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Exact Chiral Symmetry on the Lattice: QCD Applications

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Outline

- Introduction
- Meson masses
- The chiral condensate
- The topological susceptibility
- Kaon matrix elements
- Conclusions and outlook

Related Plenary Talks

- H. Wittig "Chiral Effective Lagrangian and Quark Masses"
- N. Ishizuka "K$\rightarrow \pi\pi$ Decay Amplitudes on the Lattice"
- C. Gattringer "Recent Results using Systematic Parametrizations of Ginsparg-Wilson Fermions"
- A. Feo "Supersymmetry on the lattice"

Related Topics

- Chiral gauge theories and formal developments: See parallel talks and posters by D. Adams, O. Baer, Y. Igarashi, M. Ishibashi, W. Kerler, P. Maris
- Two dimensions: See parallel talks by R. Narayanan, H. Neuberger and L. G. et al., Phys. Rev. D65(2002) 074506; F. Berruto et al., Phys. Rev. D65 (2002) 094516
- Topological charge fluctuations: See parallel talks and posters by C. Gattringer, I. Horvath and R.G. Edwards, U.M. Heller, Phys. Rev. D65 (2002) 014505

The Ginsparg-Wilson Relation

• In '82 Ginsparg and Wilson proposed the "mildest way" of breaking standard chiral symmetry on the lattice

$$\{\gamma_5, D\} = \bar{a} D \gamma_5 D \quad \Longleftrightarrow \quad \{\gamma_5, D^{-1}\} = \bar{a}\gamma_5$$

• An exact symmetry at finite cut-off is implied (M. Lüscher '98)

$$\delta q = \hat{\gamma}_5 q \qquad \delta \bar{q} = \bar{q}\gamma_5$$

$$\hat{\gamma}_5 = \gamma_5(1 - \bar{a}D)$$

where $\hat{\gamma}_5^\dagger = \hat{\gamma}_5$, $\hat{\gamma}_5^2 = 1$.

• The anomaly is recovered à la Fujikawa (M. Lüscher '98)

$$Q(x) = \frac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)]$$

$$n_{\mathrm{L}} - n_{\mathrm{R}} = \mathrm{index}(D) = \int d^4x \, Q(x)$$

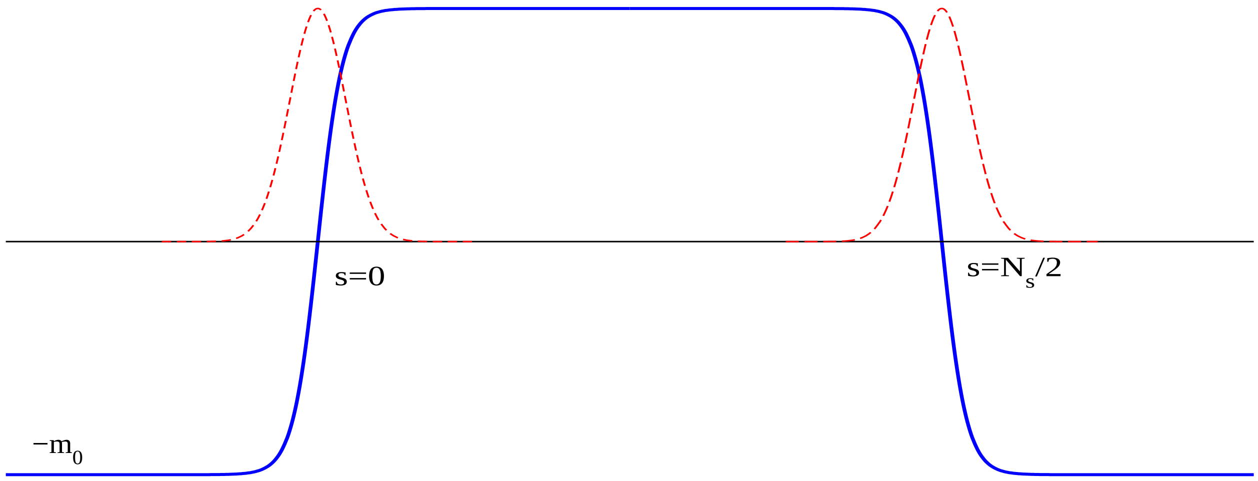
(see also H. Neuberger '97, P. Hasenfratz et al. '98)

• After more than a decade, a Dirac operator which satisfies the GW relation, is local and leads to the correct continuum limit was found

Domain-Wall and Overlap Fermions

(D. B. Kaplan '92, H. Neuberger '97)

• A five-dimensional fermion with a domain-wall mass term generates a 4D fermion with the two chiral components separated by a distance $N_s/2$ and with exponentially small overlap (Rubakov-Shaposhnikov '83, Callan-Harvey '85, Kaplan '92)



• Light 4D states interpolated by fermion fields on the walls

• In the $N_s \rightarrow \infty$ limit we expect a massless 4D fermion !

• We are studying QCD with many flavours mixed in a given way

• We can integrate out the heavy flavours and remain with a four-dimensional effective action of the light boundary fields
(R. Narayanan, H. Neuberger '92 '94; H. Neuberger '97)

$$aD_{N_s} = 1 + \gamma_5 \frac{(1+\bar{Q})^{N_s} - (1-\bar{Q})^{N_s}}{(1+\bar{Q})^{N_s} + (1-\bar{Q})^{N_s}}$$

$$Q \equiv \frac{a_s H}{2 + a_s \gamma_5 H} \qquad H \equiv \gamma_5 D_W(-m_0)$$

• The Neuberger's operator is obtained in the limit

$$D_N \equiv \lim_{a_s \to 0} \lim_{N_s \to \infty} aD_{N_s}$$

$$= 1 + \gamma_5 \frac{H}{\sqrt{H^2}}$$

• It is straightforward to verify

$$\{\gamma_5, D_N\} = aD_N \gamma_5 D_N$$

• If the gauge field is sufficiently smooth, the operator is local
(P. Hernández, K. Jansen and M. Lüscher '98)

Fixed point Dirac operator

• An operator which satisfies the GW relation can be constructed iteratively with a RG blocking procedure from the continuum
(P. Hasenfratz, F. Niedermayer '94 '98)

• In the simulations an approximate explicit solution D^{FP} of the fixed point equations can be used

• For measurements where it is crucial a very precise chiral symmetry, the residual breaking can be removed by defining

$$D_{ov}^{FP} = 1 + \frac{D^{FP}}{\sqrt{D^{FP\dagger} D^{FP}}}$$

(W. Bietenholz '99, P. Hasenfratz et al. '01)

Massive Action

- The massive action is defined as

$$S_f = a^4 \sum_x \bar{\psi}(x) \left[(D + P_- M^\dagger \hat{P}_- + P_+ M \hat{P}_+)\psi \right](x)$$

- If we define

$$\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5) \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

and

$$\psi_{R,L} = \hat{P}_\pm \psi \qquad \bar{\psi}_{R,L} = \bar{\psi} P_\mp$$

the $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$ transformations are defined as

$$\begin{array}{ll} \psi_L \to V_L \psi_L & \bar{\psi}_L \to \bar{\psi}_L V_L^\dagger \\ \psi_R \to V_R \psi_R & \bar{\psi}_R \to \bar{\psi}_R V_R^\dagger \end{array}$$

- The action is invariant if also

$$M \to V_L M V_R^\dagger$$

- The exact chiral symmetry forbids operators of $d = 5$ in the action which is $O(a)$ improved

• Chiral symmetry forbids additive quark renormalization

• The bilinears with correct chiral properties are $O(a)$ improved

$$\mathcal{O}^{\Gamma}_{\alpha\beta}(x) = \bar\psi_\alpha(x)\Gamma\tilde\psi_\beta(x) \qquad \tilde\psi_\beta(x) = \Big[(1-\frac{\bar a}{2}D)\psi_\beta\Big](x)$$

• Chiral rotations do not commute with CP. Apparently no simple transformation of $\mathcal{O}^{\Gamma}$ under CP (Lüscher '98, Fujikawa et al. '02)

• In correlations of operators at non-zero physical distance

$$\mathcal{O}^{\Gamma}_{\alpha\beta}(x) = \frac{1}{(1-\frac{\bar a}{2}m_\beta)}\bar\psi_\alpha(x)\Gamma\psi_\beta(x)$$

and therefore under CP (L.G. et al. in preparation)

$$\mathcal{O}^{\Gamma}_{\alpha\beta}(x) \xrightarrow{\text{CP}} \frac{1-\frac{\bar a}{2}m_\alpha}{1-\frac{\bar a}{2}m_\beta}\mathcal{O}^{\Gamma}_{\beta\alpha}(\tilde x)$$

• The generalization to four-fermion operators is straightforward

Ward Identities and Quark Masses

- By performing a non-singlet local rotation $[\epsilon_{V,A} = \epsilon_{V,A}(x)\delta_{xy}]$

$$-i\delta_{V,A}\psi = \Big[\hat{P}_R\epsilon_{V,A}\hat{P}_R \pm \hat{P}_L\epsilon_{V,A}\hat{P}_L\Big]\psi$$

$$i\delta_{V,A}\bar{\psi} = \bar{\psi}\Big[P_L\epsilon_{V,A} \pm P_R\epsilon_{V,A}\Big]$$

exact vector and axial WIs are obtained

$$\langle\partial^*_\mu V_\mu(x)\mathcal{O}\rangle = (m_1 - m_2)\langle S(x)\mathcal{O}\rangle + i\Big\langle\frac{\delta\mathcal{O}}{\delta\epsilon_V(x)}\Big\rangle$$

$$\langle\partial^*_\mu \mathcal{A}_\mu(x)\mathcal{O}\rangle = (m_1 + m_2)\langle P(x)\mathcal{O}\rangle + i\Big\langle\frac{\delta\mathcal{O}}{\delta\epsilon_A(x)}\Big\rangle$$

- If $\mathcal{O}(x_\alpha, x_\beta) = \tilde{\psi}_\alpha(x_\alpha)\bar{\psi}_\beta(x_\beta)$, Fourier transform and $q \to 0$

$$(m_1 + m_2)\mathrm{Tr}\Big[\gamma_5\Lambda_P(p, m_1, m_2)\Big] = \mathrm{Tr}\Big[\mathcal{S}^{-1}(p, m_1) + \mathcal{S}^{-1}(p, m_2)\Big]$$

- At variance with the Wilson case, the very same definition of the quark mass appears in the axial and vector WIs and in the quark propagator

Conserved Axial Current

(P.H. Ginsparg, K. G. Wilson '82; Y. Kikukawa, A. Yamada '99)

• Under a non-singlet local chiral rotation ($\epsilon = \epsilon^a(x) T^a \delta_{xy}$)

$$-i\delta_A \psi = (P_R \epsilon \hat{P}_R - P_L \epsilon \hat{P}_L)\psi \qquad - i\delta_A \bar{\psi} = \bar{\psi}\gamma_5 \epsilon$$

$$-i\delta_A S = -\sum_x \epsilon(x) \partial_\mu^* A_\mu(x) = \sum_x \partial_\mu \epsilon(x) A_\mu(x)$$

• By extending the gauge group $SU(N_c) \to SU(N_c) \times U(1)$ and performing a $U(1)$ gauge rotation

$$U_\mu(x) \to U_\mu^{(\alpha)} = e^{i\alpha_\mu(x)} U_\mu(x) = e^{i\epsilon(x)} U_\mu(x) e^{-i\epsilon(x+\hat{\mu})}$$

we can define the kernel

$$K_\mu = -i \frac{\delta D(U_\mu^{(\alpha)})}{\delta \alpha_\mu(x)}\Big|_{\alpha=0}$$

and the corresponding conserved axial current

$$\mathcal{A}_\mu^a(x) = \bar{\psi}\Big(P_L K_\mu(x) \hat{P}_R - P_R K_\mu(x) \hat{P}_L\Big) T^a \psi$$

• In this form it can be implemented numerically
(P. Hasenfratz et al. '02)

• For CC correlations (and generalizations) the propagator from any point to any point is required (L.G. et al. in prep.)

"Local" Axial Current

- The "local" axial current

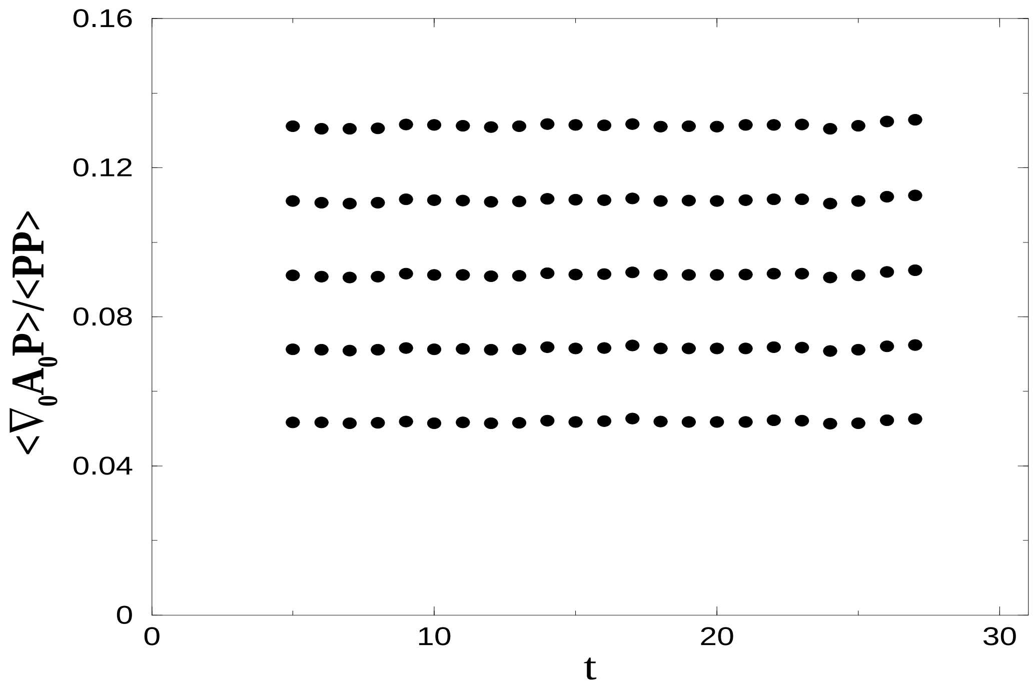
$$A_\mu(x) = \bar{\psi}_\alpha(x)\gamma_\mu\gamma_5\tilde{\psi}_\beta(x)$$

is not conserved but has the correct transformation properties

- On shell

$$Z_A\langle\nabla_\mu A_\mu(x)P(0)\rangle = (m_1 + m_2)\langle P(x)P(0)\rangle + O(a^2)$$

and Z_A can be extracted from the ratio of correlation functions (L.G., C. Hoelbling, C. Rebbi '01; S.J. Dong et al. '01, BGR Coll. '02)



- For standard overlap ($\beta = 6.0$, $\rho = 1.4$), $Z_A = 1.55(4)$ to be compared with PT $Z_A^{\rm PT} = 1.15 - 1.35$

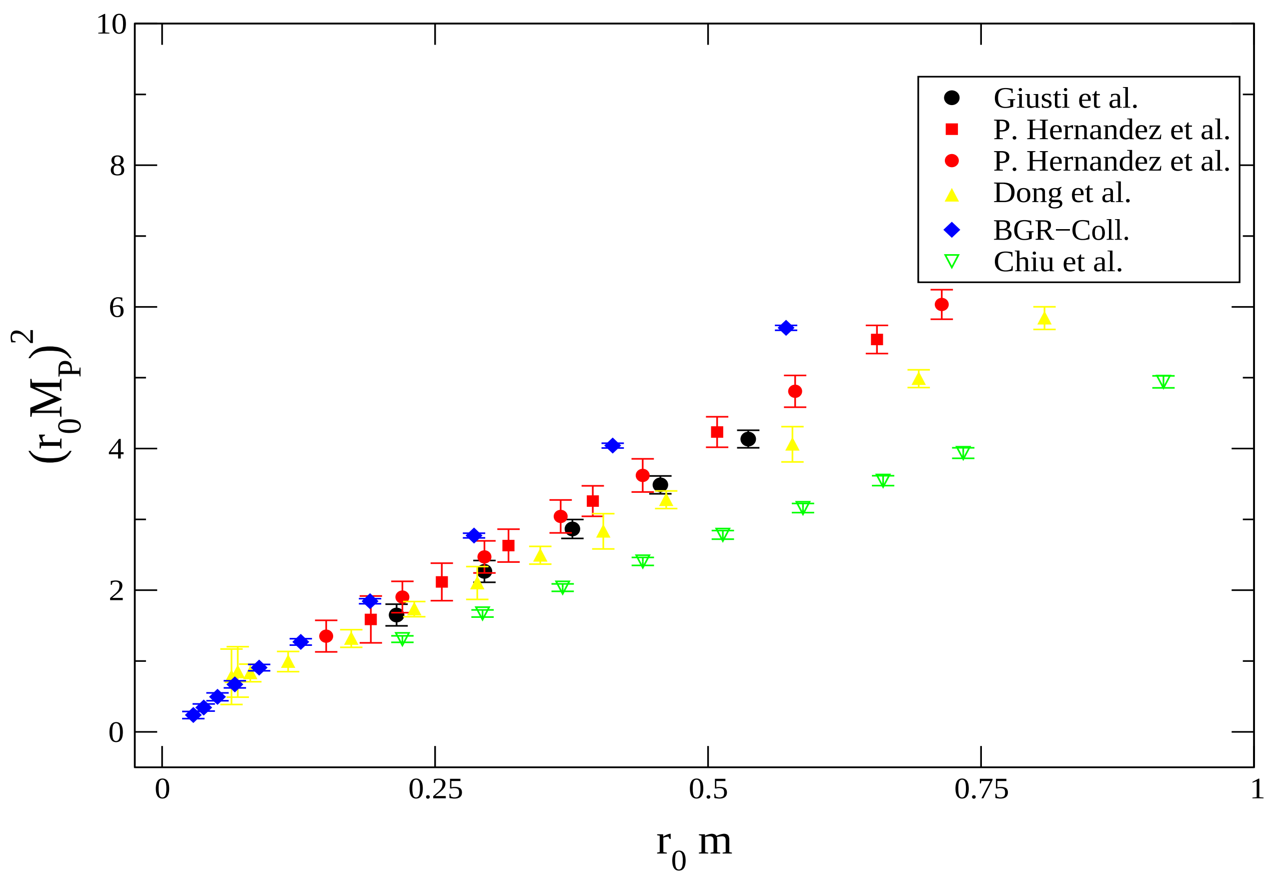
Exact Chiral Symmetry: Some Advantages

- The Dirac operator has an index at finite cut-off:
  - A natural definition for $Q(x)$
  - Identification of the topological charge

- Very light quark masses can be reached:
  - No exceptional configurations

- No mixing among operators of different chirality:
  - No additive quark renormalization
  - Simplified mixing for composite operators
  - $O(a)$ improvement straightforward

Overlap Operator: M_π^2 vs m

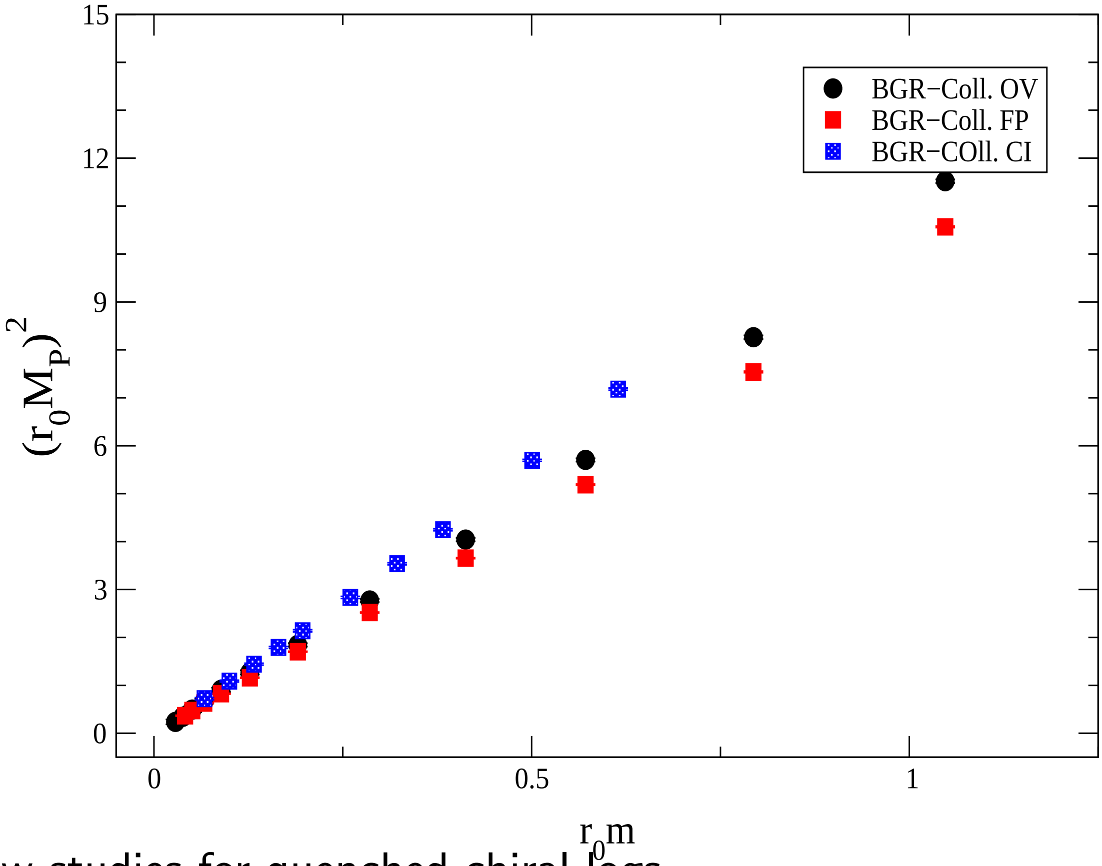


Collaboration	a (fm)	V
L. G. et al. '01	~ 0.093	$16^3 \times 32$
P. Hernández et al. '01	~ 0.093	$14^3 \times 24$
P. Hernández et al. '01	~ 0.12	$10^3 \times 24$
S. J. Dong et al. '01	~ 0.14	20^4
BGR-Coll. '02	~ 0.16	$12^3 \times 24$
Chiu-Hsieh '02	~ 0.14	$8^3 \times 24$

- Good control over chiral symmetry !

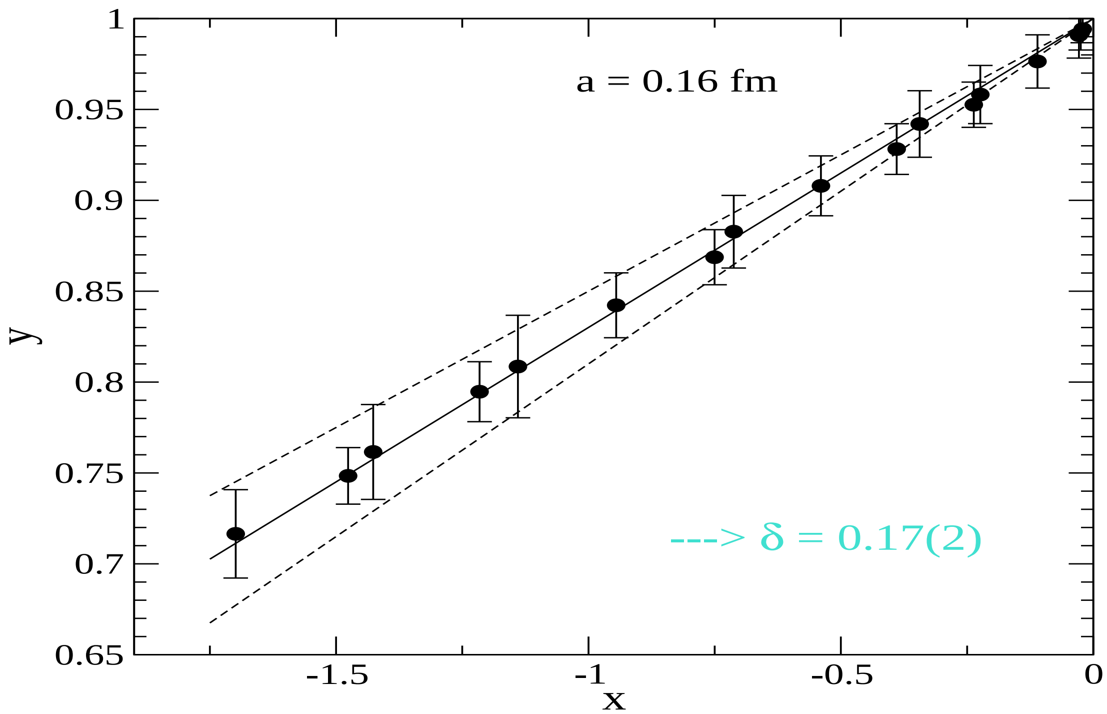
Fixed-Point and Chirally-Improved Operators: M_π^2 vs m

(See C. Gattringer talk)



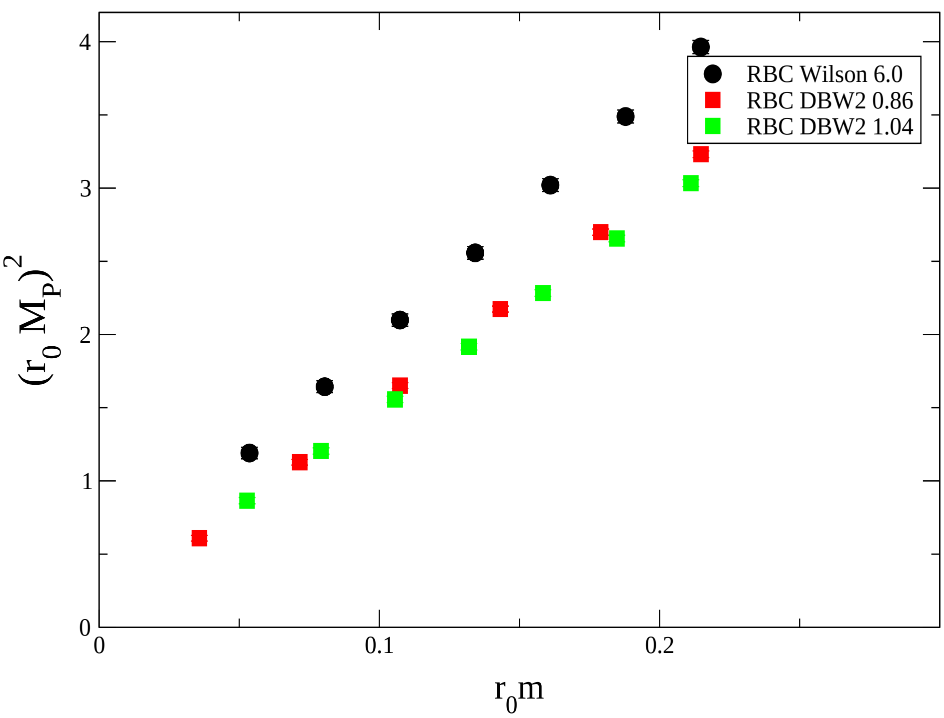
- New studies for quenched chiral logs

Collaboration	Action	δ
JLQCD '96	staggered	$0.05 - 0.10$
CP-PACS '98	Wilson	$0.10(2)$
Bardeen et al. '00	modified Wilson	$0.065(13)$
Kentucky '01	overlap	$0.23 - 0.48$
RBC '01	DW	$0.05(2)$
BGR-Coll. '02	Fixed-Point	$0.17(2)$
BGR-Coll. '02	Chirally-Improved	$0.18(4)$
Kentucky '02	overlap	$0.248(12)$
Chiu-Hsieh	overlap	0.203(14)

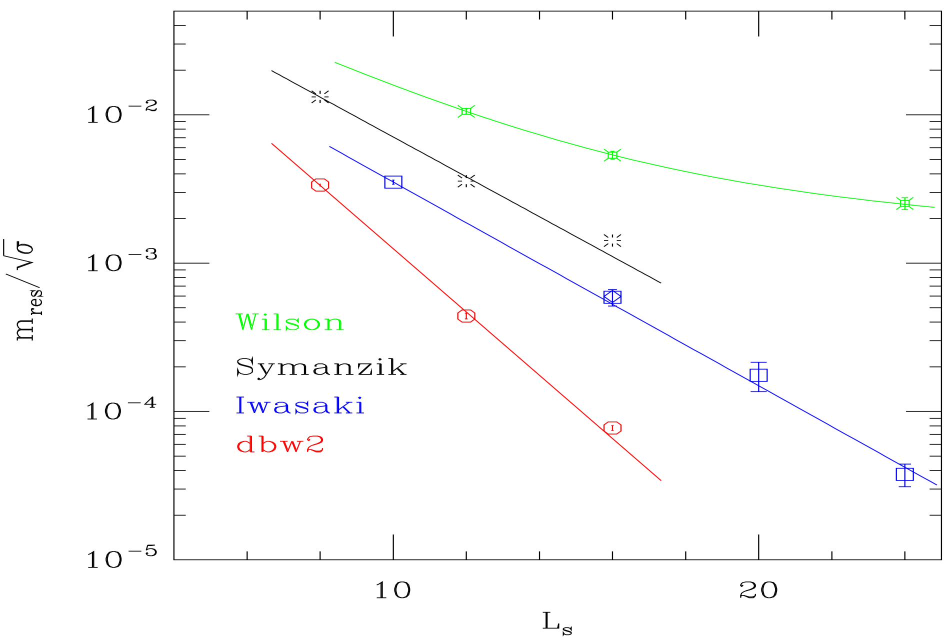


a = 0.16 fm
---> δ = 0.17(2)
y
x
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
-1.5
-1
-0.5
0

Domain-Wall Fermions with DBW2 Action: M_π^2 vs m



- With the DBW2 Action improved m_{res}

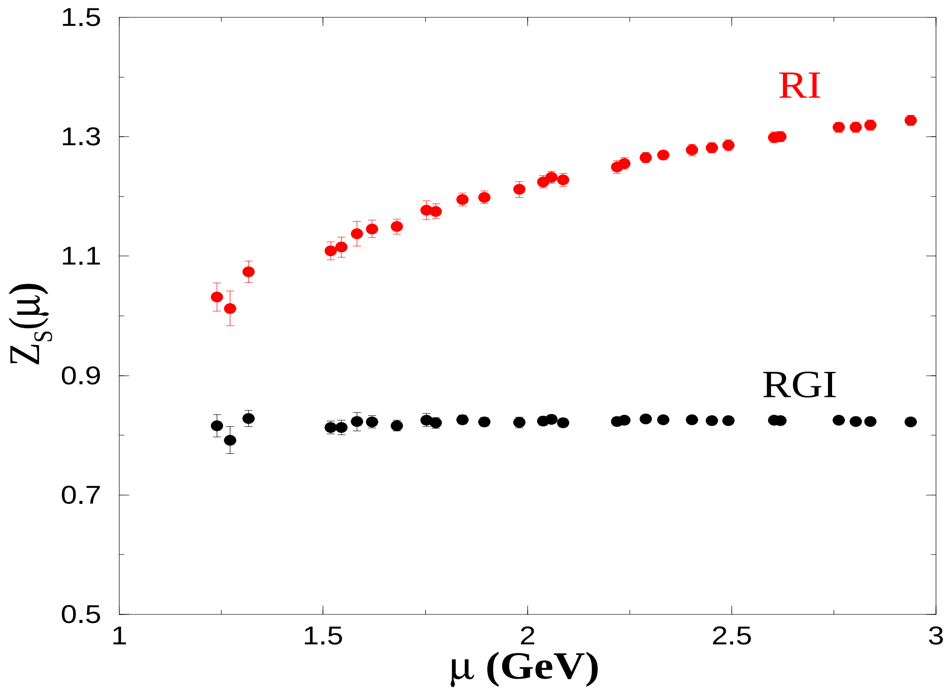


- A potential difficulty: samples with proper distribution for Q
  (see J. Noaki and S. Necco talks and K. Nagai poster)

RI/MOM Non-Perturbative Renormalization

(G. Martinelli et al. '95)

- Extensive applications for DW (T. Blum et al. '99-'01)
- First applications for overlap (L.G., C. Hoelbling, C. Rebbi '01)



- A RI/MOM renormalization condition is fixed on amputated off-shell Green's functions computed in the Landau gauge

$$Z_{\mathcal{O}}^{-1}(\mu a) Z_q(\mu a) = \lim_{m \to 0} \mathrm{Tr}\Big[P_{\mathcal{O}} \Lambda_{\mathcal{O}}(p,m)\Big]_{p^2=\mu^2}$$

- For the scalar density

$$Z_S(a\mu) = Z_A \lim_{m=0} \frac{\mathrm{Tr}\,[P_A \Lambda_A(p,m)]}{\mathrm{Tr}\,[P_S \Lambda_S(p,m)]}\Bigg|_{p^2=\mu^2}$$

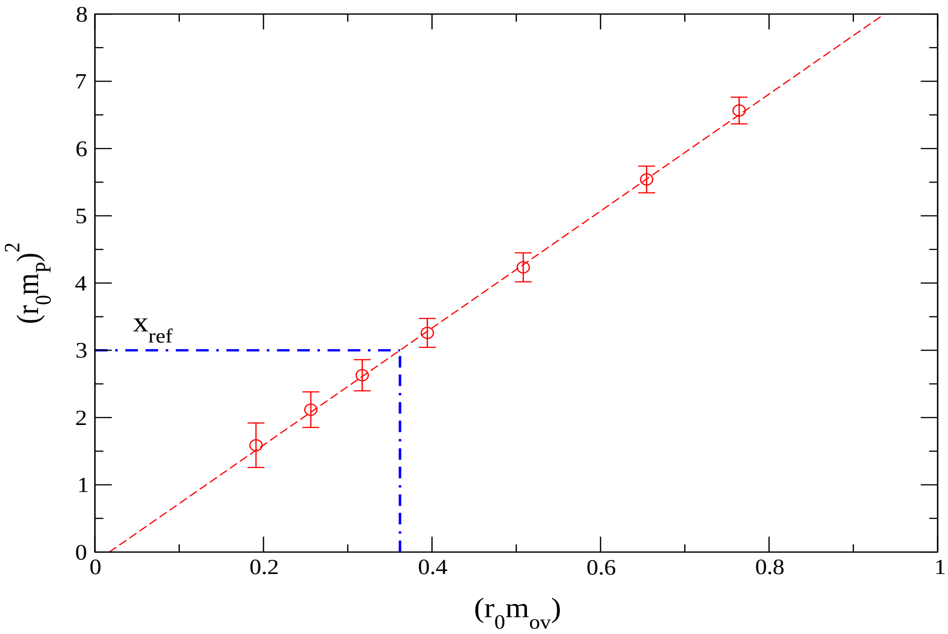
- For bilinears there is a "renormalization window"
  (see also L. Lellouch and J. Zhang talks)
- Matching with $\overline{\text{MS}}$ needs continuum PT only

Matching to Continuum Wilson Data

(P. Hernández et al. '01)

- Fixing the RGI quark mass to the continuum extrapolated Wilson value

$$M_W^{\rm RGI}\Big|_{(r_0 M_P)^2 = x_{ref}} = \frac{1}{Z_S^{\rm RGI}(a)} m_{ov}(a)\Big|_{(r_0 M_P)^2 = x_{ref}}$$



. . . or fixing

$$\langle 0|P^{\rm RGI}|\pi\rangle_W\Big|_{(r_0 M_P)^2 = x_{ref}} = Z_S^{\rm RGI}(a)\langle 0|P_{ov}(a)|\pi\rangle\Big|_{(r_0 M_P)^2 = x_{ref}}$$

- Warning: the prediction for a low energy hadronic quantity is lost in the renormalization procedure

$Z_S^{\overline{\rm MS}}$(2 GeV) Overlap, $\beta = 6.0$, $\rho = 1.4$		
PT	RI/MOM	Wilson
1.1-1.3	1.41(6)	1.43(11)

The Chiral Condensate from GMOR

- The chiral symmetry guarantees that

$$-\frac{1}{N_f}\langle\bar{\psi}\tilde{\psi}\rangle = \chi(a,m) + \alpha\frac{m(a)}{a^2}$$

renormalizes only multiplicatively in the chiral limit

- The chiral condensate is defined as

$$\Sigma(\mu) = \lim_{a\to 0} Z_S(a\mu)\Sigma(a) = \lim_{a,m\to 0} Z_S(a\mu)\chi(a,m)$$

- Large divergent subtractions for $m \neq 0$ can be avoided:

$$\frac{1}{N_f}\langle\bar{\psi}\tilde{\psi}\rangle = m\sum_x \langle P(x)P(0)\rangle$$

leads to the GMOR relation

$$\boxed{\frac{\Sigma(a)}{F^2} = \lim_{m\to 0}\frac{m}{F_P^2 M_P^2}\Big|\langle 0|P|P\rangle\Big|^2 = \lim_{m\to 0}\frac{M_P^2}{2m}}$$

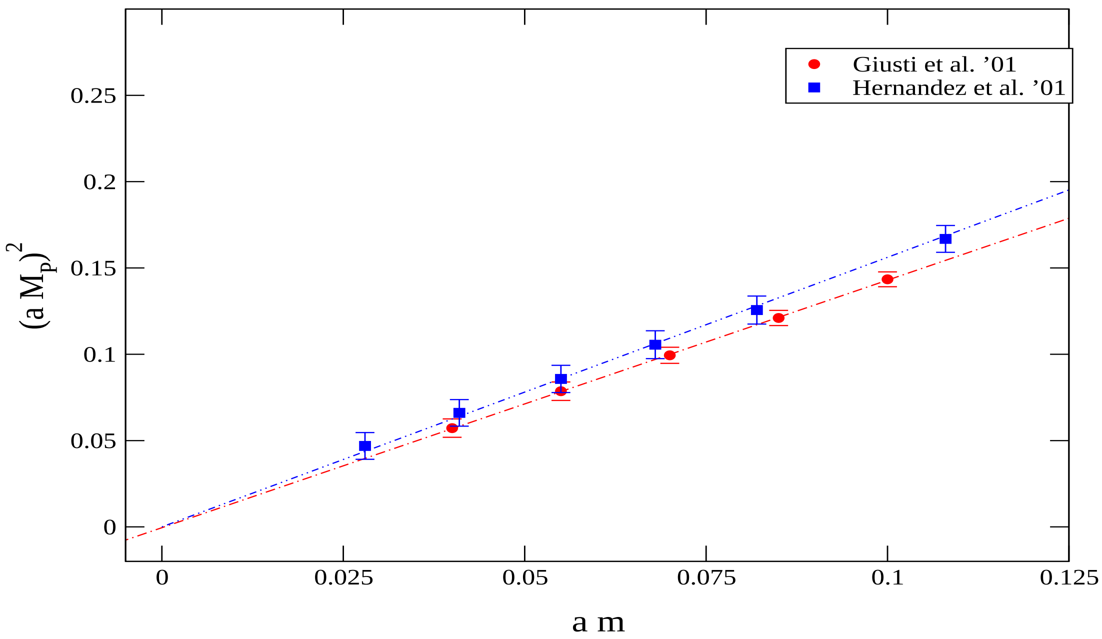
- We can extract $\Sigma(a)$ by studying M_P^2 as a function of m

- Quenched χPT at $O(p^4)$ $[M^2 = 2\Sigma m/F^2,\ \alpha_i = 8(4\pi)^2 L_i]$

$$\frac{M_P^2}{2m} = \frac{\Sigma}{F^2}\Big[1 - \frac{m_0^2}{3(4\pi F)^2}\left(1 + \log(\frac{M^2}{\mu_\chi^2})\right)$$

$$+ \frac{\alpha M^2}{3(4\pi F)^2}\left(2\log(\frac{M^2}{\mu_\chi^2}) + 1\right) + \left(2\alpha_8(\mu_\chi) - \alpha_5(\mu_\chi)\right)\frac{M^2}{(4\pi F)^2}\Big]$$

(G. Colangelo, E. Pallante '98)



- Linear behaviour in the range $m_s/2 < m < m_s$

Ref.	a (fm)	V	$\Sigma^{\overline{\rm MS}}(2\,\text{GeV})$ (GeV^3)
L. G. et al. '01	$\simeq 0.093$	$16^3 \times 32$	$0.0190(11)(33)$
P. Hernández et al. '01	$\simeq 0.093$	$14^3 \times 24$	$0.0199(8)(15)$
P. Hernández et al. '01	$\simeq 0.12$	$10^3 \times 24$	$0.0195(8)(15)$

Light Quarks in a Box: ϵ expansion

(J. Gasser, H. Leutwyler '87)

- When $F^2 M^2 L^4 \simeq 1$, $L \gg 1/(4\pi F)$ and for $p^2 \simeq 1/L^2$

$$\frac{M}{\Lambda_\chi} \sim \frac{p^2}{\Lambda_\chi^2} \sim \frac{1}{(4\pi L F)^2} = \epsilon^2$$

and QCD Green's functions can be expanded in powers of ϵ

- The chiral expansion is reordered $\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(2)} + \ldots$

$$\mathcal{S}^{(0)} = \int d^4x \frac{1}{2} \mathrm{Tr}\, [\partial_\mu \xi(x) \partial_\mu \xi(x)] - \frac{m\Sigma V}{2} \mathrm{Tr}\, [e^{i\theta/N_f} U_0 + \mathrm{h.c.}]$$

$$U = U_0 \exp\left(i\sqrt{2}\xi(x)/F\right) \qquad \int \xi(x) = 0$$

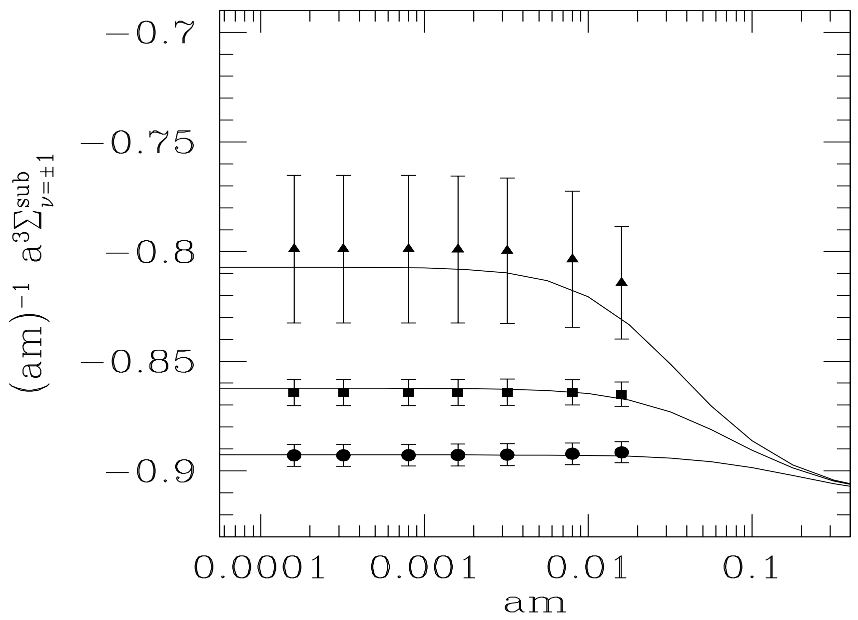
- The $O(1)$ zero-mode fluctuations have to be treated exactly

$$\int_{SU(N_f)} dU_0 \exp\left(\frac{m\Sigma V}{2} \mathrm{Tr}\, [e^{-i\theta/N_f} U_0^\dagger + U_0 e^{i\theta/N_f}]\right)$$

- The expansion can also be performed in fixed-topology sectors
- As a consequence of reordering, $L_4 \rightarrow L_8$ do not enter $\mathcal{S}^{(2)}$

- Finite-volume effects can be predicted by χPT !

Quenched Chiral Condensate in the ϵ regime



- At leading order in ϵ $(z = m\Sigma V)$
  (J.C. Osborn et al. '99, P. H. Damgaard et al. '99)

$$\frac{\Sigma_\nu}{\Sigma} = z\Big(I_\nu(z)K_\nu(z) + I_{\nu+1}(z)K_{\nu-1}(z)\Big) + \frac{\nu}{z}$$

- One-loop corrections are obtained by replacing
  (P. H. Damgaard '01, P. H. Damgaard et al. '02)

$$\Sigma \rightarrow \Sigma_{eff}(V) = \Sigma\Big[1 - \frac{m_0^2}{3(4\pi F)^2}\left(\tilde{\beta}_2 + \log\left(\frac{L_0^2}{L^2}\right)\right) - \frac{\alpha}{3(4\pi FL)^2}\tilde{\beta}_1\Big]$$

- Numerical computations performed with GW fermions
  (P. H. Damgaard et al. '99, P. Hernández et al. '99)

LO Analyses	a (fm)	L	$\Sigma^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})$ (GeV^3)
P. Hernández et al. '01	$\simeq$ 0.12	8,10,12	0.0192(22)(2)(14)
T. DeGrand '01	$\simeq$ 0.11	8,10,12	0.0224(14)
P. Hasenfratz et al. '01	$\simeq$ 0.13	8,10	0.0180(15)(8)(17)

- The method is feasible, more studies are needed to reduce and properly assess the systematic errors

Comparing Results for p and ϵ expansion

- For $M_P L \geq 1,\ L \gg 1/(4\pi F)$ [p expansion]

$$\frac{M_P^2}{2m} = \frac{\Sigma}{F^2}\Big[1 - \frac{m_0^2}{3(4\pi F)^2}\left(1 + \log\Big(\frac{M^2}{\mu_\chi^2}\Big)\right)$$

$$+ \frac{\alpha M^2}{3(4\pi F)^2}\left(2\log\Big(\frac{M^2}{\mu_\chi^2}\Big) + 1\right) + \Big(2\alpha_8(\mu_\chi) - \alpha_5(\mu_\chi)\Big)\frac{M^2}{(4\pi F)^2}\Big]$$

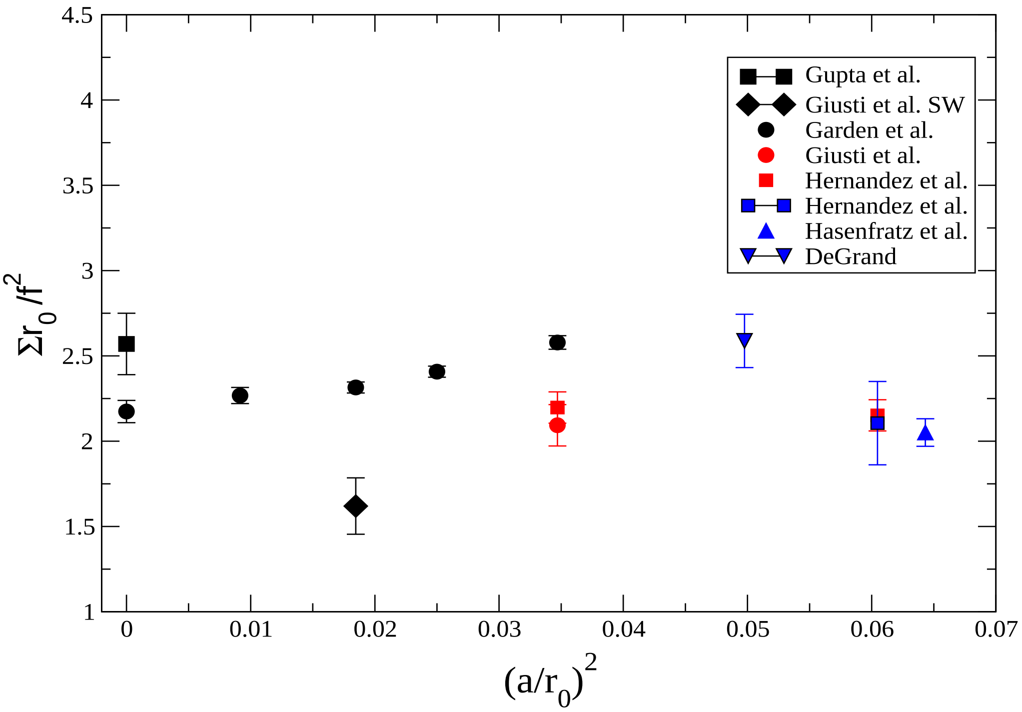
- For $F^2 M_P^2 L^4 \simeq 1,\ L \gg 1/(4\pi F)$ [ϵ expansion]

$$-\frac{1}{N_f}\langle\bar{\psi}\tilde{\psi}\rangle_\nu^{sub} = \Sigma_\nu(z)\Big[1 - \frac{m_0^2}{3(4\pi F)^2}\left(\tilde{\beta}_2 + \log\Big(\frac{L_0^2}{L^2}\Big)\right)$$

$$- \frac{\alpha}{3(4\pi F L)^2}\tilde{\beta}_1\Big]$$

- Higher-order corrections are expected to be different!

- Complementary ways to extract Σ, F, α_i, provided higher-order corrections are under control (Hernández et al. '01 and in preparation)



Topological Susceptibility in Full QCD ($m = 0$)

- For a given string of renormalized fundamental fields $\hat{O}$

$$\langle \partial^*_\mu A^0_\mu(x)\hat{O}\rangle = 2N_f \langle Q(x)\hat{O}\rangle + \langle \delta^x_A \hat{O}\rangle$$

$$2N_f \int d^4x\, \langle Q(x)\hat{O}\rangle + \langle \delta_A \hat{O}\rangle = 0$$

- As a consequence, properly renormalized operators are

$$\begin{aligned} \hat{Q}(x) &= \tfrac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)] - \tfrac{Z}{2N_f}\partial^*_\mu A^0_\mu(x) \\ \hat{A}^0_\mu(x) &= (1-Z)A^0_\mu(x) \end{aligned}$$

- The renormalized AWIs read

$$\langle \partial^*_\mu \hat{A}^0_\mu(x)\hat{O}\rangle = 2N_f \langle \hat{Q}(x)\hat{O}\rangle + \langle \delta^x_A \hat{O}\rangle$$

- Taking $\hat{O} = \hat{Q}$ and defining

$$\begin{aligned} \chi_t(p) &= \tfrac{1}{2N_f}\int d^4x\, \mathrm{e}^{-ipx}\langle \partial^*_\mu \hat{A}^0_\mu(x)\hat{Q}(0)\rangle + \mathsf{CT}(p) \\ &= \int d^4x\, \mathrm{e}^{-ipx}\langle \hat{Q}(x)\hat{Q}(0)\rangle + \mathsf{CT}(p) \end{aligned}$$

in the full theory ($\mathsf{CT}(0) = 0$)

$$\chi_t(0) = \int d^4x \langle \tfrac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)]\tfrac{1}{2}\mathrm{Tr}[\gamma_5 D(0,0)]\rangle = 0$$

(L. G., G.C. Rossi, M. Testa, G. Veneziano '01)

The Witten-Veneziano Formula

(E. Witten '79, G. Veneziano '79)

- A properly renormalized $\chi_t(p)$ satisfies
  (E. Seiler and I.O. Stamatescu '87, E. Seiler '02)

$$\chi_t(p) = b_1 + b_2 p^2 + b_3 (p^2)^2 - \frac{R^2_{\eta'}}{p^2 + m^2_{\eta'}} + (p^2)^3 \int_{\mathcal{M}^2}^{\infty} \frac{\rho(t)}{(t+p^2)t^3} dt$$

- For $p^2 \longrightarrow 0$, the "sum rule" $\chi_t(0) = 0$ $\left(R^2_{\eta'} = F^2_{\eta'} m^4_{\eta'}/2N_f\right)$

$$b_1 = \frac{R^2_{\eta'}}{m^2_{\eta'}}$$

- Under the "smooth-quenching hypothesis"

$$\frac{F_\pi^2 m^2_{\eta'}}{2N_f}\Big|_{\frac{N_f}{N_c}=0} = \int d^4x \left\langle \frac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)] \frac{1}{2}\mathrm{Tr}[\gamma_5 D(0,0)] \right\rangle\Big|_{\mathrm{YM}}$$

$$= \lim_{V\to\infty} \frac{\langle (n_{\mathrm{L}} - n_{\mathrm{R}})^2 \rangle}{V}$$

- With Wilson and staggered fermions it was argued
  (M. Bochicchio et al. '84, J. Smith and J.C. Vink '87)

$$\frac{F_\pi^2 m^2_{\eta'}}{2N_f}\Big|_{\frac{N_f}{N_c}=0} = \lim_{m\to 0} \left(\frac{2m}{2N_f}\right)^2 \int d^4x \langle P^0(x) P^0(0) \rangle \Big|^{\mathrm{ZV}}_{\mathrm{quenched}}$$

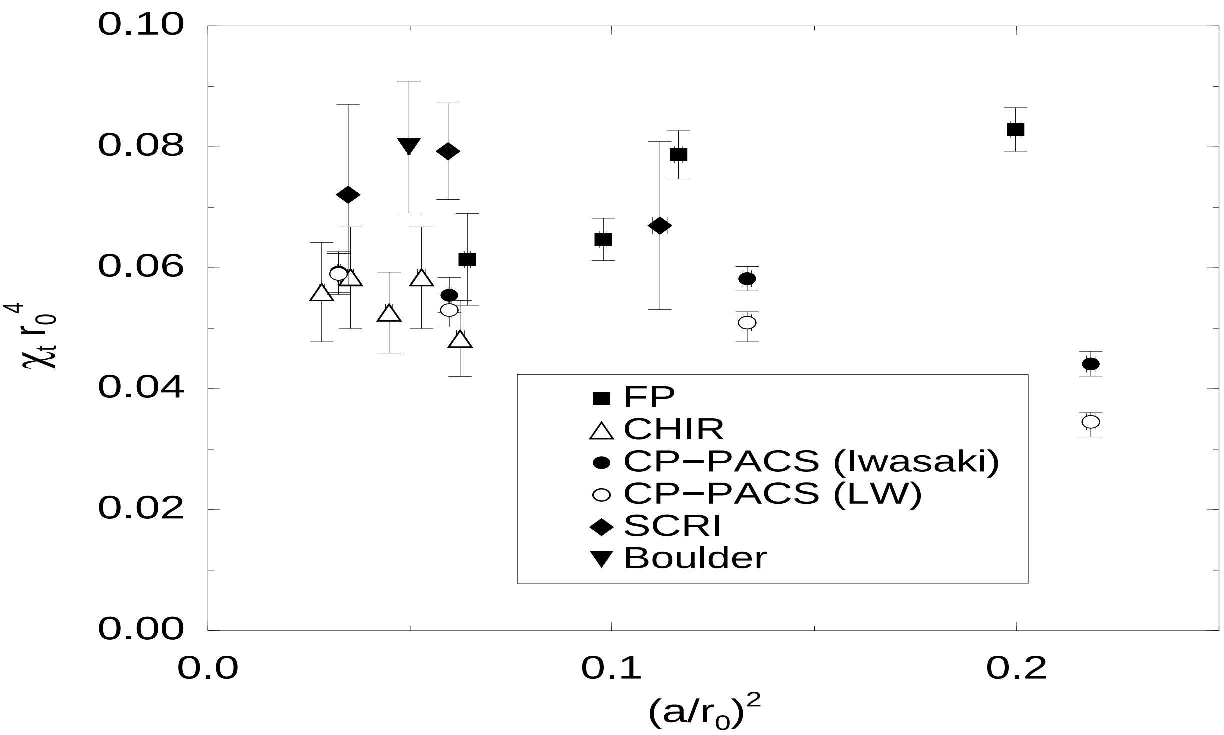
- With GW fermions obtained from the chirality of zero modes
  (S. Chandrasekharan'99, R.G. Edwards et al.'99, T. DeGrand et al. '02)

Numerical Results for the Topological Susceptibility

- From the chirality of the zero modes

$$\chi_t = \lim_{V\to\infty} \frac{\langle (n_{\rm L} - n_{\rm R})^2 \rangle}{V}$$

(Summary taken from P. Hasenfratz et al. '02)



- The WA from other determinations (M. Teper '99)

$$\chi_t \; r_0^4 = 0.067 \pm 0.009$$

gives $\chi_t^{1/4} \simeq 200 \pm 18$ MeV, to be compared with $\chi_t^{1/4} \sim 180$ MeV

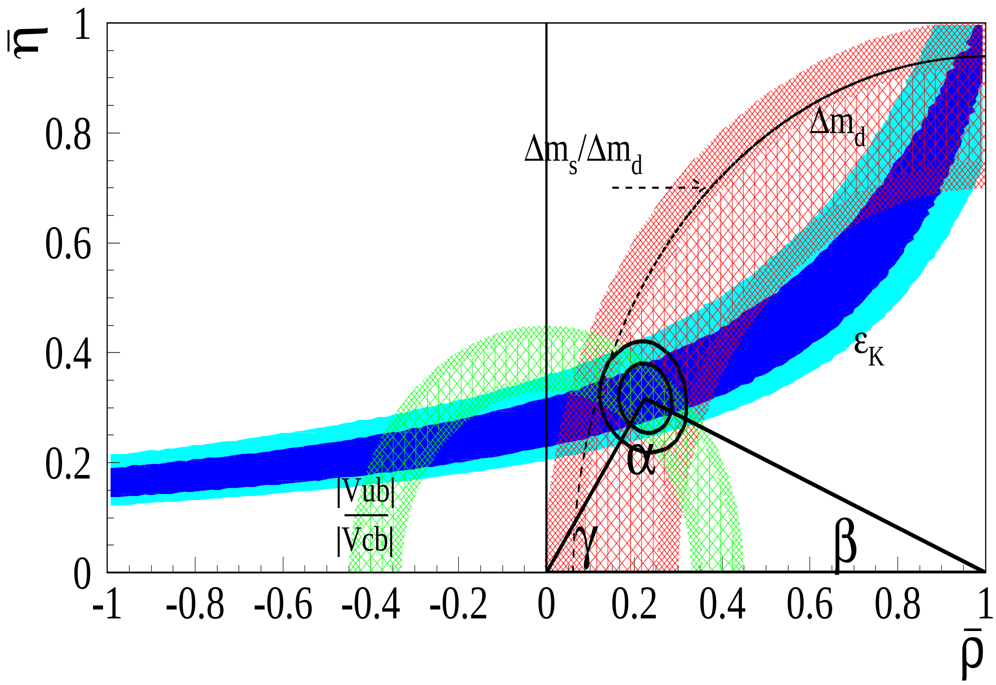
- Computations for $N_c > 3$ have been performed
  (B. Lucini M. Teper '01, L. Del Debbio et al. '02, N. Cundy et al. '02)

- Results compatible with smooth large-N_c limit and non-zero χ_t^∞

Unitarity Triangle Analysis

- Experimental input:

$$|\varepsilon_K^{\rm exp}| = (2.271 \pm 0.017) \times 10^{-3}$$
$$\Delta M_d^{\rm exp} = (0.489 \pm 0.008)\ \text{ps}^{-1}$$
$$\Delta M_s^{\rm exp} > 14.6\ \text{ps}^{-1} \quad 95\%\ \text{C.L.}$$



(Most recent analyses: M. Ciuchini et al.'01, H. Höcker et al.'01)

- Fit results:

$$\sin(2\,\beta) = 0.696 \pm 0.068$$
$$\sin(2\,\alpha) = -0.42 \pm 0.24$$
$$\gamma = (55.5 \pm 6.2)^\circ$$

- Latest experimental average:

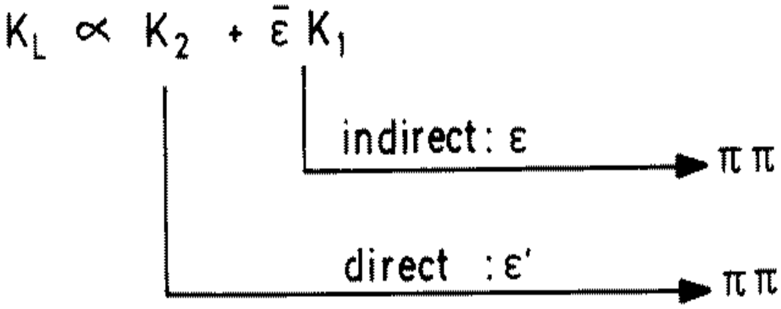
$$(\sin 2\beta)^{\rm exp} = 0.78 \pm 0.08$$

$K \to \pi\pi$ Decays

- $K \to \pi\pi$ ampl. can be parametrized $\left({}_I\langle\pi\pi|S|\pi\pi\rangle_I = e^{2i\delta_I} \right)$

$$
\begin{aligned}
A[K^+ \to \pi^+\pi^0] &= \sqrt{\tfrac{3}{2}} A_2 e^{i\delta_2} \\
A[K^0 \to \pi^+\pi^-] &= \sqrt{\tfrac{2}{3}} A_0 e^{i\delta_0} + \sqrt{\tfrac{1}{3}} A_2 e^{i\delta_2} \\
A[K^0 \to \pi^0\pi^0] &= \sqrt{\tfrac{2}{3}} A_0 e^{i\delta_0} - 2\sqrt{\tfrac{1}{3}} A_2 e^{i\delta_2}
\end{aligned}
$$

- CP violation implies $A_I \neq A_I^*$



and direct CP violation can be parametrised as

$$
\varepsilon' \simeq \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \frac{\mathrm{Re}A_2}{\mathrm{Re}A_0} \left(\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right)
$$

- From experiments:

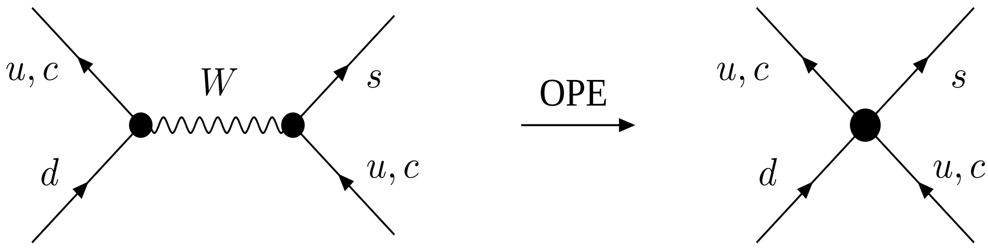
$$
\left| \frac{A_0}{A_2} \right|^{\mathrm{exp}} \simeq 22.2 \qquad \Delta I = 1/2 \text{ rule}
$$

$$
\mathrm{Re}(\epsilon'/\epsilon)^{\mathrm{exp}} = (17.3 \pm 1.7) \cdot 10^{-4}
$$

(see also talks by N. Ishizuka and E. de Rafael)

The $\Delta I = 1/2$ Rule with an Active Charm

- By using the Operator Product Expansion



$$iA_I e^{i\delta_I} = {}_I\langle \pi\pi \mid H_{\text{eff}}^{\Delta S=1} \mid K^0 \rangle$$

- The CP-conserving $\Delta S = 1$ effective Hamiltonian is

$$H_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left[C_+(\mu) \widehat{\mathcal{O}}_+(\mu) + C_-(\mu) \widehat{\mathcal{O}}_-(\mu) \right]$$

- The Wilson coefficients for the full $H_{\text{eff}}^{\Delta S=1}$ are known at NLO (see A.J. Buras et al. '92, M. Ciuchini et al. '94)

- A non-perturbative determination of ${}_I\langle \pi\pi | \widehat{\mathcal{O}}_\pm(\mu) | K^0 \rangle$ for the properly renormalized operators is needed

$$\mathcal{O}_\pm = \left[(\bar{s}^\alpha \gamma_\mu^L u^\beta)(\bar{u}^\beta \gamma_\mu^L d^\alpha) \pm (\bar{s}^\alpha \gamma_\mu^L u^\alpha)(\bar{u}^\beta \gamma_\mu^L d^\beta) \right] - (u \to c)$$

Renormalization Pattern for $\mathcal{O}_{\pm}$

- To select operators with $d \leq 6$:

  - Flavour symmetry
  - CPS (S:$d \leftrightarrow s$)
  - Chiral symmetry

- At a non-zero physical distance (on-shell) one operator is left (S. Capitani, L. G. '00; L. G. et al. in preparation)

$$\mathcal{Q}_m = (m_c^2 - m_u^2)\Big(m_d(\bar{s}P_R d) + m_s(\bar{s}P_L d)\Big)$$

- Taking into account the quadratic GIM mechanism, no power divergent subtractions are needed for GW

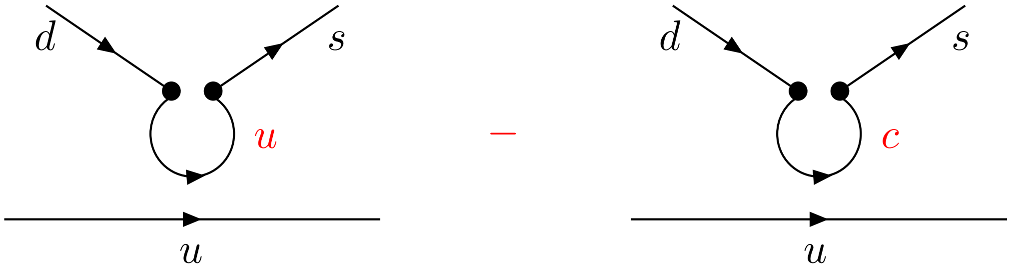
$$\boxed{\widehat{\mathcal{O}}_{\pm} = Z_{\pm}\Big[\mathcal{O}_{\pm} + C^m_{\pm}\mathcal{Q}_m\Big]}$$

- For $m_s \neq m_d$

$$\mathcal{Q}_m = \frac{1}{2}\partial_\mu\Big[\frac{m_d + m_s}{m_s - m_d}\bar{s}\gamma_\mu d - \frac{m_s - m_d}{m_s + m_d}\bar{s}\gamma_\mu\gamma_5 d\Big]$$

No contributions in MEs which preserve four-momentum

Active Charm with Wilson Fermions



- Parity-odd and parity-even components renormalize differently

- For parity conserving sector, using flavour and CPS symmetries (C. Bernard et al.'85, L. Maiani et al.'87, C. Dawson et al.'97)

$$
\begin{aligned}
\widehat{\mathcal{O}}_{\pm} &= Z_{\pm}\widetilde{\mathcal{O}}_{\pm} \\
\widetilde{\mathcal{O}}_{\pm} &= \mathcal{O}_{\pm} + \delta_6 \mathcal{O}_6 + (m_c - m_u) C_{\pm}^{\tau} \mathcal{Q}_{\tau} \\
&+ (m_c - m_u)\frac{C_{\pm}^{s}}{a^2}\mathcal{Q}_s
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{Q}_{\tau} &= \bar{s}\sigma_{\mu\nu}F_{\mu\nu}d \\
\mathcal{Q}_{s} &= \bar{s}d
\end{aligned}
$$

and $\delta_6 \mathcal{O}_6$ is a combination of dim-6 op. with wrong chirality

- With a broken chirality the GIM mechanism is only linear

The $H_{\text{eff}}^{\Delta S=1}$ with an Integrated Charm

- Either in the CP-conserving or the CP-violating case

$$H_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sum_{i=1}^{10} C_i(\mu) \widehat{\mathcal{Q}}_i(\mu)$$

where a basis for QCD-penguins operators is

$$\begin{aligned} \mathcal{Q}_{3,5} &= (\bar{s}\gamma_\mu^L d) \sum\nolimits_q (\bar{q}\gamma_\mu^{L,R} q) \\ \mathcal{Q}_{4,6} &= (\bar{s}^\alpha \gamma_\mu^L d^\beta) \sum\nolimits_q (\bar{q}^\beta \gamma_\mu^{L,R} q^\alpha) \end{aligned}$$

- At a non-zero physical distance two more operators can mix

$$\begin{aligned} \mathcal{Q}_m &= m_d(\bar{s}P_R d) + m_s(\bar{s}P_L d) \\ \mathcal{Q}_\sigma &= m_d(\bar{s}F_{\mu\nu}\sigma_{\mu\nu}P_R d) + m_s(\bar{s}F_{\mu\nu}\sigma_{\mu\nu}P_L d) \end{aligned}$$

- Without the GIM mechanism the mixing is power-divergent

$$\boxed{\widehat{\mathcal{Q}}_i = \widehat{Z}_{ij}\Big[\mathcal{Q}_j + C_j^\sigma \mathcal{Q}_\sigma + \frac{C_j^m}{a^2}\mathcal{Q}_m\Big]}$$

K$\to \pi\pi$ in Chiral Perturbation Theory

(C. Bernard et al. '85)

- At leading order

$$
\begin{aligned}
f_\pi^3 \langle \pi^+\pi^- | \mathcal{O}^{(8,1)} | K^0 \rangle &= i\alpha_1^{(8,1)} (M_K^2 - M_\pi^2) \\
f_\pi^2 \langle \pi^+(q) | \mathcal{O}^{(8,1)} | K^+(k) \rangle &= \alpha_1^{(8,1)} (q \cdot k) - \alpha_2^{(8,1)} M_K^2 \\
f_\pi \langle 0 | \mathcal{O}^{(8,1)} | K^0 \rangle &= i\alpha_2^{(8,1)} (M_K^2 - M_\pi^2)
\end{aligned}
$$

- First numerical results with DW fermions and the charm integrated out (RBC Coll. '01, CP-PACS Coll. '01, see N. Ishizuka talk)

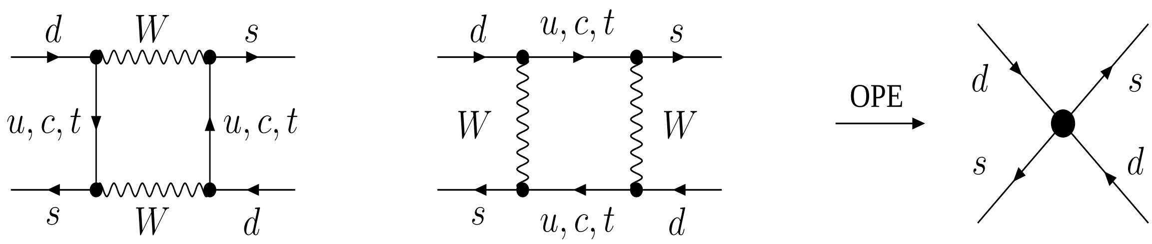
Weak Interactions in the ϵ regime

(L. G., P. Hernández, C. Hoelbling, K. Jansen, M. Laine, L. Lellouch, M. Lüscher, P. Weisz, H. Wittig in preparation)

- It is conceivable to extract the LECs of the weak chiral Lagrangian by studying the weak interactions in the ϵ regime

- A numerical study on the lattice is under way

K^0-$\bar{K}^0$ Mixing in the SM



$$\varepsilon_K \equiv \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]} \simeq \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \frac{\mathrm{Im} M_{12}}{\Delta M_K}$$

where

$$2m_K M_{12}^* = \langle \bar{K}^0 | H_{\text{eff}}^{\Delta S=2} | K^0 \rangle$$

- In the Standard Model ($\lambda_i = V_{is}^* V_{id}$)

$$H_{\text{eff}}^{\Delta S=2} = \frac{G_F^2 M_W^2}{16\pi^2} \left[\lambda_c^2 \eta_1 S_0^c + \lambda_t^2 \eta_2 S_0^t + 2\lambda_c \lambda_t \eta_3 S_0^{ct} \right] \widehat{\mathcal{O}}^{\Delta S=2} + h.c.$$

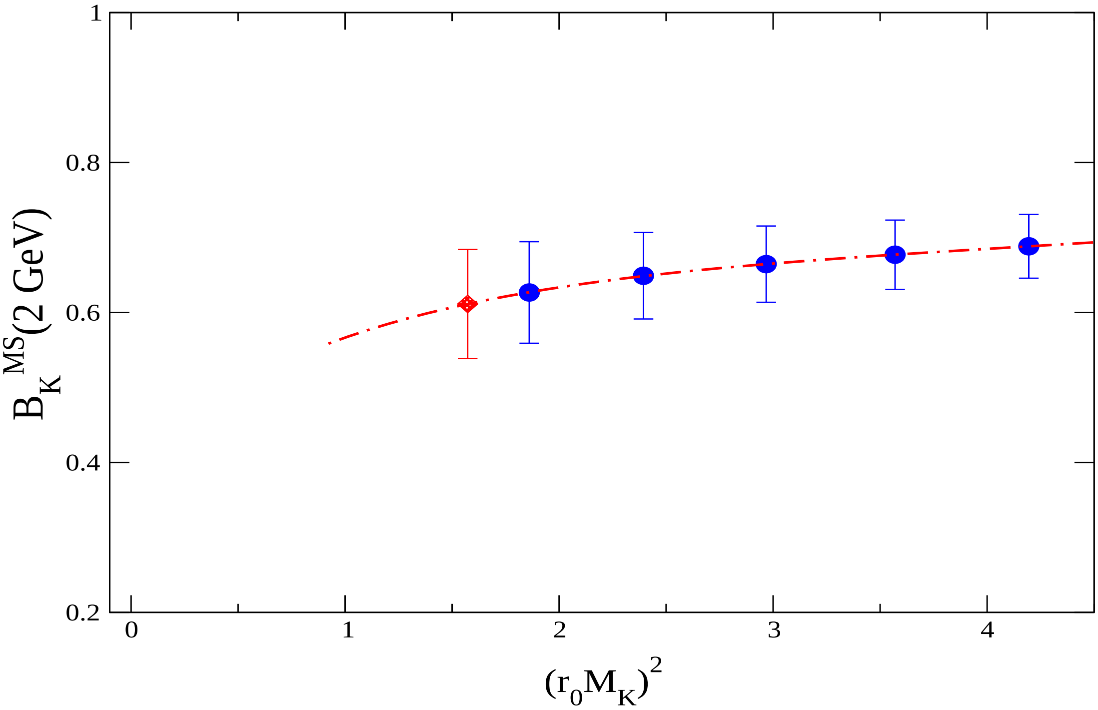
- The QCD corrections η_1, η_2, η_3 are known at NLO (S. Herrlich and U. Nierste '94 '96)

- The long-distance QCD effects are parametrized as

$$\langle \bar{K}^0 | \widehat{\mathcal{O}}^{\Delta S=2} | K^0 \rangle = \hat{Z} \langle \bar{K}^0 | 4(\bar{s}\gamma_\mu^L d)(\bar{s}\gamma_\mu^L d) | K^0 \rangle = \frac{8}{3} f_K^2 m_K^2 \hat{B}_K$$

B_K from Overlap Fermions

(N. Garron, L.G., C. Hoelbling, L. Lellouch, C. Rebbi)



- Lattice: $V = 16^3 \times 32$, $\beta = 6.0$, $m_s/2 \lesssim m \lesssim m_s$
- Renormalization: Non-Perturbative RI/MOM
- Functional form to reach the physical point

$$B_K^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = B_0\left(1 - 3\frac{M_K^2}{(8\pi^2 f_K^2)} \log\left(\frac{M_K^2}{\Lambda_\chi^2}\right) + C\frac{M_K^4}{(8\pi^2 f_K^2)^2}\right)$$

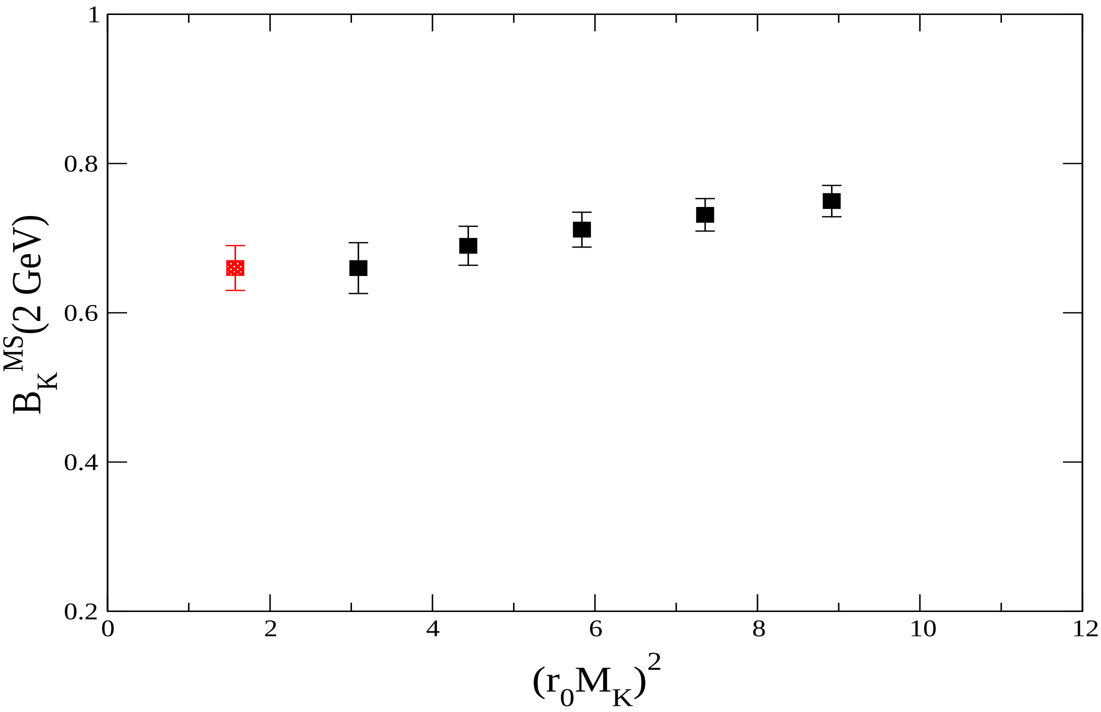
- Including the statistical error only

$$B_K^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.61 \pm 0.07 \qquad \text{Preliminary !}$$

- More work needed to properly assess the systematic error

B_K from NNC-HYP Overlap Fermions

(T. DeGrand)



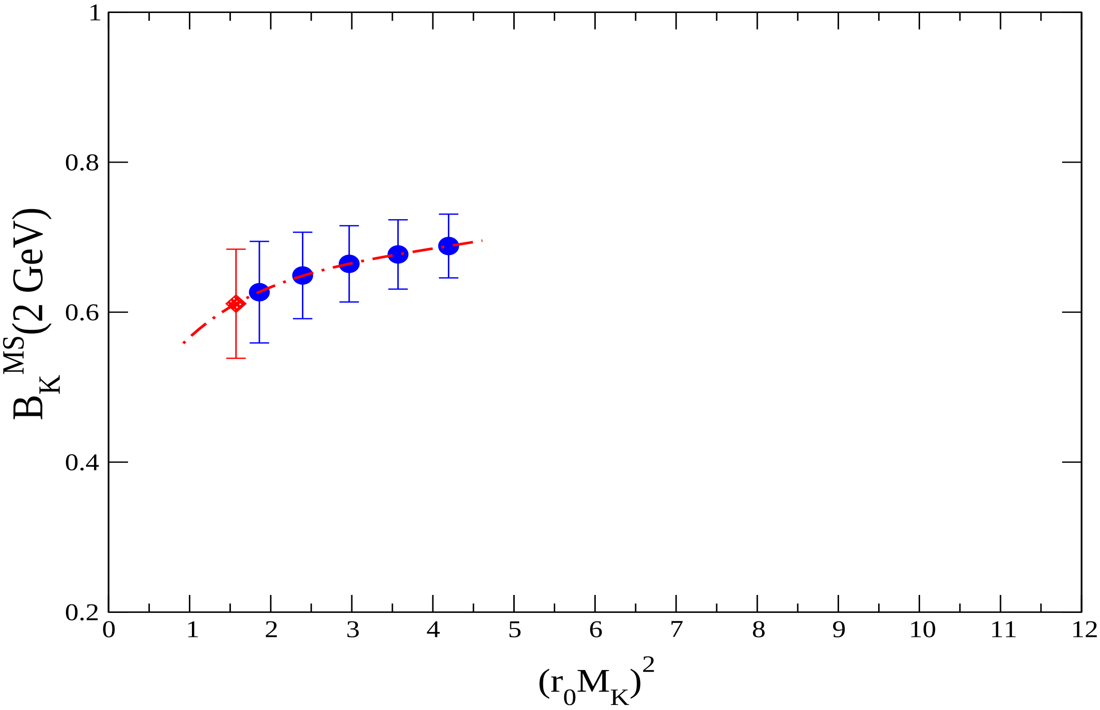
- Lattice: $V = 12^3 \times 36$, $\beta = 5.9$, $m_s \lesssim m \lesssim 2.5 m_s$
- Renormalization: Perturbative
- Functional form to reach the physical point

$$B_K^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = B_0 + D\,M_K^2$$

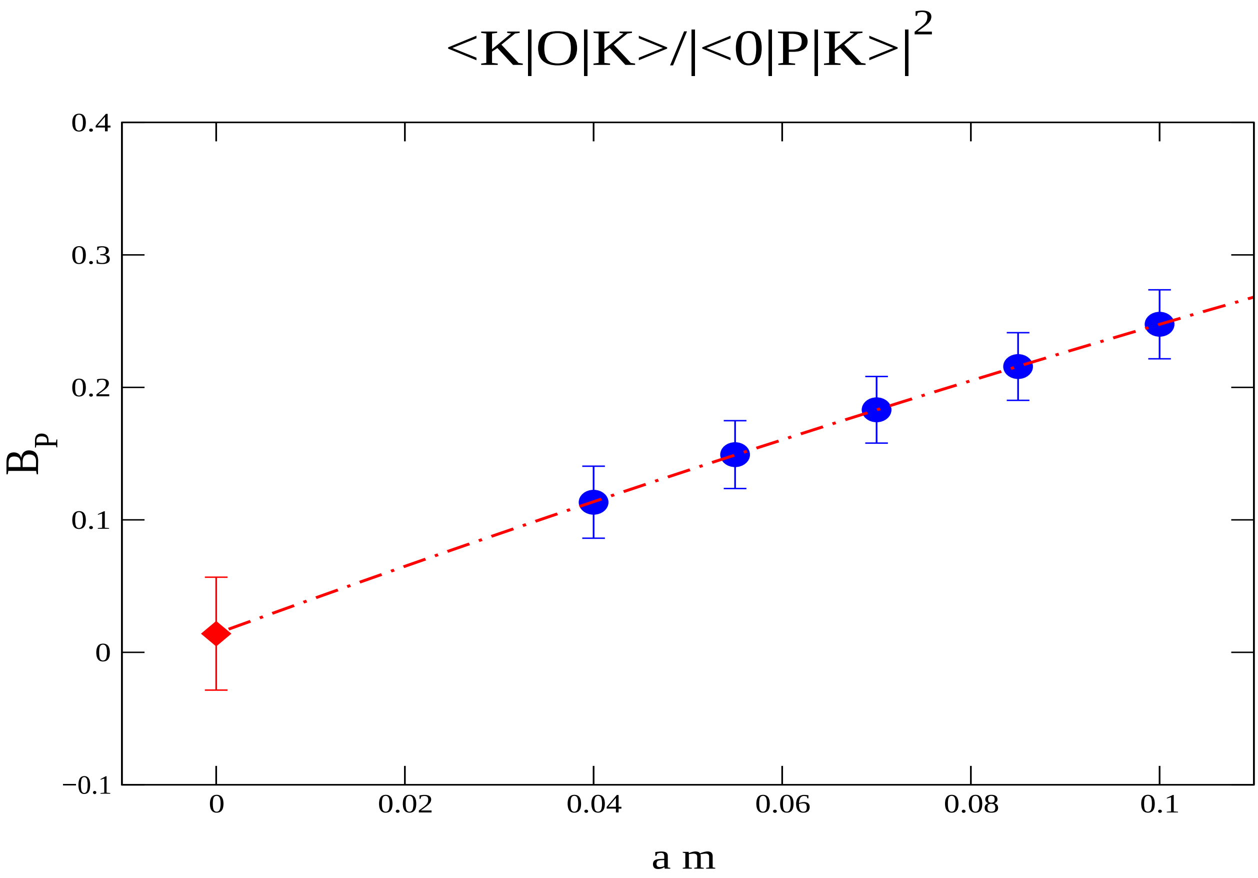
- Including the statistical error only

$$B_K^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.66 \pm 0.04 \qquad \text{Preliminary !}$$

- More to fix the systematic error

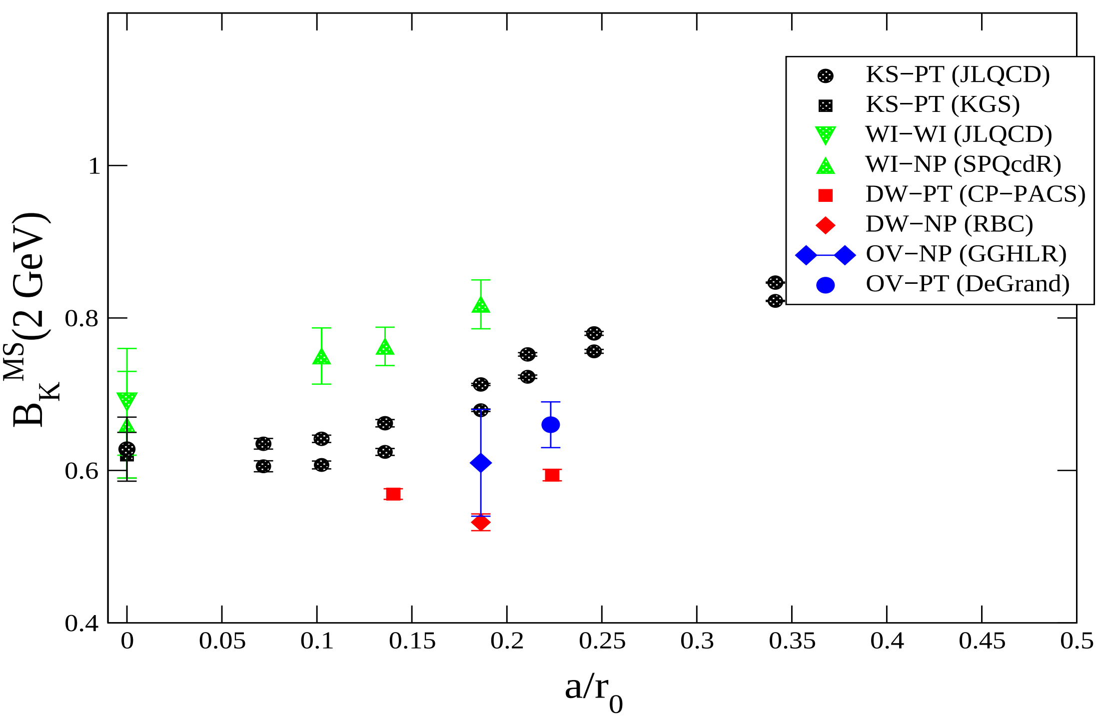


1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10
11
12
$B_K^{MS}(2\ \mathrm{GeV})$
$(r_0 M_K)^2$



<K|O|K>/|<0|P|K>|^2
0.4
0.3
0.2
0.1
0
−0.1
B_P
0
0.02
0.04
0.06
0.08
0.1
a m

B_K Summary



KS−PT (JLQCD)
KS−PT (KGS)
WI−WI (JLQCD)
WI−NP (SPQcdR)
DW−PT (CP−PACS)
DW−NP (RBC)
OV−NP (GGHLR)
OV−PT (DeGrand)
B_K^{MS}(2 GeV)
a/r_0

Conclusions

- Exact chiral symmetry on the lattice at finite cut-off
- Domain-Wall-Overlap: explicit chirally symmetric regularization
- Quenched large scale numerical simulations are feasible
  Regime of quark masses not reachable with Wilson fermions
- First phenomenological computations performed
  Results indicate small discretization errors
- QCD results in the ϵ regime are very encouraging
- Weak matrix elements:
  - First results obtained (B_K, $K \to \pi$)
  - No power divergent subtractions for the $\Delta I = 1/2$ rule
  - A study in the ϵ regime started