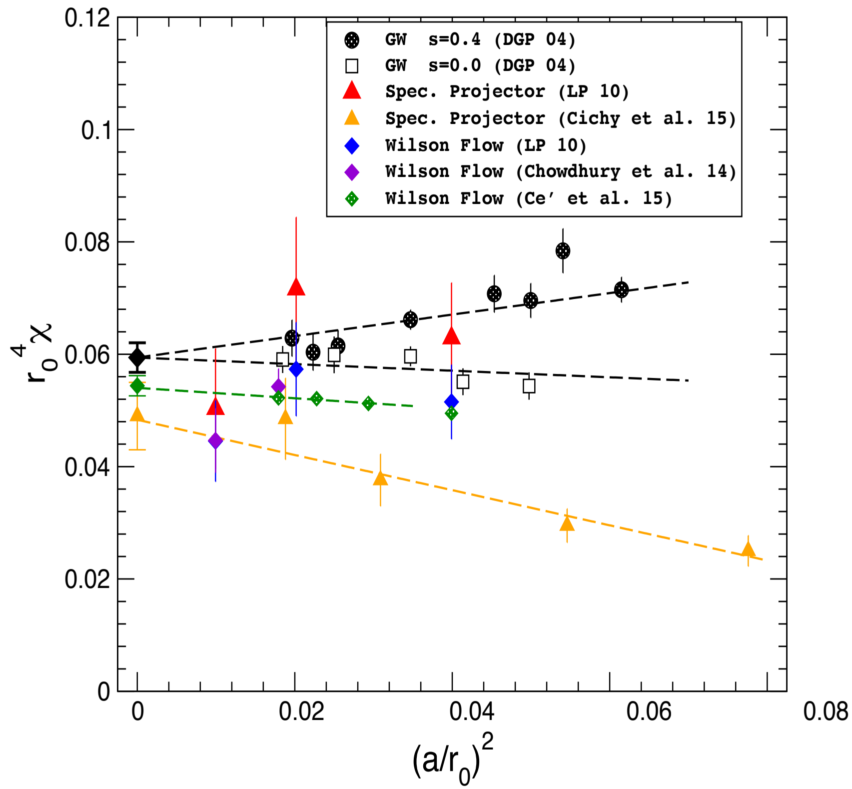
- The Wilson-flow definition gives

$$r_0^4 \chi^{\rm YM} = 0.0544 \pm 0.0018 \quad \text{[Cè et al. 15; Cè Fri 17th]}$$

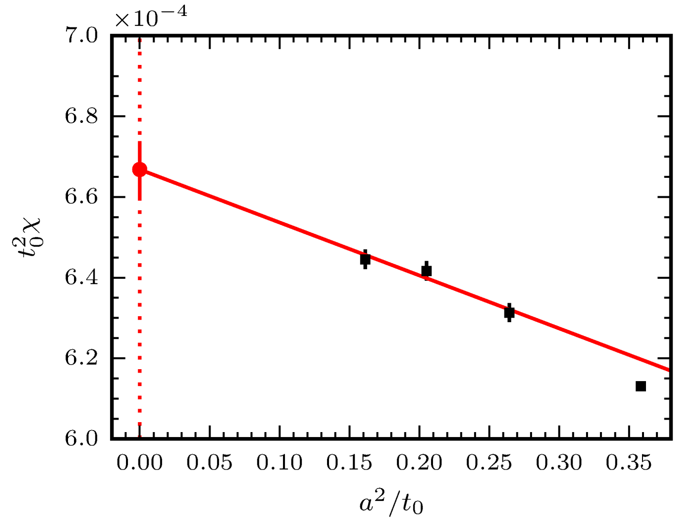


- From an unsolved problem to a precise universality test!

- The Wilson-flow definition is orders of magnitude cheaper than the others, and discretization effects are mild. The preferred choice in the Yang–Mills theory

Numerical computations in the Yang–Mills theory (II) [Cè et al. 15; Cè Fri 17th]

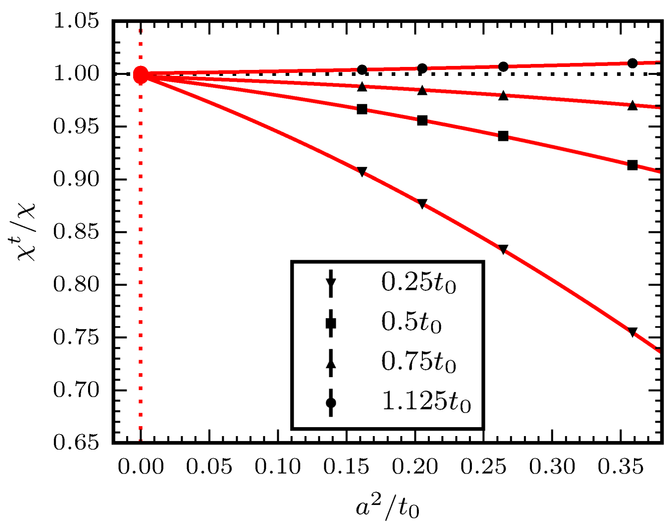
- Given the statistical precision of χ that can be easily reached with the Wilson-flow, "the all-time favorite r_0" needs to be replaced by t_0 or analogous definitions [Sommer 13]



- By restricting the linear fit to three points

$$t_0^2\chi^{\mathrm{YM}} = (6.67 \pm 0.07)\cdot 10^{-4}$$

a precision almost 3 times better than by using the present best determination of r_0



- By computing $\chi_t^{\mathrm{YM}}/\chi^{\mathrm{YM}}$ the correlations among data reduce errors to 0.1–$1.0‰$

- By extrapolating quadratically in a^2/t_0 the intercepts are all compatible with 1

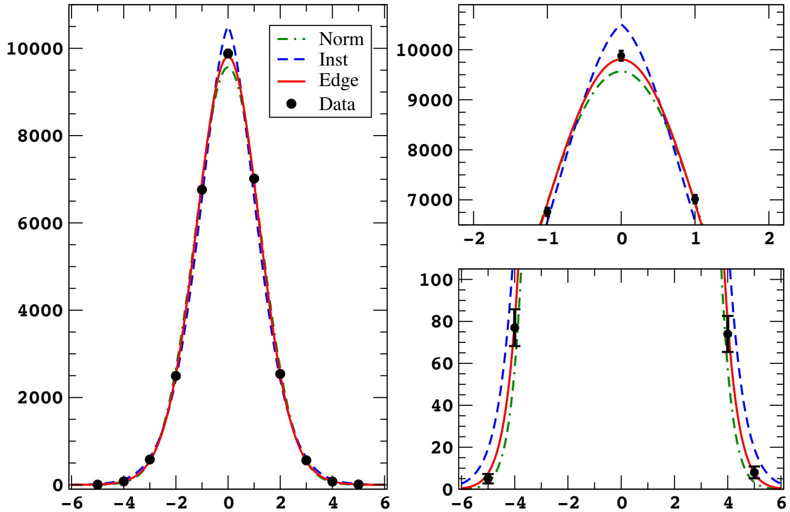
- All those numerical results are consistent with the conceptual progress made over the last decade. No signs of non-universal behaviour in the continuum limit of χ^{YM} (properly) defined on the lattice

Higher cumulants of the topological charge distribution

- Vacuum energy and charge distribution are

$$e^{-F(\theta)} = \langle e^{i\theta Q} \rangle, \; P_Q = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} e^{-F(\theta)}$$

Their behaviour is a distinctive feature of the configurations that dominate the path integr.



- Large N_c predicts ['t Hooft 74; Witten 79; Veneziano 79]

$$\frac{\langle Q^{2n} \rangle_{\rm con}}{\langle Q^2 \rangle} \propto \frac{1}{N_c^{2n-2}}$$

- Various conjectures. For example, dilute-gas instanton model gives ['t Hooft 76; Callan et al. 76; . . .]

$$F^{\rm Inst}(\theta) = -VA\{\cos(\theta) - 1\}$$

$$\frac{\langle Q^{2n} \rangle_{\rm con}}{\langle Q^2 \rangle} = 1$$

Higher cumulants of the topological charge distribution

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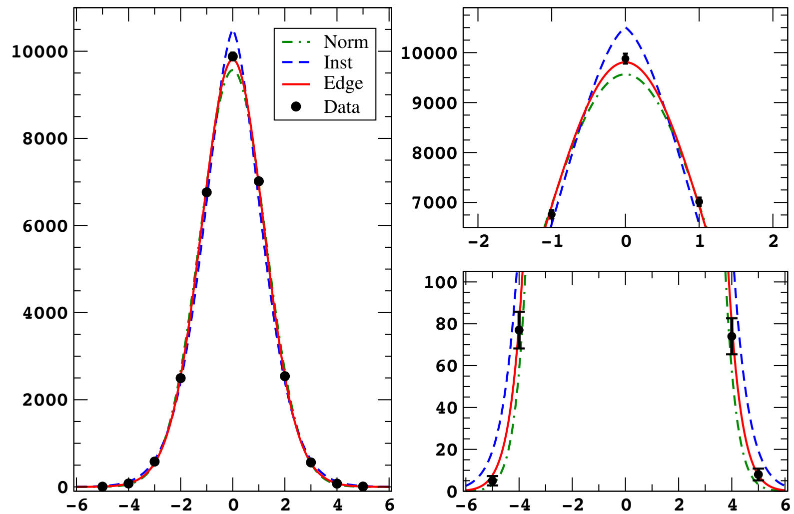
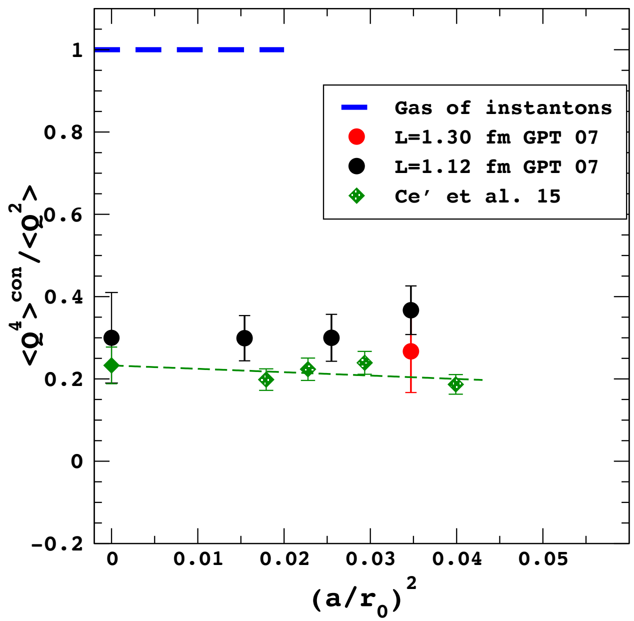


- Lattice computations give

$$\frac{\langle Q^4\rangle_{\rm con}}{\langle Q^2\rangle} \quad = \quad 0.30 \pm 0.11 \;\; \text{Ginsparg–Wilson}$$
$$= \quad 0.233 \pm 0.045 \;\; \text{Wilson Flow}$$

i.e. support large N_c and rule out a dilute gas of instantons



- The topological charge distribution is determined by the NP quantum fluctuations of Q

Outline

- Banks–Casher mechanism:
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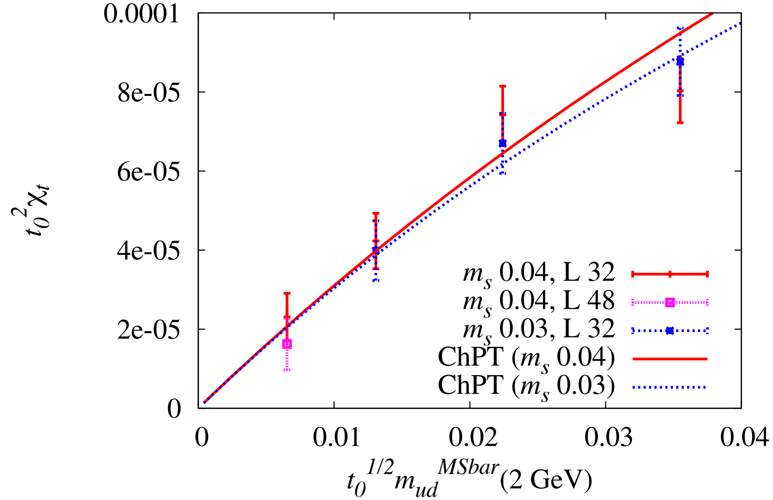
- In the presence of SSB with N_f degenerate light flavours

$$\chi = \frac{\Sigma}{N_\mathrm{f}}\, m + \mathcal{O}(m^2)$$

which implies a significant suppression with respect to the YM theory

[Fukaya et al. 14; Fukaya on Tue 14th]

$N_\mathrm{f} = 2+1$



Topological susceptibility in QCD

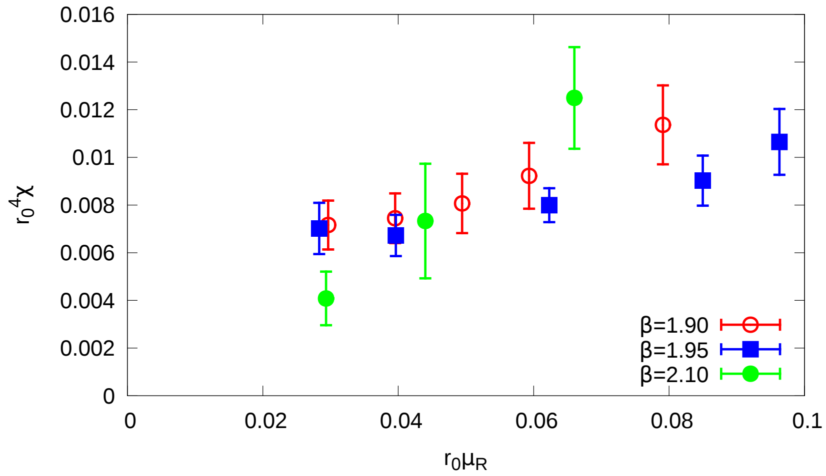
- In the presence of SSB with $N_{\rm f}$ degenerate light flavours

$$\chi = \frac{\Sigma}{N_{\rm f}}\, m + \mathcal{O}(m^2)$$

which implies a significant suppression with respect to the YM theory

[Cichy, Garcia-Ramos, Jansen 13]

$N_{\rm f} = 2+1+1$



- In the presence of SSB with N_f degenerate light flavours

$$\chi = \frac{\Sigma}{N_\mathrm{f}}\, m + \mathcal{O}(m^2)$$

which implies a significant suppression with respect to the YM theory

- Being a pure gluonic operator, the slope of χ is a measurement a posteriori of the number of flavours simulated

- χ is an expensive quantity to be computed

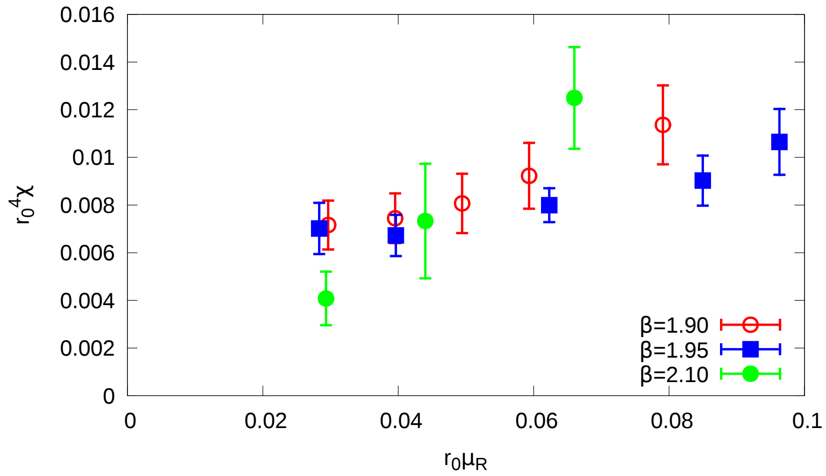
  $*$ long autocorrelation

  $$* \; \frac{\Delta\chi}{\chi} = \sqrt{\frac{2}{N_\mathrm{conf}}} + \mathcal{O}\left(\frac{1}{V}\right)$$

- $O(1000)$ independent configurations are needed for a precision of $\sim 5\%$. Not competitive to extract Σ

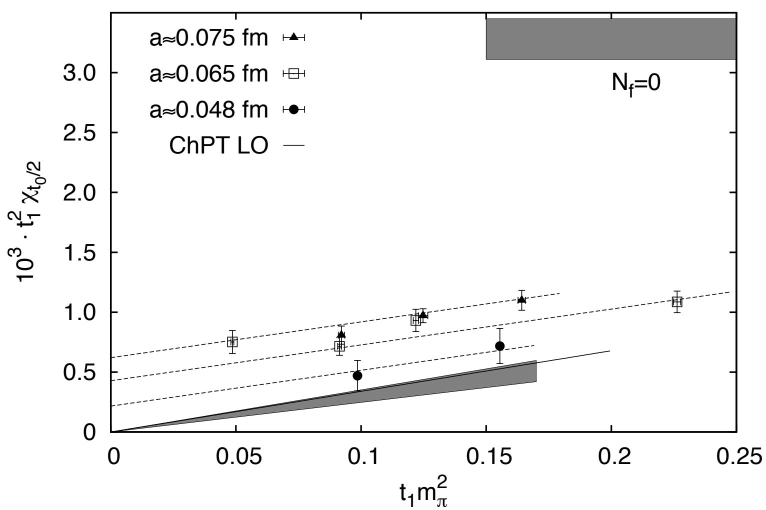
[Cichy, Garcia-Ramos, Jansen 13]

$N_\mathrm{f} = 2+1+1$



[Bruno, Schaefer, Sommer 14]

$N_\mathrm{f} = 2$



Topological susceptibility in QCD

- In the presence of SSB with N_f degenerate light flavours

$$\chi = \frac{\Sigma}{N_\mathrm{f}}\, m + \mathcal{O}(m^2)$$

which implies a significant suppression with respect to the YM theory

- In general no reason for χ to vanish in the chiral limit at finite lattice spacing

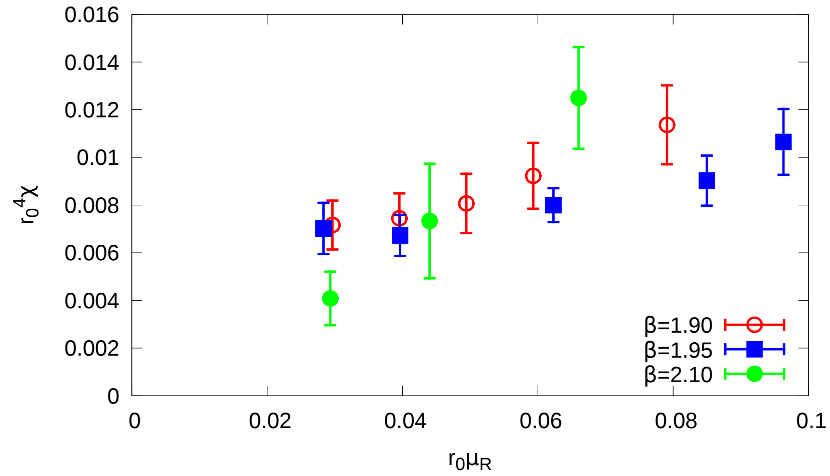
$$t_1^2\chi = c t_1 M_\pi^2 + b\,\frac{a^2}{t_1}$$

a LO (W)ChPT functional form which fits well data within (large) statistical errors [Bruno et al. 14]

- By taking at face value the result of the fit $c = 2.8(5)\cdot 10^{-3}$, I get $N_\mathrm{f} = 2.06 \pm 0.38$. Error still large but the result is encouraging

[Cichy, Garcia-Ramos, Jansen 13]

$N_\mathrm{f} = 2+1+1$



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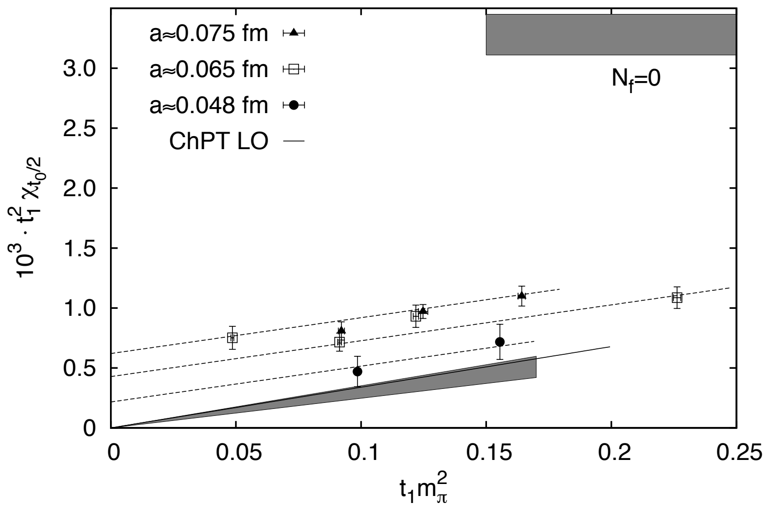
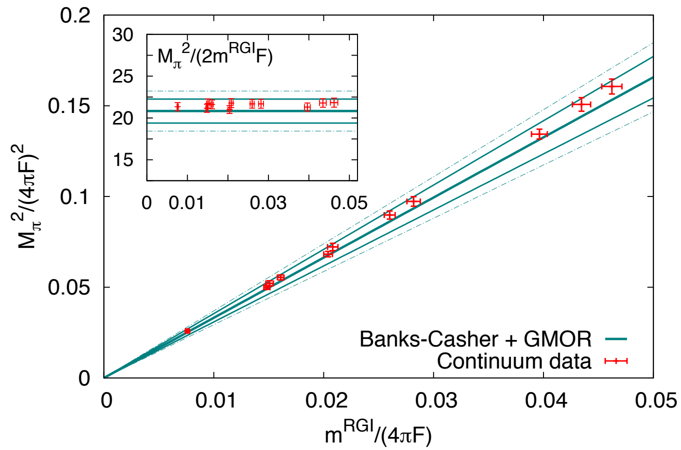


Conclusions

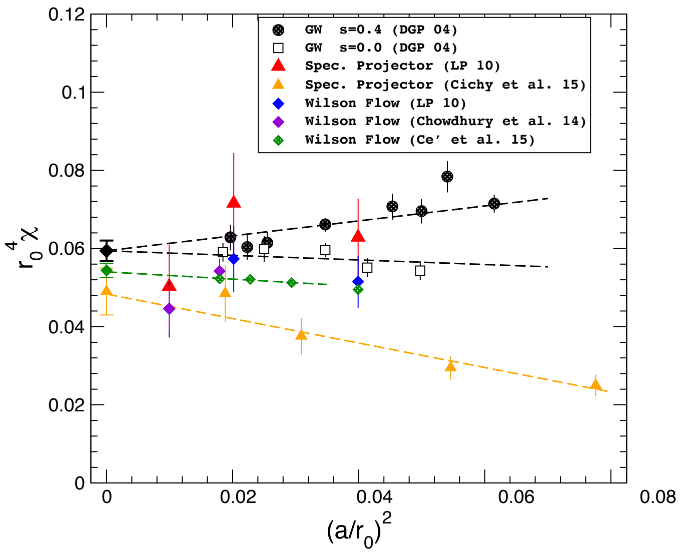
- An impressive global (lattice) community effort to reach a precise quantitative understanding of the behaviour of QCD in the chiral regime $(\Sigma, F, \bar{l}_3, \bar{l}_4, \ldots)$ from first principles

- The spectral density of the Dirac operator in the continuum and chiral limits is $\neq 0$ at the origin. The rate of condensation explains the bulk of the pion mass up to $M_\pi \leq 500$ MeV



- The topological susceptibility defined with the gradient flow satisfies the AWIs in the continuum limit. It is the right quantity to be inserted in the Witten-Veneziano formula

- All numerical results for χ^{YM} are consistent with the conceptual progress made over the last decade. A percent precision reached. Universality is at work if χ is (properly) defined on the lattice!



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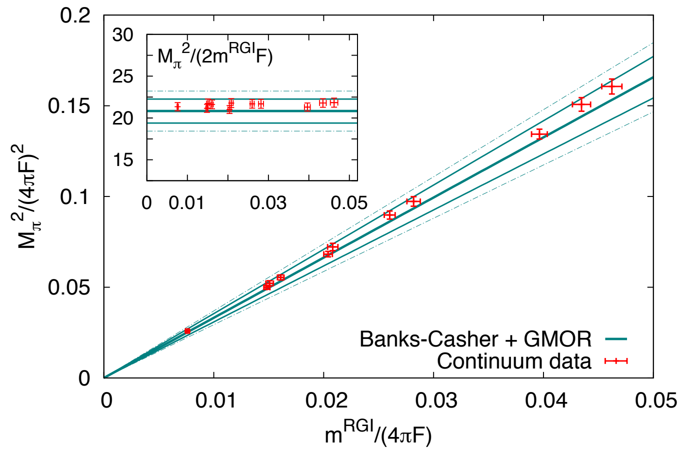
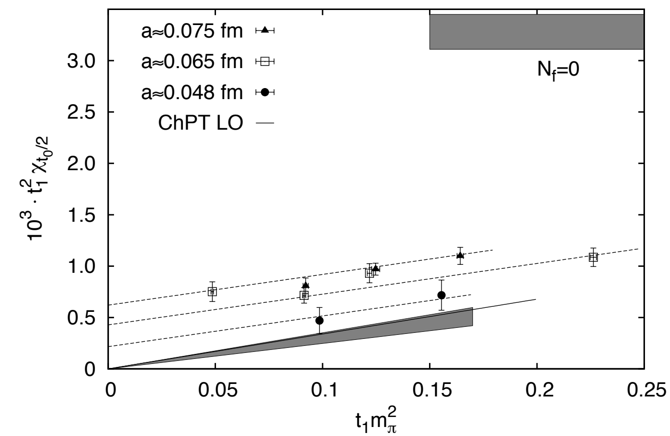
- The spectral density of the Dirac operator in the continuum and chiral limits is $\neq 0$ at the origin. The rate of condensation explains the bulk of the pion mass up to $M_\pi \leq 500$ MeV



- In full QCD χ shows the expected suppression with respect to its value in the Yang–Mills theory. Within the (so far) large errors, results are compatible with LO ChPT



- Our theoretical femtoscope can explore the chiral regime of QCD with higher and higher precision. This was just a dream only 10-15 years ago!

BACKUP SLIDES

- By defining

$$\chi(p) = \int d^4x\, \mathrm{e}^{-ipx} \langle q(x) q(0) \rangle \;, \qquad q(x) = \frac{1}{32\pi^2}\, \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\Big[F_{\mu\nu}(x) F_{\rho\sigma}(x)\Big]$$

in the chiral limit the integrated anomalous singlet AWI guarantees

$$\lim_{m\to 0} \chi(0) = - \lim_{m\to 0} \frac{m}{N_{\rm f}} \int d^4x\, \langle P^0(x) q(0) \rangle = 0$$

only if $\chi(0)$ is defined so to satisfy the AWI (crucial!)

- By expanding in $1/N_{\rm c}$

$$\chi(p) = \chi_0(p) + \chi_1(p) + \chi_2(p) + \dots$$

- Non-perturbatively $\chi^{\rm YM} = \chi_0(0) \neq 0$ even if it vanishes order by order in PT

- How is it possible that terms sub-leading in $N_{\rm f}/N_{\rm c}$ cancel the leading one ?

The Witten–Veneziano mechanism in a nutshell (II) [Witten 79; Veneziano 79]

- The Euclidean $\chi(p)$ satisfies a three-times subtracted dispersion relation

$$\chi(p) = b_1 + b_2 p^2 + b_3 (p^2)^2 - \frac{R^2_{\eta'}}{p^2 + m^2_{\eta'}} + (p^2)^3 \int_{\mathcal{M}^2}^{\infty} \frac{\rho(t)}{(t+p^2)t^3} dt$$

- For $p^2 \to 0$, the condition $\chi(0) = 0$ implies

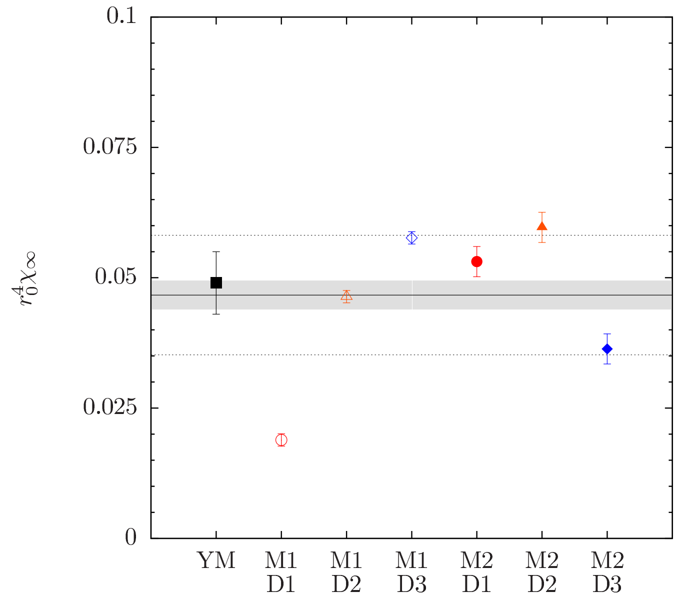
$$b_1 = \frac{R^2_{\eta'}}{m^2_{\eta'}} \quad \text{with} \quad R^2_{\eta'} = \frac{F^2_{\eta'} m^4_{\eta'}}{2N_{\rm f}} \quad \Longrightarrow \quad \left.\frac{F^2_{\eta'} M^2_{\eta'}}{2N_{\rm f}}\right|^{m=0}_{\frac{1}{N_{\rm c}}=0} = \left.\chi^{\rm YM}\right|_{\frac{1}{N_{\rm c}}=0}$$

- Note that in the limit $1/N_{\rm c} \to 0$:
  1. $\mathrm{U}(1)_{\rm A}$ is restored
  2. η' is a Nambu–Goldstone boson $\Longrightarrow m_{\eta'} = 0$
  3. At first order in $1/N_{\rm c}$, $m^2_{\eta'} = \mathcal{O}(1/N_{\rm c})$

- ChPT with a simultaneous expansion in powers of p^2 and $1/N_{\rm c}$
  [Coleman, Witten 80; Di Vecchia et al. 81; Kaiser, Leutwyler 00]

Witten–Veneziano in full QCD

$$r_0^4 \frac{F_{\eta'}^2}{2N_{\rm f}}(M_\eta^2 + M_{\eta'}^2 - 2M_K^2) = 0.047 \pm 3 \pm 11$$



- The HISQ action consists of a Fat7 smearing of the gauge links, then a projection of each smeared link onto a unitary matrix, followed by an "asq" smearing with twice the Lepage term and including the Naik term, a third nearest-neighbor coupling.

- Several of the largest ensembles were run with the RHMD algorithm, which is identical to the RHMC algorithm, except that the accept-reject step at the end of each trajectory is omitted.

- In the continuum limit the definition of χ suffers from ultraviolet singularities that require regularization. Such complications are unimportant at our range of lattice spacings

[Bazavov et al. 12]