Solving the $\mathrm{U}(1)_A$ problem in QCD with a non-perturbative implementation of the Witten-Veneziano mechanism

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Outline

- Introduction to the $\mathrm{U}(1)_\mathrm{A}$ problem
- The Witten–Veneziano mechanism

- Chiral anomaly with Ginsparg–Wilson fermions
- Non-perturbative definition of the topological susceptibility
- Witten–Veneziano formula

- Algorithm for zero-mode counting
- Lattice computation and analysis of systematics

- Results and conclusions

- Octet compatible with SSB pattern

$$SU(3)_{\rm L} \times SU(3)_{\rm R} \to SU(3)_{\rm L+R}$$

and soft explicit symmetry breaking

$$m_u, m_d \ll m_s < \Lambda_{\rm QCD}$$

- $m_u, m_d \ll m_s \Longrightarrow m_\pi \ll m_{\rm K}$

- A $9^{\rm th}$ pseudoscalar with $m_{\eta'} \sim \mathcal{O}(\Lambda_{\rm QCD})$

I	I_3	S	Meson	Quark Content	Mass (MeV)
1	1	0	π^+	$u\bar{d}$	140
1	-1	0	π^-	$d\bar{u}$	140
1	0	0	π^0	$(d\bar{d} - u\bar{u})/\sqrt{2}$	135
$\frac{1}{2}$	$\frac{1}{2}$	+1	K^+	$u\bar{s}$	494
$\frac{1}{2}$	-$\frac{1}{2}$	+1	K^0	$d\bar{s}$	498
$\frac{1}{2}$	-$\frac{1}{2}$	-1	K^-	$s\bar{u}$	494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\overline{\mathrm{K}}^0$	$s\bar{d}$	498
0	0	0	η	$\cos\vartheta\eta_8 + \sin\vartheta\eta_0$	547
0	0	0	η'	$-\sin\vartheta\eta_0 + \cos\vartheta\eta_8$	958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\vartheta \simeq -11^\circ$$

The $\mathrm{U}(1)_{\mathrm{A}}$ problem in QCD

- The (Euclidean) QCD Lagrangian is

$$S_{\mathrm{QCD}} = S^{\theta}_{\mathrm{YM}} + \int \{\bar\psi_{\mathrm{L}}\gamma_\mu D_\mu \psi_{\mathrm{L}} + \bar\psi_{\mathrm{R}}\gamma_\mu D_\mu \psi_{\mathrm{R}} + \bar\psi_{\mathrm{L}} M^\dagger \psi_{\mathrm{R}} + \bar\psi_{\mathrm{R}} M \psi_{\mathrm{L}}\}$$

- For $N_{\mathrm{f}} = 3$ and $M = 0$ the action is invariant under the group $U(3)_{\mathrm{L}} \times U(3)_{\mathrm{R}}$

$$\begin{array}{ll} \psi_L \to V_L \psi_L & \qquad \bar\psi_L \to \bar\psi_L V_L^\dagger \\ \psi_R \to V_R \psi_R & \qquad \bar\psi_R \to \bar\psi_R V_R^\dagger \end{array}$$

- The η' quantum numbers compatible with $U(1)_{\mathrm{A}}$ but no parity partner in Nature

- If we assume a SSB pattern $U(3)_{\mathrm{L}} \times U(3)_{\mathrm{R}} \to U(3)_{\mathrm{R+L}}$, a 9^{th} pseudo GB in QCD

$$m_{\eta'} < \sqrt{3} m_\pi$$

[Weinberg 75]

The Adler–Bell–Jackiw $U(1)_A$ anomaly does play a rôle

- In the chiral limit the AWIs read

$$\langle \partial_\mu A^a_\mu(x) P^a(0)\rangle = -\delta(x)\frac{\langle\bar\psi\psi\rangle}{N_f}$$

$$\langle \partial_\mu A^0_\mu(x) P^0(0)\rangle = 2\frac{N_f}{N_c}N_c\langle Q(x)P^0(0)\rangle - 2\delta(x)\langle\bar\psi\psi\rangle$$

with

$$Q(x) = -\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,[F_{\mu\nu}(x)F_{\rho\sigma}(x)]$$

- Despite of $Q(x) = \partial_\mu K_\mu(x)$, for topologically non-trivial configurations [Belavin et al. 75]

$$\int d^4x\, Q(x) = Q \neq 0$$

- A semiclassical approximation shows that instantons of radius R contribute $\propto \exp\{-8\pi^2/g^2(1/R)\}$ ['t Hooft 76, Jackiw Rebbi 76, Callan Dashen Gross 76]

- No IR bound on the radius of instantons: a non-perturbative approach needed

- By defining (formally) the topological susceptibility as

$$\chi(p) = \int d^4x\, \mathrm{e}^{-ipx} \langle Q(x)Q(0)\rangle$$

in the chiral limit the AWIs guarantee

$$\lim_{p\to 0} \chi(p) = 0$$

- By expanding in number of fermion loops (i.e. in $N_\mathrm{f}/N_\mathrm{c}$)

$$\chi(p) = \chi_0(p) + \chi_1(p) + \chi_2(p) + \dots$$

- Non-perturbatively $\chi_0(0) \neq 0$ is natural even if it vanishes order by order in PT
- At leading order in $1/N_\mathrm{c}$ by saturating the corr. with one-hadron intermediate states

$$\chi_0(0) = \lim_{p^2\to 0} \left[\mathcal{P}_0(p^2) - \sum_{\mathrm{glue}} \frac{R_{0\mathrm{g}}^2}{p^2 + m_{0\mathrm{g}}^2} \right]$$

The Witten–Veneziano mechanism [Witten 79, Veneziano 79]

- How is it possible that terms sub-leading in $N_\mathrm{f}/N_\mathrm{c}$ cancel the leading one ?
- Again at leading order in $1/N_\mathrm{c}$

$$\chi(p) = \mathcal{P}(p^2) - \sum_{\mathrm{glue}} \frac{R_\mathrm{g}^2}{p^2 + m_\mathrm{g}^2} - \frac{R_{\eta'}^2}{p^2 + m_{\eta'}^2} - \sum_{\mathrm{mes}} \frac{R_\mathrm{m}^2}{p^2 + m_\mathrm{m}^2} \cdots$$

where $R_\mathrm{g}^2 = \mathcal{O}(1)$, $R_{\eta'}^2 = \mathcal{O}(N_\mathrm{f}/N_\mathrm{c})$ and $R_\mathrm{m}^2 = \mathcal{O}(N_\mathrm{f}/N_\mathrm{c})$

- For $p^2 \to 0$ the fermionic contribution can be of $\mathcal{O}(1)$ if also $m_{\eta'}^2 = O(N_\mathrm{f}/N_\mathrm{c})$

$$\left.\frac{F_\pi^2 m_{\eta'}^2}{2N_\mathrm{f}}\right|_{\frac{N_\mathrm{f}}{N_\mathrm{c}}=0}^{M=0} = \chi_0(0) = \chi^{\mathrm{YM}}$$

- Note that in the limit $N_\mathrm{f}/N_\mathrm{c} \to 0$:
  1. $\mathrm{U}(1)_\mathrm{A}$ is restored
  2. η' is a Nambu–Goldstone boson $\Longrightarrow m_{\eta'} = 0$
  3. At first order in $N_\mathrm{f}/N_\mathrm{c}$, $m_{\eta'}^2 = \mathcal{O}(N_\mathrm{f}/N_\mathrm{c})$

The WV formula from dispersion relation [L.G., Rossi, Testa, Veneziano 01]

- The Euclidean $\chi(p)$ satisfies a three-times subtracted dispersion relation

$$\chi(p) = b_1 + b_2 p^2 + b_3 (p^2)^2 - \frac{R^2_{\eta'}}{p^2 + m^2_{\eta'}} + (p^2)^3 \int_{\mathcal{M}^2}^{\infty} \frac{\rho(t)}{(t+p^2)t^3} dt$$

- For $p^2 \to 0$, the “sum rule” $\chi(0) = 0$ implies

$$b_1 = \frac{R^2_{\eta'}}{m^2_{\eta'}} \qquad \text{with} \qquad R^2_{\eta'} = \frac{F^2_{\eta'} m^4_{\eta'}}{2N_\mathrm{f}}$$

- Under the “smooth-quenching hypothesis”, if we take $N_\mathrm{f}/N_\mathrm{c} \to 0$ and then $p^2 \to 0$

$$\left.\frac{F^2_\pi m^2_{\eta'}}{2N_\mathrm{f}}\right|^{M=0}_{\frac{N_\mathrm{f}}{N_\mathrm{c}}=0} = \chi^{\mathrm{YM}}$$

Challenges for implementing the formula beyond the formal level

- E. Witten 79: “We cannot ask whether the formula is correct, because it involves $\chi^{\rm YM}$, which we can neither measure nor calculate”

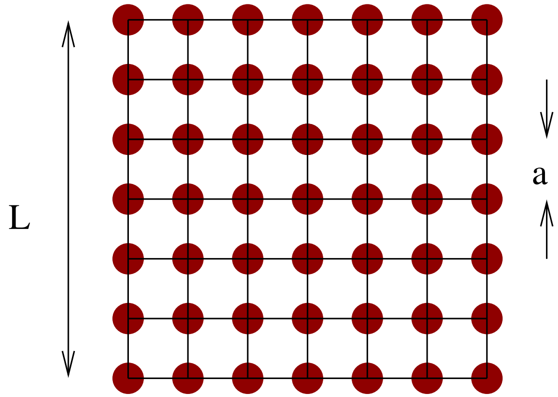
- We need a non-perturbative regularization of QCD

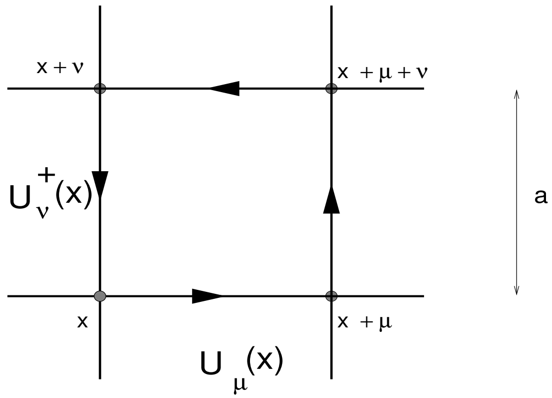
- We need an unambiguous implementation of the WV formula (if any) at the non-perturbative level. Naively with an UV cut-off Λ

$$\chi(0) = \int d^4x \, \langle Q(x) Q(0) \rangle \propto \Lambda^4$$

- We have to be able to compute non-perturbatively $\chi^{\rm YM}$

Lattice regularization





- The lattice Wilson action for the Yang–Mills theory is

$$S_{\rm YM} = \frac{2N_{\rm c}}{g^2} \sum_{x,\mu<\nu} \left\{ 1 - \frac{1}{2N_{\rm c}} {\rm Tr} \left[U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x) \right] \right\}$$

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\mu) U^\dagger_\mu(x+\nu) U^\dagger_\nu(x)$$

- For small fields (PT) $U_\mu(x) = 1 - agA_\mu(x) + \ldots$
- Correlation functions computed non-perturbatively via Monte Carlo techniques

$$\langle O_1(x) O_2(0) \rangle = \int \mathcal{D}U \, e^{-S_{\rm YM}(U)} O_1(U;x) O_2(U;0)$$

- If fermion action is invariant under standard chiral symmetry $\Longrightarrow$ no chiral anomaly
- Wilson solution: chirality broken explicitly

$$D_W = \tfrac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \tfrac{a}{2}\nabla^*_\mu \nabla_\mu$$

- At the classical level, chiral symmetry naively restored for $a \to 0$
- . . . but at the quantum level chiral symmetry is broken

$$\frac{1}{a}\mathcal{O}(a) \simeq \mathcal{O}(1)$$

- A problem for more than two decades: with a naive lattice implementation of $Q(x)$

$$\chi(0) = \sum_x \langle Q(x)Q(0)\rangle \propto \frac{1}{a^4}$$

Ginsparg–Wilson fermions

- The "mildest way" of breaking standard chiral symmetry [Ginsparg Wilson 82]

$$\gamma_5 D + D\gamma_5 = \bar{a} D \gamma_5 D$$

- An exact symmetry at finite cut-off implied [Lüscher 98]

$$\delta q = \epsilon \hat{\gamma}_5 q \quad \delta \bar{q} = \epsilon \bar{q} \gamma_5 \qquad \hat{\gamma}_5 = \gamma_5 (1 - \bar{a} D)$$

- $U(1)_{\rm A}$ anomaly from the Jacobian

$$J = \exp\{\epsilon \bar{a} \sum_x \text{Tr}\,[\gamma_5 D(x,x)]\}$$

- The topological charge density defined as [Neuberger 97, Hasenfratz et al. 98, Lüscher 98]

$$a^4 Q(x) = -\frac{\bar{a}}{2} \text{Tr}[\gamma_5 D(x,x)] \qquad n_+ - n_- = \text{index}(D) = \sum_x Q(x)$$

and for smooth gauge configurations $Q(x) \rightarrow -\frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}\,[F_{\mu\nu}(x) F_{\rho\sigma}(x)]$

- After 15 years from the GW relation, a Dirac operator that satisfies the GW relation, is local and leads to the correct continuum limit was found [Neuberger 97]

$$D = \frac{1}{\bar{a}}\left(1 + \frac{X}{\sqrt{X^\dagger X}}\right)$$

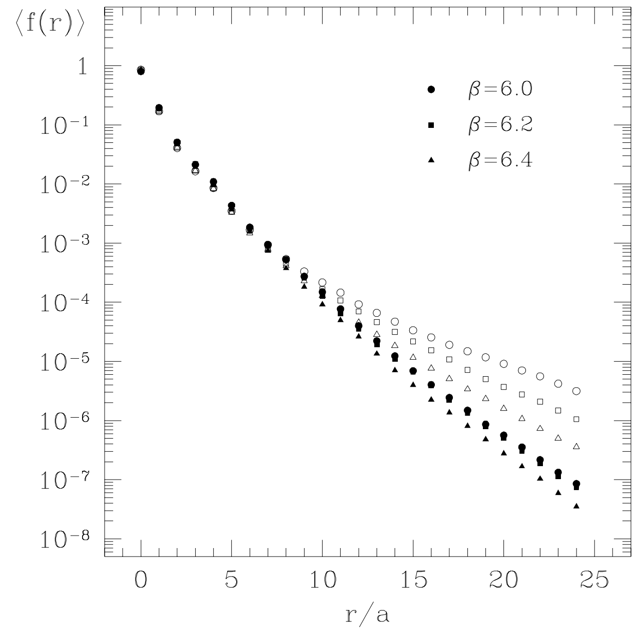
with

$$X = D_W - 1/\bar{a} \qquad \bar{a} = a/(1+s)$$

- A family of GW regularizations for $0 < s < 2$

- Numerical treatment challenging

Hernández, Jansen, Lüscher 99



$$f(r) = \max\left\{ \left|\left| \frac{X}{\sqrt{X^\dagger X}}(x,y) \right|\right| \;\middle|\; ||x-y|| = r \right\}$$

$$\langle f(r) \rangle \propto e^{-\mu r/a} \qquad r/a \gg 1$$

QCD fermion action with GW fermions

- QCD massive action defined as

$$S_{\rm F} = \sum_x \bar\psi(x) \left[(D + P_+ M^\dagger \hat P_+ + P_- M \hat P_-)\psi \right](x)$$

- If we define

$$\hat P_\pm = \frac{1}{2}(1 \pm \hat\gamma_5) \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

and

$$\psi_{\rm R,L} = \hat P_\pm \psi \qquad \bar\psi_{\rm R,L} = \bar\psi P_\mp$$

the $U(N_{\rm f})_{\rm L} \times U(N_{\rm f})_{\rm R}$ transformations are defined as

$$\begin{array}{ll} \psi_{\rm L} \to V_{\rm L}\psi_{\rm L} & \bar\psi_{\rm L} \to \bar\psi_{\rm L} V_{\rm L}^\dagger \\ \psi_{\rm R} \to V_{\rm R}\psi_{\rm R} & \bar\psi_{\rm R} \to \bar\psi_{\rm R} V_{\rm R}^\dagger \end{array}$$

- The action is invariant if also

$$M \to V_{\rm R} M V_{\rm L}^\dagger$$

- Exact chiral symmetry $\Rightarrow$ no operators of $d = 5$ in $S_{\rm F}$ $\Rightarrow$ cut-off effects $O(a^2)$

- No additive quark renormalization

- Bilinears with correct chiral properties

$$O^{\Gamma}_{\alpha\beta}(x) = \bar\psi_\alpha(x)\Gamma\tilde\psi_\beta(x) \qquad \tilde\psi_\beta(x) = \Big[(1 - \tfrac{\bar a}{2} D)\psi_\beta\Big](x)$$

- Apparently no simple transformation under CP. In corr. at non-zero physical distance

$$O^{\Gamma}_{\alpha\beta}(x) = \frac{1}{(1 - \frac{\bar a}{2} m_\beta)} \bar\psi_\alpha(x)\Gamma\psi_\beta(x)$$

and therefore under CP

$$O^{\Gamma}_{\alpha\beta}(x) \xrightarrow{\rm CP} \frac{1 - \frac{\bar a}{2} m_\alpha}{1 - \frac{\bar a}{2} m_\beta} O^{\Gamma}_{\beta\alpha}(\tilde x)$$

- In the chiral limit for a given string of renormalized fundamental fields $\hat{O}$

$$\langle \partial^*_\mu A^0_\mu(x) \hat{O} \rangle = 2N_\mathrm{f} \langle Q(x) \hat{O} \rangle - \langle \delta^x_A \hat{O} \rangle$$

- Neither $Q(x)$ or $A^0_\mu(x)$ are finite operators:
  1. $\delta^x_A \hat{O}$ is finite
  2. $A^0_\mu(x)$ is multiplicatively renormalizable
  3. $Q(x)$ can mix with $d \leq 4$ operators $\Longrightarrow$ only with $\partial^*_\mu A^0_\mu(x)$

$$\hat{Q}(x) = Q(x) - \frac{Z}{2N_\mathrm{f}} \partial^*_\mu A^0_\mu(x) \qquad \hat{A}^0_\mu(x) = (1-Z) A^0_\mu(x)$$

Singlet axial Ward identities in the chiral limit

- In the chiral limit for a given string of renormalized fundamental fields $\hat{O}$

$$\langle \partial^*_\mu A^0_\mu(x)\hat{O}\rangle = 2N_\mathrm{f}\langle Q(x)\hat{O}\rangle - \langle \delta^x_A \hat{O}\rangle$$

- Distinctive features of GW fermions [L. G., Rossi, Testa, Veneziano 01]

  1. No further multiplicative renormalization required for $\hat{Q}(x)$
  2. No mixing of $Q(x)$ with $P^0(x)$
  3. No extra contact terms in the AWIs

$$\hat{Q}(x) = Q(x) - \frac{Z}{2N_\mathrm{f}}\partial^*_\mu A^0_\mu(x) \qquad \hat{A}^0_\mu(x) = (1-Z)A^0_\mu(x)$$

Topological susceptibility with GW fermions in the chiral limit [L. G., Rossi, Testa, Veneziano 01]

- The renormalized AWIs read

$$\langle \partial^*_\mu \hat{A}^0_\mu(x)\hat{O}\rangle = 2N_{\rm f}\langle \hat{Q}(x)\hat{O}\rangle - \langle \delta^x_A \hat{O}\rangle$$

- Taking $\hat{O} = \hat{Q}$

$$\begin{aligned}\chi(p) &= \sum_x \mathrm{e}^{-ipx}\langle \hat{Q}(x)\hat{Q}(0)\rangle + \mathrm{CT}(p)\\ &= \frac{1}{2N_{\rm f}}\sum_x \mathrm{e}^{-ipx}\langle \partial^*_\mu \hat{A}^0_\mu(x)\hat{Q}(0)\rangle + \mathrm{CT}(p)\end{aligned}$$

in the full theory the requirement is

$$\chi(0) = 0$$

- $\mathrm{CT}(p)$ is a $2^{\rm nd}$ order polynomial in p^2 (counter-terms) to make the int. corr. fnc. finite
- For GW fermions, the AWI $\Longrightarrow$ $\mathrm{CT}(0) = 0$

Topological susceptibility with massive fermions

- With massive Ginsparg-Wilson fermions

$$
\begin{array}{ll}
N_{\rm f}=2 & \chi(0)=m_1 m_2 \sum_{x_1} \langle P_{11}(x_1) P_{22}(0)\rangle \\
\dots & \dots\dots \\
\dots & \dots\dots \\
N_{\rm f}=5 & \chi(0)=m_1 \dots m_5 \sum_{x_1,\dots,x_4} \langle P_{31}(x_1) S_{12}(x_2) S_{23}(x_3) P_{54}(x_4) S_{45}(0)\rangle
\end{array}
$$

- No UV divergences for $\chi(0)$ with massive quarks [L. G., Rossi, Testa 04; Lüscher 04]
- A definition of $\chi(0)$ even if the regularization breaks chiral symmetry [Lüscher 04]
- $N_{\rm f}/N_{\rm c} \to 0$ without going to the chiral limit $\Longrightarrow$ No UV surprises for $\chi^{\rm YM}$
- For a precise argumentation with pseudofermions see Lüscher 04

Witten–Veneziano formula with GW fermions [L. G., Rossi, Testa, Veneziano 01]

- In the Euclidean $\chi(p)$ satisfies

$$\chi(p) = b_1 + b_2 p^2 + b_3 (p^2)^2 - \frac{R^2_{\eta'}}{p^2 + m^2_{\eta'}} + (p^2)^3 \int_{\mathcal{M}^2}^{\infty} \frac{\rho(t)}{(t+p^2)t^3} dt$$

- For $p^2 \to 0$, the "sum rule" $\chi(0) = 0$ implies

$$b_1 = \frac{R^2_{\eta'}}{m^2_{\eta'}} \qquad \text{with} \qquad R^2_{\eta'} = \frac{F^2_{\eta'} m^4_{\eta'}}{2N_\mathrm{f}}$$

- If we take $N_\mathrm{f}/N_\mathrm{c} \to 0$ and then $p^2 \to 0$, under the "smooth-quenching hypothesis"

$$b_1 = \chi^{\mathrm{YM}} \qquad \Longrightarrow \qquad \left.\frac{F^2_\pi m^2_{\eta'}}{2N_\mathrm{f}}\right|_{\frac{N_\mathrm{f}}{N_\mathrm{c}}=0} = \chi^{\mathrm{YM}}$$

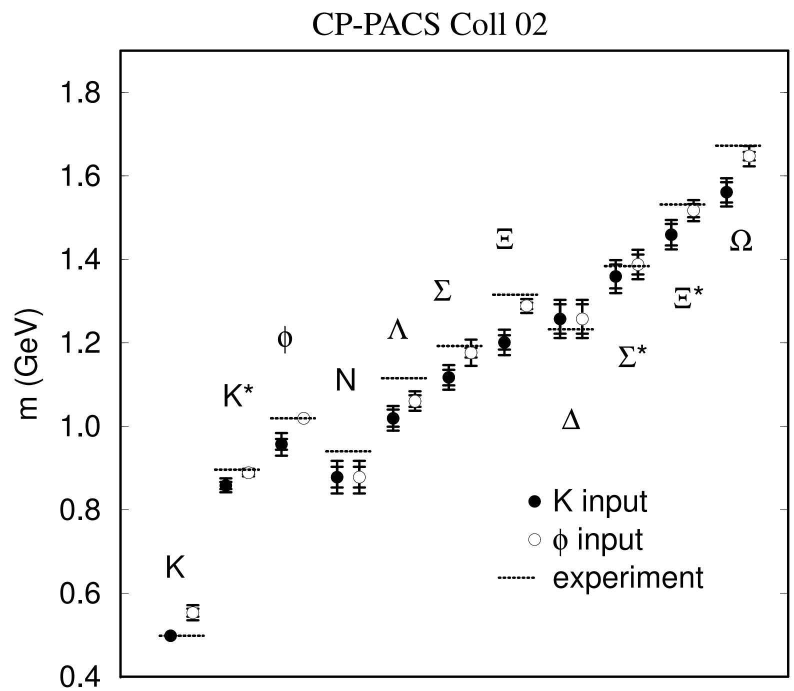
- For Ginsparg–Wilson fermions

$$\left.\frac{F^2_\pi m^2_{\eta'}}{2N_\mathrm{f}}\right|_{\frac{N_\mathrm{f}}{N_\mathrm{c}}=0}^{M=0} = \lim_{\substack{V\to\infty \\ a\to 0}} \frac{1}{V} \Big\langle (n_+ - n_-)^2 \Big\rangle^{\mathrm{YM}}$$

Some remarks on the $N_\mathrm{f}/N_\mathrm{c}$ expansion

- Perturbation theory: QCD β-function

$$\beta_0 = \frac{11}{3} N_\mathrm{c} \left(1 - \frac{2}{11} \frac{N_\mathrm{f}}{N_\mathrm{c}}\right)$$



- Quenched light hadron spectrum $\sim 10\%$ discrepancy with experiment

Numerical challenge

- A Monte Carlo computation of

$$\chi_{\rm L}^{\rm YM} = \frac{1}{V}\Big\langle (n_+ - n_-)^2 \Big\rangle^{\rm YM}$$

is challenging for several reasons

- $L \sim 1$ fm and $a \sim 0.08$ fm $\Longrightarrow$ $\dim[D] \sim 2.5\ 10^5$: computing and diagonalizing the full matrix not feasible

- A standard minimization would require high precision to beat contamination from quasi-zero modes

- At large V the probability distribution has a width which increases linearly with V

$$P_Q = \frac{1}{\sqrt{2\pi V \chi_{\rm L}^{\rm YM}}} e^{-\frac{Q^2}{2V\chi_{\rm L}^{\rm YM}}} \{1 + O(V^{-1})\}$$

$\Longrightarrow$ computing $\chi_{\rm L}^{\rm YM}$ requires very high statistics

Algorithm for zero-mode counting [L. G., Hoelbling, Lüscher and Wittig 02]

- In finite V null prob. for $n_+ \neq 0$ and $n_- \neq 0$

- Simultaneous minimization of Ritz functionals associated to

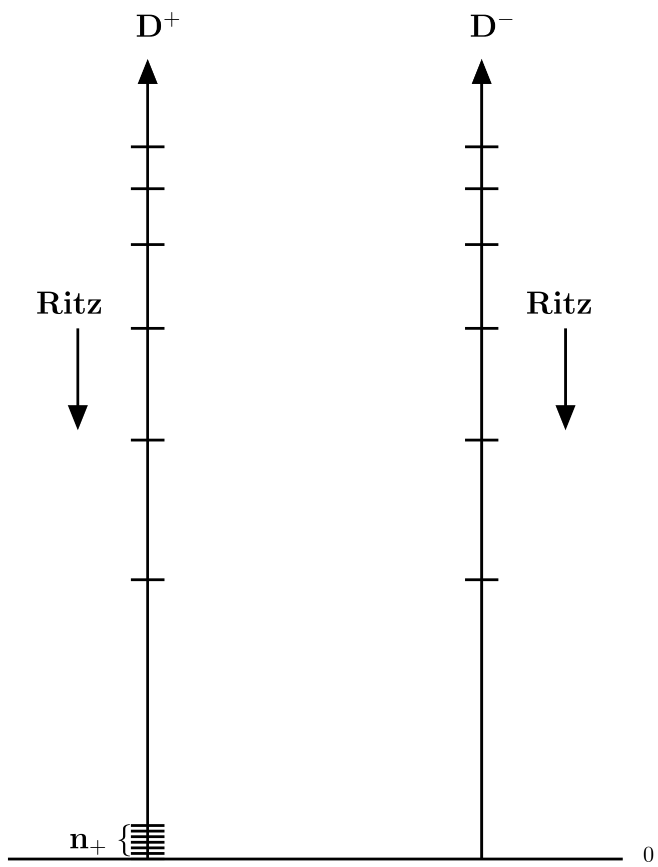
$$D^{\pm} = P_{\pm} D P_{\pm} \qquad P_{\pm} = \frac{1 \pm \gamma_5}{2}$$

to find the gap in one of the sectors

- Run again the minimization in the sector without gap and count zero modes

- No contamination from quasi-zero modes

- Adaptive precision for computing D during the minimization [see also L. G., Hoelbling, Rebbi 01]



Monte Carlo computation of χ_L^{YM} [L.G., Lüscher, Weisz, Wittig 03; Del Debbio, Pica 03; Del Debbio, L. G., Pica 04]

$s = 0.4$

β	L/a	L [fm]	N_{conf}	$\langle Q^2 \rangle$	$r_0^4 \chi_L^{YM}$
6.0	12	1.12	2452	1.63(5)	0.065(2)
6.1791	16	1.12	1138	1.59(8)	0.063(3)
5.8989	10	1.12	1460	1.74(7)	0.070(3)
6.0938	14	1.12	1405	1.54(6)	0.062(3)
5.8458	12	1.49	2918	5.6(2)	0.072(2)
6.0	16	1.49	1001	5.6(3)	0.071(4)
6.1366	20	1.49	963	4.8(2)	0.060(3)
5.9249	14	1.49	1284	5.6(2)	0.071(3)
5.8784	16	1.86	1109	15.0(7)	0.078(4)
6.0	20	1.86	931	13(1)	0.066(5)
6.0	14	1.30	1577	3.0(1)	0.065(3)

$s = 0.0$

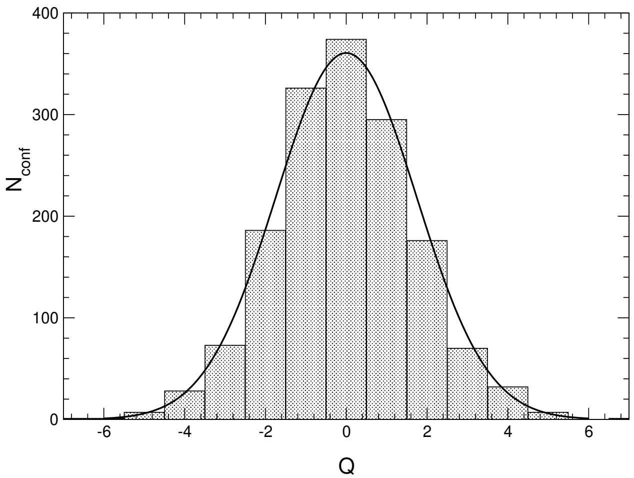
β	L/a	L[fm]	N_{conf}	$\langle Q^2 \rangle$	$r_0^4 \chi_L^{YM}$
5.9	12	1.34	1349	2.8(1)	0.054(2)
5.95	12	1.22	1291	1.96(8)	0.055(2)
6.0	12	1.12	3586	1.49(4)	0.060(2)
6.1	16	1.26	962	2.5(1)	0.060(3)
6.2	18	1.22	1721	2.11(8)	0.059(2)

- r_0 is a lattice reference scale of ≈ 0.5 fm, $\beta = 6/g^2$
- To keep stat. err. under control $N_{conf} \gtrsim 1000 \Longrightarrow \Delta\chi_L^{YM}/\chi_L^{YM} \lesssim 5\%$ for every lattice
- To keep systematic errors under control:
  1. Finite volume corrections: $L > 1$ fm
  2. Finite lattice spacing effects: $a = 0.068 \div 0.124$ fm, two values of s

Volume dependence [L.G., Lüscher, Weisz, Wittig 03]

- Probability distribution at large volume

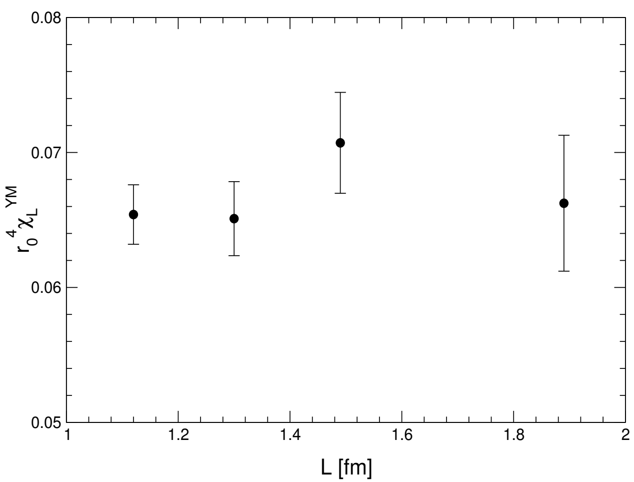
$$P_Q = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} \mathrm{e}^{-\frac{Q^2}{2\langle Q^2\rangle}} \{1 + O(V^{-1})\}$$



- Mass gap in the pure gauge theory $m_g \sim 1.5$ GeV

- $\chi_{\mathrm{L}}^{\mathrm{YM}}$ goes to the infinite-volume limit as $e^{-m_g L}$

- For $L \gtrsim 1$ fm, $\chi_{\mathrm{L}}^{\mathrm{YM}}$ is indep. of L within stat. errors



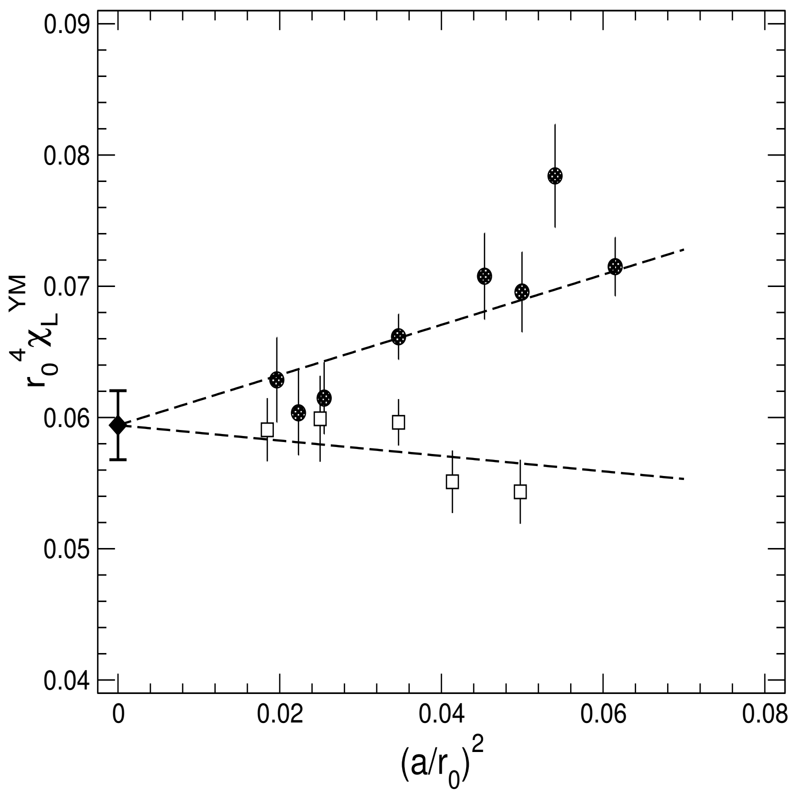
Continuum extrapolation [Del Debbio, L. G., Pica 04]

- Combined fit of the form [$\chi^2_{\rm dof} = 0.73$]

$$r_0^4 \chi_{\rm L}^{\rm YM}(s) = r_0^4 \chi^{\rm YM} + c_1(s) \frac{a^2}{r_0^2}$$

gives

$$r_0^4 \chi^{\rm YM} = 0.059 \pm 0.003$$



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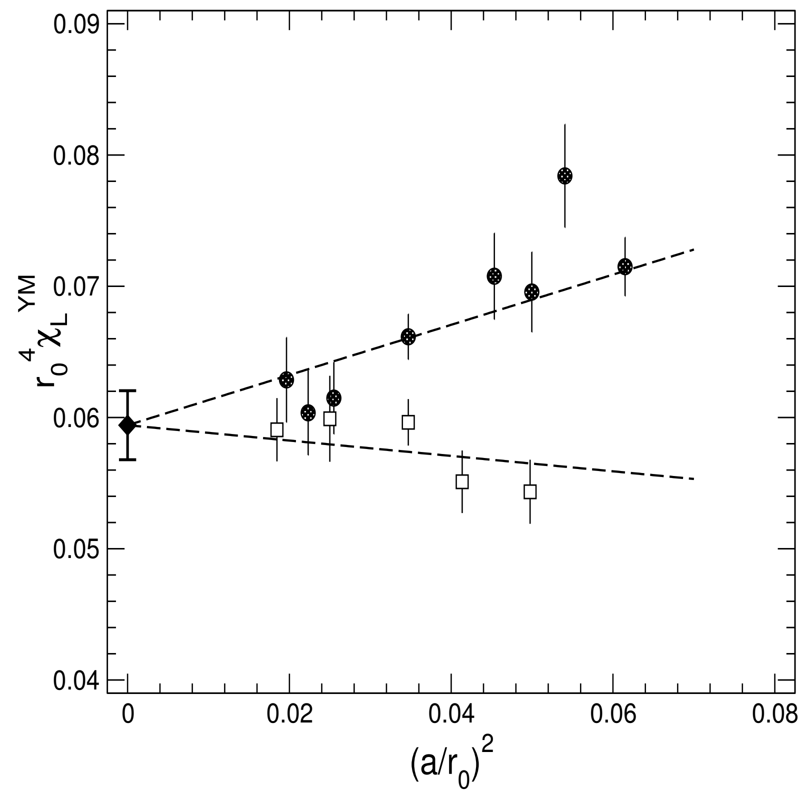
$$r_0^4 \chi_{\rm L}^{\rm YM}(s) = r_0^4 \chi^{\rm YM} + c_1(s)\frac{a^2}{r_0^2}$$

gives

$$r_0^4 \chi^{\rm YM} = 0.059 \pm 0.003$$

- By setting the scale $F_{\rm K} = 113(1)$ MeV

$$\boxed{\chi^{\rm YM} = (191 \pm 5~{\rm MeV})^4}$$



Continuum extrapolation [Del Debbio, L. G., Pica 04]

- Combined fit of the form [$\chi^2_{\rm dof} = 0.73$]

$$r_0^4 \chi_{\rm L}^{\rm YM}(s) = r_0^4 \chi^{\rm YM} + c_1(s) \frac{a^2}{r_0^2}$$

gives

$$r_0^4 \chi^{\rm YM} = 0.059 \pm 0.003$$

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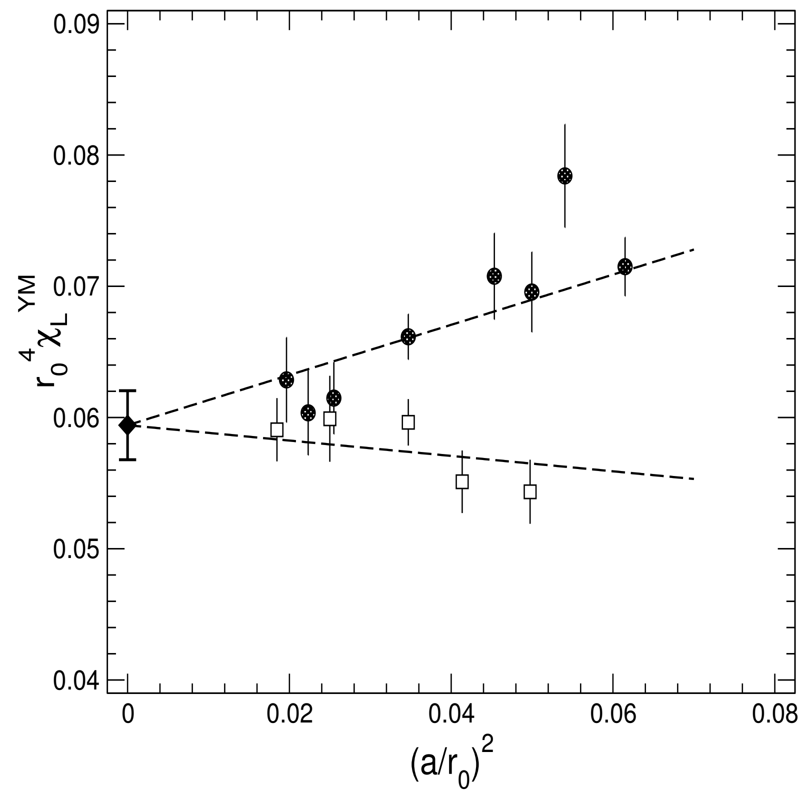

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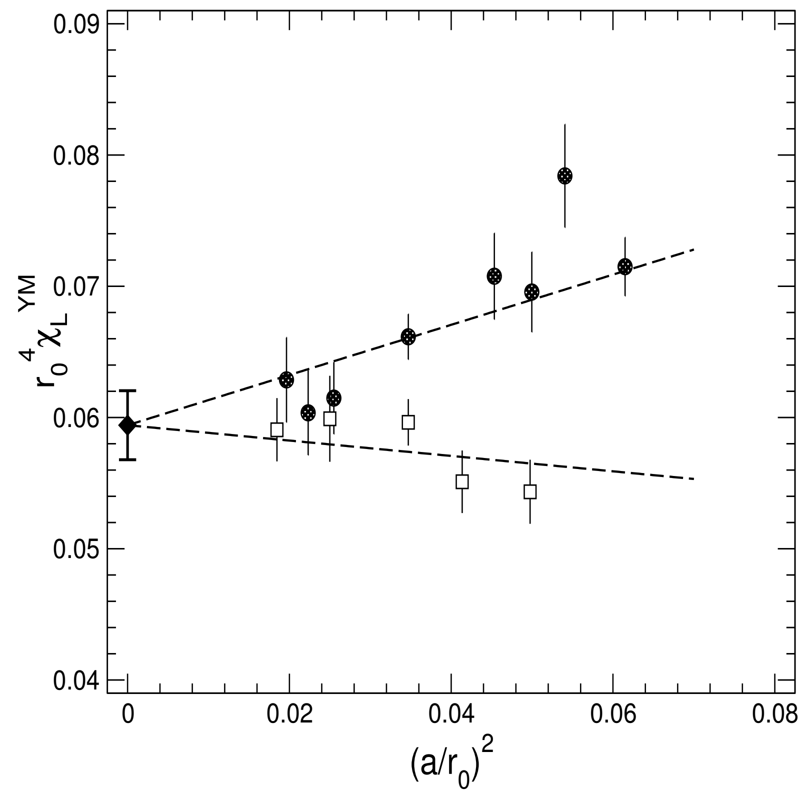
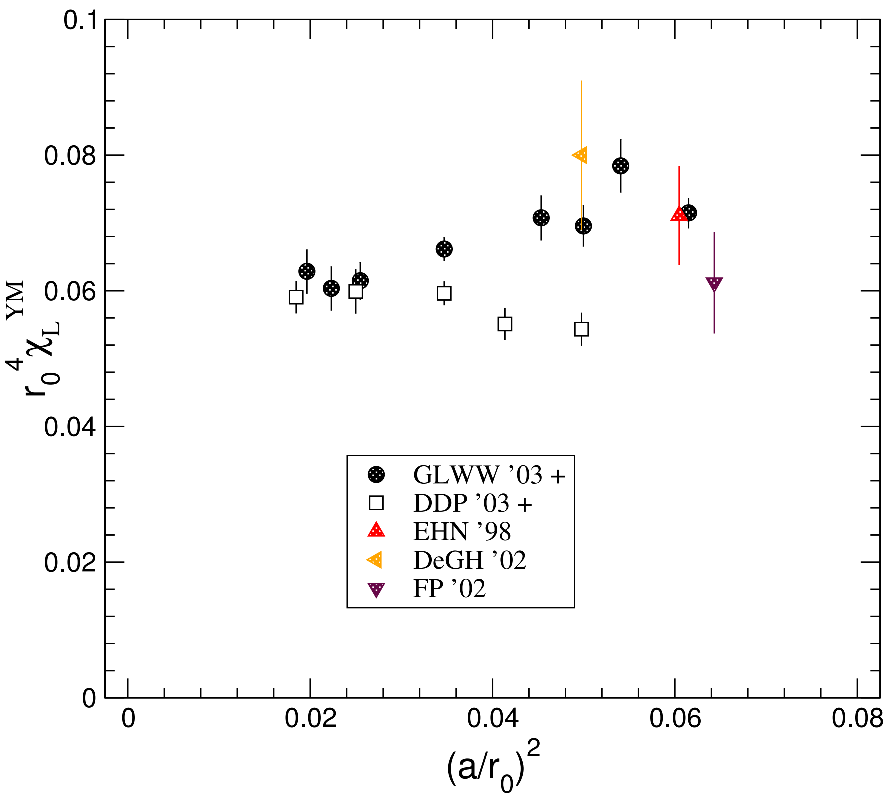


- The (leading) QCD anomalous contribution to $m^2_{\eta'}$, explains the bulk of its large experimental value as conjectured by Witten and Veneziano

Comparison with previous numerical studies

- From Ginsparg–Wilson fermions



- Cooling-type det. not from first-principles: systematic errors not under control

- A precise and unambiguous implementation of the Witten–Veneziano formula can be derived at the non-perturbative level in QCD
- Ultraviolet power-divergent subtractions fixed (avoided) without ambiguities
- Under the "smooth-quenching hypothesis", formula can be derived from dispersion relation (no reference to large N_c)
- With Ginsparg–Wilson fermions

$$\frac{F_\pi^2 m_{\eta'}^2}{2N_f}\Bigg|_{\frac{N_f}{N_c}=0}^{M=0} = \lim_{\substack{V\to\infty \\ a\to 0}} \frac{1}{V}\Big\langle (n_+ - n_-)^2 \Big\rangle^{\rm YM}$$

- A Monte Carlo non-perturbative computation with the Neuberger operator gives

$$\frac{F_\pi^2 m_{\eta'}^2}{2N_f}\Bigg|_{\frac{N_f}{N_c}=0}^{M=0} = (191 \pm 5\ \mathrm{MeV})^4$$

- The (leading) QCD anomalous contribution to $m_{\eta'}^2$ explains the bulk of its large experimental value as conjectured by Witten and Veneziano

Determinations before GW operators