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Spectral density of the QCD Dirac operator at the NLO in chiral perturbation theory

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1 Introduction

We compute the spectral density of the QCD Dirac operator by using partially quenched chiral perturbation theory (PQChPT) at the next-to-leading order (NLO). We focus on the theory with two degenerate dynamical flavors supplemented with an additional quenched one. The definition of the partially quenched (PQ) chiral theory and the computational strategy adopted here are slightly different from those that can be found in the literature [1, 2]. The spectral density is extracted from the analytic continuation of the PQ chiral condensate following Refs. [3, 4], where some of the results obtained here can be found.

2 Spectral density of the QCD Dirac operator

In this note fermions are assumed to be discretized with a massless Dirac operator D which satisfies the Ginsparg–Wilson (GW) relation

$$\gamma_5 D + D\gamma_5 = \bar{a} D\gamma_5 D \ , \tag{1}$$

where $\bar{a}$ is a parameter proportional to the lattice spacing (for notations not defined here see Ref. [5]). This relation implies that the eigenvalues γ_k of D are of the form

$$\gamma_k = \frac{1}{\bar{a}}(1 - e^{-i\omega_k}) \ , \tag{2}$$

and that for every gauge configuration complex eigenvalues appear in conjugate pairs. The spectral density of the Dirac operator can be defined as

$$\rho(\lambda) \equiv \frac{1}{V}\sum_k \langle \delta(\lambda - \lambda_k)\rangle \ , \qquad \lambda_k \equiv \frac{2}{\bar{a}}\tan\left(\frac{\omega_k}{2}\right) \tag{3}$$

where $\langle \ldots \rangle$ as usual indicates the path-integral average. The GW relation (chiral symmetry) implies that $\rho(\lambda) = \rho(-\lambda)$. At finite lattice spacing the bare spectral density can be extracted, for example, from the discontinuity of the bare PQ chiral condensate in the complex plane [3, 4]. The analytic continuation of the condensate

$$\Sigma(z) \equiv -\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle = \int_{-\infty}^{\infty} d\lambda\, \rho(\lambda)\, \frac{1}{i\lambda + z}\;, \qquad z \in \mathbb{C} \tag{4}$$

has a cut on the imaginary axis, and its discontinuity is given by

$$2\pi\rho(\lambda) = \lim_{\epsilon\to 0} \Big[\Sigma(i\lambda+\epsilon) - \Sigma(i\lambda-\epsilon)\Big]\;. \tag{5}$$

Thanks to the chiral symmetry $\Sigma(-z) = -\Sigma(z)$, Eq. (5) can also be written as

$$2\pi\rho(\lambda) = \lim_{\epsilon\to 0} \Big[\Sigma(i\lambda+\epsilon) + \Sigma(-i\lambda+\epsilon)\Big]\;, \tag{6}$$

a formula which can be directly applied in the chiral theory. The renormalization of the chiral condensate in presence of massive fermions requires two subtractions proportional to the linear and to the cubic power of the mass respectively, and an overall renormalization constant Z_S. Since the subtraction coefficients can be chosen to be mass independent, the counter-terms do not contribute to the discontinuity and the renormalized spectral density $\hat{\rho}$ reads [6]

$$\hat{\rho}(\lambda) = Z_S \rho(Z_S \lambda)\;. \tag{7}$$

In the following we are also interested in the integrated spectral density defined as

$$\hat{R}(\Lambda) = \int_{-\Lambda}^{\Lambda} \hat{\rho}(\lambda)\, d\lambda\;. \tag{8}$$

3 The chiral model at the leading order

We are interested in a non-linear sigma model with an underlying graded symmetry group $SU(3|1)_L \otimes SU(3|1)_R$ spontaneously broken to the graded vector subgroup $SU(3|1)$.
For every space-time point the unitary fundamental fields of the theory are defined to be

$$U \equiv \exp\left\{\frac{2i}{F}\Phi\right\}\;, \qquad \Phi = \sum_a \phi^a T^a \tag{9}$$

where T^a represent the generators of the underlying Lie superalgebra (see Appendix C). The field U satisfies the constraint

$$UU^\dagger = 1\!\!1 \qquad \Longleftrightarrow \qquad \Phi = \Phi^\dagger \tag{10}$$

which in turn implies $\phi^a = \phi^{a\dagger}$, and

$$\mathrm{Sdet}\, U = 1 \qquad \Longleftrightarrow \qquad \mathrm{Str}\, \Phi = 0\;. \tag{11}$$

Under the $SU(3|1)_L \otimes SU(3|1)_R$ transformations

$$V_L \;\equiv\; \exp\{i\alpha_L\}\;, \qquad \alpha_L = \alpha_L^\dagger\;, \qquad \mathrm{Str}\, \alpha_L = 0 \tag{12}$$

$$V_R \;\equiv\; \exp\{i\alpha_R\}\;, \qquad \alpha_R = \alpha_R^\dagger\;, \qquad \mathrm{Str}\, \alpha_R = 0 \tag{13}$$

the fundamental fields rotate as

$$U \to V_R U V_L^\dagger\,, \qquad\qquad U^\dagger \to V_L U^\dagger V_R^\dagger\,. \tag{14}$$

If we adopt the analogous of the standard power-counting of chiral perturbation theory [7, 8], i.e. U is a quantity of order 1, $\partial_\mu U$ is of order p and M is of order p^2, the Euclidean Lagrangian

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\Big\{\mathrm{Str}\,\Big[\partial_\mu U^\dagger \partial_\mu U\Big] - 2B\,\mathrm{Str}\,\Big[M U^\dagger + M^\dagger U\Big]\Big\} \tag{15}$$

is the most general one at order p^2 invariant under the $SU(3|1)_L \otimes SU(3|1)_R$ symmetry, if also the mass operators M and $M^\dagger$ are transformed as

$$M \to V_R M V_L^\dagger\,, \qquad\qquad M^\dagger \to V_L M^\dagger V_R^\dagger\,, \tag{16}$$

and which encodes the spontaneous symmetry breaking to the vector sub-group $SU(3|1)$. At this order the equations of motion of the model are

$$\Big[\partial_\mu\partial_\mu U\Big]U^\dagger - U\Big[\partial_\mu\partial_\mu U^\dagger\Big] + 2B\Big[MU^\dagger - UM^\dagger\Big] - B\,\mathrm{Str}\,\Big[MU^\dagger - UM^\dagger\Big]\mathbb{1} = 0\,. \tag{17}$$

In the following we are interested in a mass term of the form

$$M = M^\dagger = \begin{bmatrix} m_{sea} & 0 & 0 & 0 \\ 0 & m_{sea} & 0 & 0 \\ 0 & 0 & m_{val} & 0 \\ 0 & 0 & 0 & m_{val} \end{bmatrix}\,, \tag{18}$$

which breaks explicitly the residual flavor symmetry $SU(3|1)$ to $SU(2)\otimes U(1|1)$. The parameter m_{sea} is identified with the quark mass of the sea flavors and m_{val} is the quark mass of the quenched flavor. The residual $SU(2)$ may be identified with the physical isospin symmetry group.

By expanding U and keeping up to the second order in Φ we obtain

$$\begin{aligned}\mathcal{L}^{(2)}_{2\phi} &= \mathrm{Str}\,\Big[\partial_\mu\Phi\partial_\mu\Phi\Big] + 2B\,\mathrm{Str}\,[M\Phi^2] \\ &= \frac{g^{ab}}{2}\Big[\partial_\mu\phi^a\partial_\mu\phi^b + M_a^2\,\phi^a\phi^b\Big] - \frac{B}{3}\,(m_{val} - m_{sea})k^{ab}\phi^a\phi^b\,,\end{aligned} \tag{19}$$

which is the Lagrangian of a free theory of bosons and pseudo-fermions with the following propagators

$$\boldsymbol{\Delta}^{ab}(x) \equiv \langle\phi^a(x)\phi^b(0)\rangle = g^{ab}\,D_1^4(x,M_a^2) + \frac{2B}{3}\,(m_{val}-m_{sea})\,h^{ab}\,D_2^4(x,M_a^2)\,, \tag{20}$$

where

$$M_a^2 = \begin{cases} M_{ss}^2 & a = 1,\dots,3 \\ M_{sv}^2 & a = 4,\dots,7,9,\dots,12 \\ M_{vv}^2 & a = 8,13,14,15 \end{cases}\,, \tag{21}$$

with $M_{ss}^2 = 2Bm_{sea}$, $M_{sv}^2 = B(m_{sea}+m_{val})$, $M_{vv}^2 = 2Bm_{val}$, $\langle\ldots\rangle$ indicates the average in the Euclidean functional integral, g^{ab}, k^{ab} and h^{ab} are defined in appendix C and D, and $D_1^4(x, M^2)$ and $D_2^4(x, M^2)$ are defined in appendix F. The Lagrangian in Eq. (19) is quite non-standard and requires some comments. The Grassman fields have a bosonic action which in turn implies that the theory is not unitary. One of the consequences is that the bosonic fields ϕ^8 and ϕ^{15} have double poles in their propagators when $m_{val} \neq m_{sea}$. The boson ϕ^{15} has a kinematical term with the wrong sign which spoils the convergence of the standard Gaussian functional integral. This problem can be solved by choosing properly the integration domain for this field [9, 2], and the resulting functional integral is convergent if

$$m_{val} > \frac{m_{sea}}{4} \tag{22}$$

Since in the following we are interested in the perturbative expansion around $U = \mathbb{1}$, the result will be independent of the particular integration domain of ϕ^{15}.

4 The action at the next to leading order

When considering the model beyond tree-level, the Weinberg power-counting theorem can be applied to the momentum expansion as in standard ChPT [7, 8]. Since the model is non-renormalizable, the Lagrangian in Eq. (15) must be supplemented with a NLO order term which contains operators at order p^4.

In the following we are interested in identifying the chiral model defined above as the effective low-energy theory of partially quenched QCD , i.e. the on-shell correlation functions of the chiral model are interpreted to be asymptotic expansions in p^2 of on-shell correlation functions of partially quenched QCD. Therefore $\mathcal{L}^{(2)}$ has to be considered the Lagrangian of the effective low-energy theory of partially quenched QCD, and the NLO action has to be interpreted as operator insertions in on-shell correlation functions. For this purpose we can use the leading-order (LO) equations of motion in Eq. (17) for eliminating redundant operators in the NLO Lagrangian to obtain [1]

$$\begin{aligned}
\mathcal{L}^{(4)} \quad = \quad & -L_0 \,\mathrm{Str}\left[\partial_\mu U^\dagger \partial_\nu U \partial_\mu U^\dagger \partial_\nu U\right] \\
& -\left\{L_1 - \frac{L_0}{2}\right\} \mathrm{Str}\left[\partial_\mu U^\dagger \partial_\mu U\right] \mathrm{Str}\left[\partial_\nu U^\dagger \partial_\nu U\right] \\
& -\left\{L_2 - L_0\right\} \mathrm{Str}\left[\partial_\mu U^\dagger \partial_\nu U\right] \mathrm{Str}\left[\partial_\mu U^\dagger \partial_\nu U\right] \\
& -\left\{L_3 + 2L_0\right\} \mathrm{Str}\left[\partial_\mu U^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\nu U\right] \\
& +2\,B\,L_4 \,\mathrm{Str}\left[\partial_\mu U^\dagger \partial_\mu U\right] \mathrm{Str}\left[M U^\dagger + M^\dagger U\right] \\
& +2\,B\,L_5 \,\mathrm{Str}\left[\partial_\mu U^\dagger \partial_\mu U (U^\dagger M + M^\dagger U)\right] \\
& -4B^2 L_6 \,\mathrm{Str}\left[U^\dagger M + M^\dagger U\right] \mathrm{Str}\left[U^\dagger M + M^\dagger U\right]
\end{aligned}$$

[1] We discard the CP-violation Wess–Zumino contribution to the NLO Lagrangian since we are not interested in the θ dependence of correlation functions nor in insertions of singlet operators.

$$-4B^2 L_7 \,\mathrm{Str}\left[M^\dagger U - M U^\dagger\right] \mathrm{Str}\left[M^\dagger U - M U^\dagger\right]$$
$$-4B^2 L_8 \,\mathrm{Str}\left[M U^\dagger M U^\dagger + M^\dagger U M^\dagger U\right]$$
$$-4B^2 H_2 \,\mathrm{Str}\left[M^\dagger M\right] , \qquad (23)$$

where also in this case [2] we will be interested in the case $M = M^\dagger$.
Since the isospin chiral group $SU(2)_L \otimes SU(2)_R$ is a sub-group of $SU(3|1)_L \otimes SU(3|1)_R$, the LO low-energy constants (LECs) F and B, and some particular combinations of NLO LECs L_i of partially quenched QCD are equal to those of QCD with $N_f = 2$. Following the notation in Refs. [8, 11]

$$\begin{aligned} l_1 &= 4L_1 + 2L_3 \\ l_2 &= 4L_2 \\ l_3 &= 16L_6 + 8L_8 - 8L_4 - 4L_5 \\ l_4 &= 8L_4 + 4L_5 \,. \end{aligned} \qquad (24)$$

If we choose a mass matrix M that breaks the group $SU(3|1)$ to $SU(2) \otimes U(1)$ and we integrate out the pseudo-fermion fields, we obtain an effective theory with the same structure of the one of QCD with three flavors but with different values for the LECs. In this limit the convention adopted in Eq. (23) are the same as those in Ref. [11].

5 Partially quenched chiral condensate at NLO

The PQ chiral condensate can be defined as

$$\Sigma_V \equiv -\frac{1}{V} \left\langle \frac{\partial}{\partial J} \int d^4x \, \mathcal{L}(x) \right\rangle \Big|_{J=0} , \qquad (25)$$

where the Lagrangian is $\mathcal{L}(x) = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$, the external source J modifies the mass matrix as

$$M_J = M_J^\dagger \equiv M + J \cdot P_J , \qquad P_J \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \qquad (26)$$

and $\langle \ldots \rangle$ as usual indicates the path-integral average. The part of the Lagrangian relevant to the computation of the chiral condensate at the NLO order is given by

$$\mathcal{L}^{\mathrm{nlo}} = \mathcal{L}_0^{(2)} + \mathcal{L}_{2\phi}^{(2)} + \mathcal{L}_0^{(4)} , \qquad (27)$$

which gives

$$\frac{\partial \mathcal{L}_0^{(2)}}{\partial J}\Big|_{J=0} = -BF^2 , \qquad (28)$$

[2] Notice that in PQChPT there is the extra coupling L_0 with respect to the $SU(3)$ NLO Lagrangian. It was neglected for many years in the literature and was considered for the first time in Ref. [10].

$$\frac{\partial \mathcal{L}_{2\phi}^{(2)}}{\partial J}\Big|_{J=0} = 2B\,\mathrm{Str}\left[P_J \Phi^2\right] , \tag{29}$$

$$\frac{\partial \mathcal{L}_{0}^{(4)}}{\partial J}\Big|_{J=0} = -64\,B^2\,L_6\,m_{sea} - 8\,B^2\,(H_2 + 2L_8)\,m_{val} \ . \tag{30}$$

By regularizing the ultraviolet divergences in dimensional regularization, the bare chiral condensate in finite volume is given by

$$\Sigma_V^{\rm nlo} = \Sigma\Big\{1 + \frac{1}{2F^2}\Big[D_1^d(M_{vv}^2) - 4D_1^d(M_{sv}^2) - \frac{2\Sigma}{F^2}(m_{val} - m_{sea})D_2^d(M_{vv}^2) + \frac{128\Sigma}{F^2}L_6 m_{sea} + \frac{16\Sigma}{F^2}(H_2 + 2L_8)m_{val}\Big]\Big\} \tag{31}$$

where $\Sigma = BF^2$ and the integrals $D_r^d(M^2)$ are defined in Appendix F.

5.1 The infinite volume result

In the infinite volume limit $D_r^d(M^2) \to \Delta_r^d(M^2)$, with $\Delta^d(M^2)$ being also defined in Appendix F, and the chiral condensate is given by

$$\Sigma^{\rm nlo} = \Sigma\Big\{1 + \frac{1}{2F^2}\Big[\Delta_1^d(M_{vv}^2) - 4\Delta_1^d(M_{sv}^2) - \frac{2\Sigma}{F^2}(m_{val} - m_{sea})\Delta_2^d(M_{vv}^2) + \frac{128\Sigma}{F^2}L_6 m_{sea} + \frac{16\Sigma}{F^2}(H_2 + 2L_8)m_{val}\Big]\Big\} \tag{32}$$

If we define the renormalized NLO LECs to be

$$\hat{L}_6 = L_6 + \frac{3}{64}\frac{\lambda}{(4\pi)^2} \ , \qquad \hat{H}_2 + 2\hat{L}_8 = H_2 + 2L_8 \ , \tag{33}$$

then

$$\Sigma^{\rm nlo} = \Sigma\Big\{1 + \frac{\Sigma}{(4\pi)^2 F^4}\Big[m_{sea}\left(64\,(4\pi)^2\hat{L}_6 - 1\right) + m_{val}\left(8\,(4\pi)^2(\hat{H}_2 + 2\hat{L}_8) + 1\right) + (2m_{val} - m_{sea})\ln\left(\frac{2\Sigma\, m_{val}}{F^2\mu^2}\right) - 2\,(m_{sea} + m_{val})\ln\left(\frac{\Sigma\,(m_{sea} + m_{val})}{F^2\mu^2}\right)\Big]\Big\} \ . \tag{34}$$

5.2 Finite volume correction

The finite volume correction to the chiral condensate

$$\Delta\Sigma_V^{\rm nlo} \equiv \Sigma_V^{\rm nlo} - \Sigma^{\rm nlo} \tag{35}$$

is ultraviolet finite and is given by

$$\Delta\Sigma_V^{\rm nlo} = \frac{\Sigma}{2F^2}\Big\{g_1^4(M_{vv}^2) - 4g_1^4(M_{sv}^2) - 2(M_{vv}^2 - M_{sv}^2)g_2^4(M_{vv}^2)\Big\} \ , \tag{36}$$

where $g_r^d(M^2)$ is given in Appendix F.

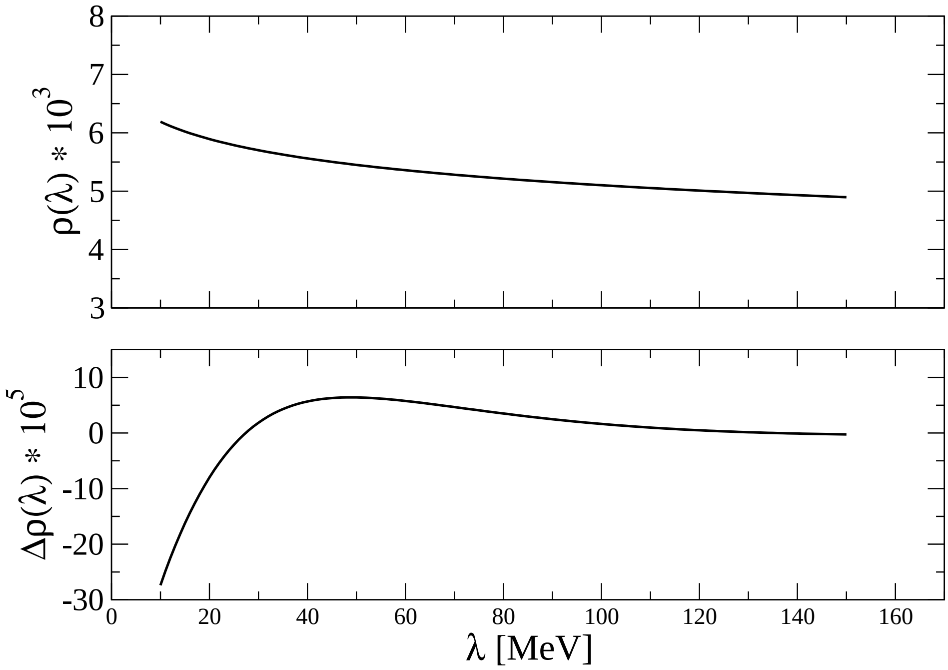

Figure 1: The spectral density (top) and its finite volume correction (bottom) as a function of λ. The upper plot is a representation of Eq. (41) with $\Sigma = (260\text{ MeV})^3$, $F = 86.2$ MeV, $\bar{l}_6 = 2.76$, $M = 139.6$ MeV and $m_{sea} = 15$ MeV. The lower plot is a reprentation of Eq. (46) for a box of volume $V = T \times L^3$ with $T = 2\,L$, $L/a = 32$ and $a = 0.0784$ fm.

6 Spectral density at NLO

The spectral density of the QCD Dirac operator can be extracted from the results of the previous section by using Eq. (6), i.e.

$$2\pi\rho_V^{\rm nlo}(\lambda) = \lim_{\epsilon\to 0}\left[\Sigma_V^{\rm nlo}(m_{val} = i\lambda+\epsilon) + \Sigma_V^{\rm nlo}(m_{val} = -i\lambda+\epsilon)\right]. \qquad (37)$$

The integrated spectral density can be defined analogously to Eq. (8)

$$R_V^{\rm nlo}(\Lambda) = \int_{-\Lambda}^{\Lambda} \rho_V^{\rm nlo}(\lambda)\, d\lambda\,. \qquad (38)$$

6.1 The infinite volume result

In the infinite volume limit the spectral density reads

$$\rho^{\rm nlo}(\lambda) = \frac{\Sigma}{\pi}\Big\{1 + \frac{\Sigma}{(4\pi)^2F^4}\Big[m_{sea}\left(64(4\pi)^2\hat{L}_6 - 1\right) + \qquad (39)$$

$$2\lambda\arctan\Big(\frac{\lambda}{m_{sea}}\Big) - \pi|\lambda| - 2\,m_{sea}\ln\left(\frac{\Sigma\sqrt{\lambda^2+m_{sea}^2}}{F^2\mu^2}\right) - m_{sea}\ln\left(\frac{2\Sigma|\lambda|}{F^2\mu^2}\right)\Big]\Big\}\,.$$

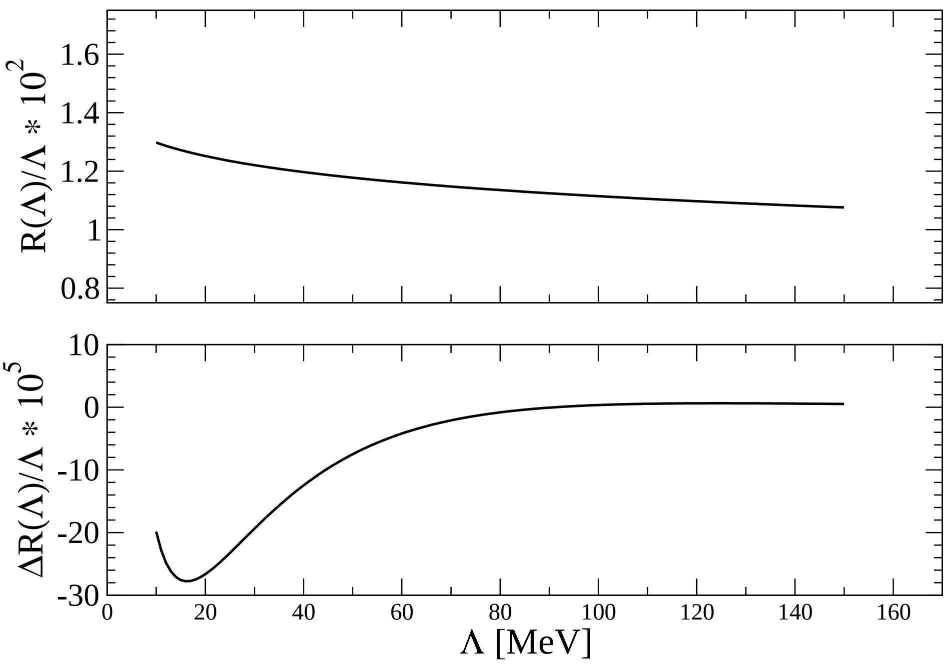

Figure 2: The integrated spectral density (top) and its finite volume correction (bottom) as a function of Λ. The upper plot is a representation of Eq. (42) with $\Sigma = (260\text{ MeV})^3$, $F = 86.2$ MeV, $\bar{l}_6 = 2.76$, $M = 139.6$ MeV and $m_{sea} = 15$ MeV. The lower plot is a reprentation of Eq. (49) for a box of volume $V = T \times L^3$ with $T = 2\,L$, $L/a = 32$ and $a = 0.0784$ fm.

With a proper choice of the reference scale μ, $\rho^{\rm nlo}(\lambda)$ agrees with the result in Eq. (83) of Ref. [4] where, however, the explicit dependence on $\hat{L}_6$ was not derived. If we define a scale independent LEC $\bar{l}_6$ as

$$\hat{L}_6 \equiv \frac{3}{64(4\pi)^2}\left[\bar{l}_6 + \ln\left(\frac{M^2}{\mu^2}\right)\right] \tag{40}$$

then

$$\begin{aligned}\rho^{\rm nlo}(\lambda) \quad &= \quad \frac{\Sigma}{\pi}\Big\{1 + \frac{\Sigma}{(4\pi)^2 F^4}\Big[m_{sea}\left(3\,\bar{l}_6 - 1\right) + 2\lambda \arctan\Big(\frac{\lambda}{m_{sea}}\Big) - \pi|\lambda| \\ &\quad -2\,m_{sea}\ln\left(\frac{\Sigma\sqrt{\lambda^2 + m_{sea}^2}}{F^2 M^2}\right) - m_{sea}\ln\left(\frac{2\Sigma|\lambda|}{F^2 M^2}\right)\Big]\Big\}\,. \end{aligned}\tag{41}$$

By integrating $\rho^{\rm nlo}(\lambda)$ we obtain

$$\begin{aligned}R^{\rm nlo}(\Lambda) \quad &= \quad \frac{2\Sigma\Lambda}{\pi}\Big\{1 + \frac{\Sigma}{(4\pi)^2 F^4}\Big[m_{sea}\left(3\,\bar{l}_6 + 1\right) + \left(\frac{\Lambda^2 - m_{sea}^2}{\Lambda}\right)\arctan\Big(\frac{\Lambda}{m_{sea}}\Big) \\ &\quad -\frac{\pi}{2}\Lambda - 2m_{sea}\ln\left(\frac{\Sigma\sqrt{\Lambda^2 + m_{sea}^2}}{F^2 M^2}\right) - m_{sea}\ln\left(\frac{2\Sigma\Lambda}{F^2 M^2}\right)\Big]\Big\}\,. \end{aligned}\tag{42}$$

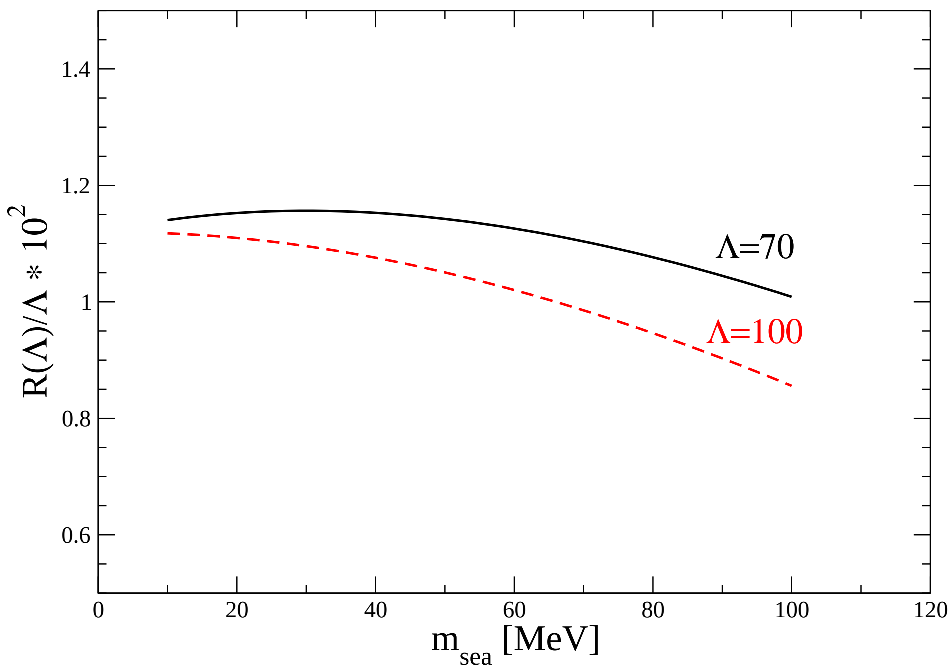

Figure 3: The integrated spectral density as a function of the sea quark mass m_{sea}. The upper plot is a representation of Eq. (42) with $\Sigma = (260\text{ MeV})^3$, $F = 86.2$ MeV, $\bar{l}_6 = 2.76$, $M = 139.6$ MeV and $\Lambda = 70, 100$ MeV.

If we expand the function arctan as

$$\arctan(|x|) = \frac{\pi}{2} - \frac{1}{|x|} + \dots\,, \qquad |x| \gg 1 \tag{43}$$

then

$$R^{\rm nlo}(\Lambda) = \frac{2\Sigma\Lambda}{\pi}\left\{1 + \frac{\Sigma\, m_{sea}}{(4\pi)^2 F^4}\left[3\,\bar{l}_6 - 3\ln\left(\frac{\Sigma\Lambda}{F^2M^2}\right) - \ln(2) - \frac{\pi}{2}\frac{m_{sea}}{\Lambda} + O\left(\frac{m_{sea}^2}{\Lambda^2}\right)\right]\right\}\,.$$

The functions $\rho^{\rm nlo}(\lambda)$ and $R^{\rm nlo}(\Lambda)$ are plotted in Figs. 1-3 for a given choice (see captions) of the parameter values.

6.2 Finite volume correction

The finite volume correction to the spectral density

$$\Delta\rho_V^{\rm nlo}(\lambda) \equiv \rho_V^{\rm nlo}(\lambda) - \rho^{\rm nlo}(\lambda) \tag{44}$$

is given by

$$2\pi\Delta\rho_V^{\rm nlo}(\lambda) = \lim_{\epsilon\to 0}\left[\Delta\Sigma_V^{\rm nlo}(m_{val} = i\lambda + \epsilon) + \Delta\Sigma_V^{\rm nlo}(m_{val} = -i\lambda + \epsilon)\right]\,. \tag{45}$$

By using Eq. (36) we obtain

$$\Delta\rho_V^{\rm nlo}(\lambda) = \frac{\Sigma}{\pi}\frac{\Sigma}{(4\pi)^2F^4}\sideset{}{'}\sum_{\{n_1,\ldots,n_4\}}\lim_{\epsilon\to 0}\left\{{\rm Re}\Big[F_{-1}\Big(\frac{\Sigma q_n^2}{2F^2}, i\lambda+\epsilon\Big)\Big] - \right. \tag{46}$$

$$\left. 2{\rm Re}\Big[F_{-1}\Big(\frac{\Sigma q_n^2}{4F^2}, i\lambda+m_{sea}+\epsilon\Big)\Big] + m_{sea}{\rm Re}\Big[F_0\Big(\frac{\Sigma q_n^2}{2F^2}, i\lambda+\epsilon\Big)\Big] + \lambda{\rm Im}\Big[F_0\Big(\frac{\Sigma q_n^2}{2F^2}, i\lambda+\epsilon\Big)\Big]\right\} ,$$

where $F_\nu(a,z)$, $\sum'$ and q_n are defined in Appendices E and F. The finite volume corrections to the integrated spectral density can be defined analogously

$$\Delta R_V^{\rm nlo} \equiv R_V^{\rm nlo} - R^{\rm nlo} , \tag{47}$$

where

$$\Delta R_V^{\rm nlo} = \int_{-\Lambda}^{\Lambda} \Delta\rho_V^{\rm nlo}(\lambda)\, d\lambda . \tag{48}$$

By using the results in Appendix E we obtain

$$\Delta R_V^{\rm nlo} = \frac{2\Sigma\Lambda}{\pi}\frac{\Sigma}{(4\pi)^2F^4}\sideset{}{'}\sum_{\{n_1,\ldots,n_4\}}\lim_{\epsilon\to 0}\left\{\frac{2}{\Lambda}{\rm Im}\Big[F_{-2}\Big(\frac{\Sigma q_n^2}{4F^2}, i\Lambda+m_{sea}+\epsilon\Big)\Big] - \right. \tag{49}$$

$$\left. \frac{m_{sea}}{\Lambda}{\rm Im}\Big[F_{-1}\Big(\frac{\Sigma q_n^2}{2F^2}, i\Lambda+\epsilon\Big)\Big] + {\rm Re}\Big[F_{-1}\Big(\frac{\Sigma q_n^2}{2F^2}, i\Lambda+\epsilon\Big)\Big]\right\} .$$

The functions $R^{\rm nlo}(\Lambda)$ and $R_V^{\rm nlo}(\Lambda)$ are plotted in Figs. 1 and 2 for a given choice (see captions) of the parameter values.

Appendix A

In this appendix we define a *square even supermatrix*, which in short will be called supermatrix, and report some of its properties (for further readings see [12]).

Given m and n any two positive integers, a supermatrix M is defined as a $(m+n)\times(m+n)$ matrix with the partitioning

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} . \tag{50}$$

The matrices A and D have dimensions $m\times m$ and $n\times n$ respectively with the elements which are members of the *even subspace of a Grassman algebra*[3] (commuting elements). The matrices B and C have dimensions $m\times n$ and $n\times m$ respectively with the elements which are members of the *odd subspace of a Grassman algebra* (anti-commuting elements). The matrix product is defined by the usual combination

$$(UV)_{ij} \equiv \sum_{k=1}^{m+n} (U)_{ik}(V)_{kj} . \tag{51}$$

[3]Grassman algebras are particular examples of associative superalgebras, for their definition and properties see [12]

The *adjoint* of the supermatrix U is defined to be

$$(U^\dagger)_{ji} \equiv (U_{ij})^\dagger \qquad \Longrightarrow \qquad U^\dagger \equiv \left[\begin{array}{cc} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{array} \right] , \tag{52}$$

where the adjoint of an element g of a Grassman algebra is defined to be

$$g^\dagger \equiv \left\{ \begin{array}{ll} g^* & \text{even} \\ -ig^* & \text{odd} \end{array} \right. , \tag{53}$$

and g^* is the complex conjugate defined in the usual way [12].

A supermatrix is defined to be *invertible* if it exists an *inverse matrix* U^{-1} such that

$$UU^{-1} = U^{-1}U = 1\!\!1 , \tag{54}$$

where $1\!\!1$ is the unit supermatrix (all diagonal elements equal to the identity). A supermatrix partitioned as in Eq. (50) is invertible if and only if the matrices A and D are invertible.

The *supertrace* of a supermatrix is an even element of the Grassman algebra defined as

$$\text{Str}\, U \equiv \text{Tr}\, A - \text{Tr}\, D , \tag{55}$$

and has the properties

$$\text{Str}\,[U+V] = \text{Str}\, U + \text{Str}\, V , \tag{56}$$

$$\text{Str}\,[UV] = \text{Str}\,[VU] . \tag{57}$$

The *superdeterminant* of an invertible supermatrix is an even element of the Grassman algebra defined as

$$\text{Sdet}\, U \equiv \text{Det}\,[A - BD^{-1}C]/\text{Det}\,[D] . \tag{58}$$

It satisfies

$$\text{Sdet}\, U^{-1} = (\text{Sdet}\, U)^{-1} , \tag{59}$$

$$\text{Sdet}\, U = \text{Det}\,[A]/\text{Det}\,[D - CA^{-1}B] , \tag{60}$$

and therefore

$$\text{Str}\,[VUV^{-1}] = \text{Str}\, U , \tag{61}$$

$$\text{Sdet}\,[VUV^{-1}] = \text{Sdet}\, U . \tag{62}$$

By using the property

$$\text{Sdet}\, U \equiv \exp[\text{Str}\, \ln U] , \tag{63}$$

it is straightforward to verify that

$$\text{Sdet}\,[UV] = [\text{Sdet}\, U][\text{Sdet}\, V] , \tag{64}$$

$$\text{Sdet}\,[\exp U] = \exp[\text{Str}\, U] , \tag{65}$$

$$\text{Sdet}\, U^\dagger = [\text{Sdet}\, U]^\dagger = [\text{Sdet}\, U]^* . \tag{66}$$

Appendix B

In this appendix we define associative superalgebras of *square complex matrices*. For definitions of the concept of grading and of associative superalgebras used here see [12].

Given m and n any two positive integers, let consider a $(m+n)\times(m+n)$ matrix M with *complex entries* and with the partitioning

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{67}$$

where A, B, C and D are matrices of dimensions $m\times m$, $m\times n$, $n\times m$ and $n\times n$ respectively. M is said to be *even* if $B = C = 0$ whereas it is *odd* if $A = D = 0$, and the *degree* or *parity* is defined to be

$$deg\; M = \begin{cases} 0 & \text{if } M \text{ is even} \\ 1 & \text{if } M \text{ is odd} \end{cases} \;. \tag{68}$$

The set of all complex linear combinations of these matrices forms an associative superalgebra [12]. The *supertrace* is defined to be

$$\text{Str } M \equiv \text{Tr } A - \text{Tr } D \,. \tag{69}$$

A Lie superalgebra of square complex matrices can be constructed by supplementing the associative superalgebra with the commutators

$$\Big[M, N\Big]_s = MN - (-1)^{(deg\, M)(deg\, N)} NM \,. \tag{70}$$

In the following it will be useful to define anti-commutators as

$$\Big\{M, N\Big\}_s = MN + (-1)^{(deg\, M)(deg\, N)} NM \,, \tag{71}$$

and the matrices

$$I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad \bar{I} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} . \tag{72}$$

Appendix C

The Lie supergroups $SU(m|n)$ are defined to be the set of $(m+n)\times(m+n)$ even supermatrices U that satisfy the conditions

$$U^\dagger U = 1\!\!1 \,, \tag{73}$$

$$\text{Sdet } U = 1 \,. \tag{74}$$

If we parameterize U as

$$U \equiv e^{i\Phi} \tag{75}$$

then $\Phi = \Phi^\dagger$ and $\text{Str } \Phi = 0$. It should be noted that $SU(m|n)$ are *non-compact* Lie groups for $n > 0$ [12].

The Lie superalgebra of $SU(m|n)$ is the subset of square complex matrices with dimensions $(m+n)\times(m+n)$ partitioned as in Eq. (67) which are Hermitean and satisfy the condition

$$\mathrm{Str}\; M = 0 \; . \tag{76}$$

A basis of matrices for the Lie superalgebra can be chosen to be the set of Hermitean matrices $T^a = T^{a\dagger}$ $(a = 1, \ldots, (m+n)^2 - 1)$ each one with a definite parity

$$\deg(a) \equiv deg\, T^a \; , \tag{77}$$

and such that

$$\mathrm{Str}\; T^a = 0 \; . \tag{78}$$

The normalization is fixed to be

$$\mathrm{Str}\; [T^a T^b] = \frac{g^{ab}}{2} \; , \tag{79}$$

where

$$g^{ab} \;=\; \delta^{ab} \qquad a, b = 1, \ldots, m^2 - 1 \tag{80}$$

$$g^{ab} \;=\; \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad \begin{array}{lcl} a & = & m^2, m^2+2, \ldots, m^2+2mn-2 \\ b & = & a+1 \end{array} \tag{81}$$

$$g^{ab} \;=\; -\delta^{ab} \qquad a, b = m^2 + 2mn, \ldots, (m+n)^2 - 1 \; . \tag{82}$$

and $g^{ab} = (-)^{\deg(a)\,\deg(b)} g^{ba}$.

The structure constants are defined to be

$$\left[T^a, T^b\right]_s = \sum_c F_c^{ab}\, T^c \; , \tag{83}$$

and the anti-commutators are given by

$$\left\{T^a, T^b\right\}_s = \frac{g^{ab}}{m-n} + \sum_c D_c^{ab}\, T^c \; . \tag{84}$$

With these conventions it is possible to show that

$$\sum_{a,b} g^{ab}\; T^a_{\alpha\beta} T^b_{\gamma\delta} = \frac{1}{2} I_{\alpha\delta} \bar{I}_{\gamma\beta} - \frac{1}{2\,(m-n)} I_{\alpha\beta} I_{\gamma\delta} \; , \tag{85}$$

which implies

$$\sum_{a,b} g^{ab}\; (T^a T^b) \;=\; \frac{(m-n)^2 - 1}{2\,(m-n)} I \; , \tag{86}$$

$$\sum_{a,b} g^{ab}\; (T^a T^c T^b)_{\alpha\beta} (-1)^{\deg(a)\,\deg(c)} \;=\; -\frac{1}{2\,(m-n)} T^c \; . \tag{87}$$

Appendix D

In this appendix miscellaneous properties of the $SU(3|1)$ group are reported. When the $SU(3|1)$ group is broken to $SU(2)\otimes U(1|1)$ the following identities are useful

$$\sum_{a,b=1}^{3} g^{ab}\, T^a_{\alpha\beta}T^b_{\gamma\delta} \;=\; \frac{1}{2}I^2_{\alpha\delta}I^2_{\gamma\beta} - \frac{1}{4}I^2_{\alpha\beta}I^2_{\gamma\delta} \tag{88}$$

$$\sum_{a,b=4}^{7,9,\ldots,12} g^{ab}\, T^a_{\alpha\beta}T^b_{\gamma\delta} \;=\; \frac{1}{2}\Big[(I-I^2)_{\alpha\delta}I^2_{\gamma\beta} + I^2_{\alpha\delta}(\bar{I}-I^2)_{\gamma\beta}\Big] \tag{89}$$

$$\sum_{a,b=8,13}^{15} g^{ab}\, T^a_{\alpha\beta}T^b_{\gamma\delta} \;=\; \frac{1}{2}(I-I^2)_{\alpha\delta}(\bar{I}-I^2)_{\gamma\beta} + \frac{1}{4}\Big[I^2_{\alpha\beta}I^2_{\gamma\delta} - I_{\alpha\beta}I_{\gamma\delta}\Big] \tag{90}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \bar{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad I^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{91}$$

These relations imply

$$\sum_{a,b=1}^{3} g^{ab}\,(T^aT^b) \;=\; \frac{3}{4}I^2 \tag{92}$$

$$\sum_{a,b=4}^{7,9,\ldots,12} g^{ab}\,(T^aT^b) \;=\; I - I^2 \tag{93}$$

$$\sum_{a,b=8,13}^{15} g^{ab}\,(T^aT^b) \;=\; -\frac{1}{4}\Big[I - I^2\Big]\,. \tag{94}$$

It is also useful to define the following tensors

$$h^{ab} \equiv \left(g^{a8} - g^{a15}\right)\left(g^{b8} - g^{b15}\right)\,, \qquad k^{ab} \equiv \left(g^{a8} + g^{a15}\right)\left(g^{b8} + g^{b15}\right) \tag{95}$$

which satisfy

$$\sum_{c=1}^{15} h^{ac}k^{cb} = \sum_{c=1}^{15} k^{ac}h^{cb} = 0\,. \tag{96}$$

The following identities are also useful

$$\sum_{a,b=1}^{3} h^{ab}\,(T^aT^b) \;=\; 0 \tag{97}$$

$$\sum_{a,b=4}^{7,9,\ldots,12} h^{ab}\,(T^aT^b) \;=\; 0 \tag{98}$$

$$\sum_{a,b=8,13}^{15} h^{ab}\,(T^aT^b) \;=\; \frac{3}{4}\Big[I - I^2\Big]\,. \tag{99}$$

Appendix E

In this appendix various integrals relevant to the finite volume computation are reported. The basic integral we are interested in is[4][14]

$$F_\nu(a,z) \equiv \int_0^\infty dy\, y^{\nu-1}\, e^{-yz}\, e^{-\frac{a}{y}} = 2\,\left(\frac{a}{z}\right)^{\nu/2}\, K_\nu(2\sqrt{az})\,, \tag{100}$$

with $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, z > 0$, and K_ν is a modified Bessel functions [13]. If we restrict to the case in which ν is real, then it is straightforward to prove from its definition that

$$F_\nu(a^*, z^*) = \Big[F_\nu(a,z)\Big]^*\,. \tag{101}$$

We are also interested in the integrals

$$G_\nu(a,x_1,x_2) \;\equiv\; \mathrm{Re}\Big[\int_0^{x_2} dy\, F_\nu(a, iy+x_1)\Big]\,, \tag{102}$$

$$H_\nu(a,x_1,x_2) \;\equiv\; \mathrm{Im}\Big[\int_0^{x_2} dy\, y\, F_\nu(a, iy+x_1)\Big]\,, \tag{103}$$

with $x_1 \geq 0$, $x_2 > 0$. A straightforward computation leads to

$$G_\nu(a,x_1,x_2) \;=\; -\mathrm{Im}\Big[F_{\nu-1}(a, ix_2+x_1)\Big]\,, \tag{104}$$

$$H_\nu(a,x_1,x_2) \;=\; x_2\,\mathrm{Re}\Big[F_{\nu-1}(a, ix_2+x_1)\Big] + \mathrm{Im}\Big[F_{\nu-2}(a, ix_2+x_1)\Big]\,. \tag{105}$$

The asymptotic expansion of the Bessel function $K_\nu(z)$ for large arguments, i.e. $|z| \gg 1$, leads to [13]

$$F_\nu(a,z) \sim \sqrt{\pi}\,(az)^{-1/4}\left(\frac{a}{z}\right)^{\nu/2} e^{-2\sqrt{az}}\,. \tag{106}$$

If we define $z = |z|e^{i\theta}$, and we assume a to be real and positive, then

$$F_\nu(a,z) \sim \sqrt{\pi}\,(a|z|)^{-1/4}\left(\frac{a}{|z|}\right)^{\nu/2} e^{-2\sqrt{a|z|}\cos(\theta/2)}\, e^{-i\left[\theta(\nu/2+1/4)+2\sqrt{a|z|}\sin(\theta/2)\right]}\,. \tag{107}$$

[4]The integral representation of $F_\nu(a,z)$ in Eq. (100) holds also for $\mathrm{Re}\, z = 0$. This can be proved by rotating the integration path in the complex plane.

Appendix F

In this appendix some integrals and the corresponding finite volume sums are computed in dimensional regularization.

We are interested in the propagators of the form

$$\Delta_r^d(x,M^2) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{e^{ipx}}{(p^2+M^2)^r} \qquad r \geq 1\,, \tag{108}$$

and in particular in their value for $x=0$

$$\Delta_r^d(M^2) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+M^2)^r} \tag{109}$$

$$= \frac{(M^2)^{2-r}}{(4\pi)^2} \frac{\Gamma(r-d/2)}{\Gamma(r)} \left(\frac{M}{2\sqrt{\pi}}\right)^{d-4} \tag{110}$$

where $\Gamma(z)$ is the Euler Gamma function, and r is an integer $r \geq 1$ [13]. For $d \geq 2r$ the integral $\Delta_r^d(M^2)$ is divergent, and for $\epsilon = (4-d)/2 \to 0$

$$\Delta_1^{4-2\epsilon}(M^2) = \frac{M^2}{(4\pi)^2}\left\{-\lambda + \ln\left(\frac{M^2}{\mu^2}\right)\right\} + \mathcal{O}(\epsilon)\,, \tag{111}$$

$$\Delta_2^{4-2\epsilon}(M^2) = \frac{1}{(4\pi)^2}\left\{\lambda - \ln\left(\frac{M^2}{\mu^2}\right) - 1\right\} + \mathcal{O}(\epsilon) \tag{112}$$

where

$$\lambda = \frac{1}{\epsilon} + \ln(4\pi) - \ln(\mu^2) - \gamma + 1\,, \tag{113}$$

and $\gamma = 0.5772156649\ldots$ [13].

In a finite d-dimensional volume V the corresponding sums are defined to be

$$D_r^d(x,M^2) \equiv \frac{1}{V}\sum_p \frac{e^{ipx}}{(p^2+M^2)^r}\,, \qquad D_r^d(M^2) \equiv \frac{1}{V}\sum_p \frac{1}{(p^2+M^2)^r}\,, \tag{114}$$

where p runs over a d-dimensional momentum space lattice $p_\mu = 2\pi\, n_\mu/L_\mu$ $(\mu = 1,\ldots,d)$. The difference between the discrete sum (114) and the corresponding integral (109) is ultraviolet finite and therefore universal. If we define

$$D_r^d(M^2) = \Delta_r^d(M^2) + g_r^d(M^2)\,, \tag{115}$$

the volume dependence is contained in the function $g_r^d(M^2)$ given by [15]

$$g_r^d(M^2) = \frac{1}{\Gamma(r)}\int_0^\infty d\lambda\, \frac{\lambda^{r-1}}{(4\pi\lambda)^{d/2}}\, e^{-\lambda M^2} \sum_{\{n_1,\ldots,n_d\}}{}' \exp\left\{-\sum_{\mu=1}^d \frac{(n_\mu L_\mu)^2}{4\lambda}\right\}\,, \tag{116}$$

where $\sum'_{\{n_1,\ldots,n_d\}}$ denotes the sum over all integers without $n=(0,\ldots,0)$. By using the definition in Eq. (100), $g_r^d(M^2)$ can be written as

$$g_r^d(M^2) = \frac{1}{\Gamma(r)\,(4\pi)^{d/2}} \sum_{\{n_1,\ldots,n_d\}}{}' F_{r-d/2}\Big(\frac{q_n^2}{4}, M^2\Big)\,, \tag{117}$$

where $q_n^2 = \sum_{\mu=1}^{d} (n_\mu L_\mu)^2$. By inserting the r.h.s. of Eq. (100) in Eq. (117), $g_r^d(M^2)$ can then be written as

$$g_r^d(M^2) = \frac{2}{\Gamma(r)\,(4\pi)^{d/2}} \sum_{\{n_1,\dots,n_d\}}{}' \left(\frac{|q_n|}{2M}\right)^{r-d/2} K_{r-d/2}\Big(|q_n| M\Big)\,, \tag{118}$$

and by using the the asymptotic expansion of $K_\nu(z)$ for large arguments it can be shown that $g_r^d(M^2) \to 0$ exponentially fast when $ML_\mu \to \infty$.

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