FACTORIZATION OF FERMIONS AND MULTI-LEVEL INTEGRATION:

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- FERMION FACTORIZATION

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1) WILSON ACTIONS AND PATH INTEGRAL

- FOR GLUONS $(a=1)$

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{2N_c} \mathrm{Tr} \left\{ U_{\mu\nu}(x) + U^+_{\mu\nu}(x) \right\} \right]$$

$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U^+_\mu(x+\hat{\nu})\, U^+_\nu(x) \quad ; \quad \beta = \frac{2N_c}{g_0^2}$$

- FOR FERMIONS $(a=1)$

$$S_F = \sum_x \bar{\psi}(x) \left[D_W + m_0 \right] \psi(x)$$

$$D_W = \frac{1}{2} \left\{ \gamma_\mu \left(\nabla^*_\mu + \nabla_\mu \right) - \nabla^*_\mu \nabla_\mu \right\}$$

WHERE

$$[\nabla_\mu \psi](x) = U_\mu(x)\, \psi(x+\hat{\mu}) - \psi(x) \qquad \text{FWD}$$

$$[\nabla^*_\mu \psi](x) = \psi(x) - U^+_\mu(x-\hat{\mu})\, \psi(x-\hat{\mu}) \qquad \text{BWD}$$

NOTICE

$$\bar{\psi}(x) [D_W \psi](x) = 4\, \bar{\psi}(x)\psi(x)$$

$$- \bar{\psi}(x) \left(\frac{1-\gamma_\mu}{2} \right) U_\mu(x)\, \psi(x+\hat{\mu}) - \bar{\psi}(x) \left(\frac{1+\gamma_\mu}{2} \right) U^+_\mu(x-\mu) \cdot \psi(x-\mu)$$

IN PARTICULAR IN THE KINETIC TERM $(\mu=0)$ FERMIONS ON DIFFERENT TIME SLICES INTERACT ONLY IF NEAREST NEIGHBOR

FROM NOW ON $x_0 = i \cdot a$, AND WE WILL CONSIDER THE HERMITEAN DIRAC OPERATOR

$$Q = \gamma_5 \{ D_W + m_0 \}$$

IT CAN BE REPRESENTED IN MATRIX FORM

$$Q = \begin{pmatrix} \ddots & & & 0 & \\ Q_{i,i-1} & Q_{i,i} & Q_{i,i+1} & 0 & \\ 0 & Q_{i+1,i} & Q_{i+1,i+1} & Q_{i+1,i+2} & 0 \\ 0 & 0 & Q_{i+2,i+1} & Q_{i+2,i+2} & \\ & & & & \ddots \end{pmatrix}$$

· NOTE: CONSIDERATIONS IN THESE LECTURES APPLY TO OTHER DISCRETIZATIONS AS WELL, SUCH AS $O(a)$ IMPROVED WILSON FERMIONS ETC..

· IN NUMERICAL LATTICE QCD GRASSMANN VARIABLES ARE INTEGRATED ANALYTICALLY

$$Z = \int DU \, e^{-S_G[U]} \{\det[Q]\}^{N_f}$$

$$\langle O_1 O_2 \rangle = \frac{1}{Z} \int DU \, e^{-S_G[U]} \{\det[Q]\}^{N_f} \, W_{O_1 O_2}$$

WHERE $W_{O_1 O_2}$ IS THE SUM OF WICK CONTRACTIONS

- LARGELY USED TO RE-EXPRESS $\det[Q]$ AS INTEGRAL OVER AUXILIARY SCALAR FIELDS (SIMPLEST REALIZATION)

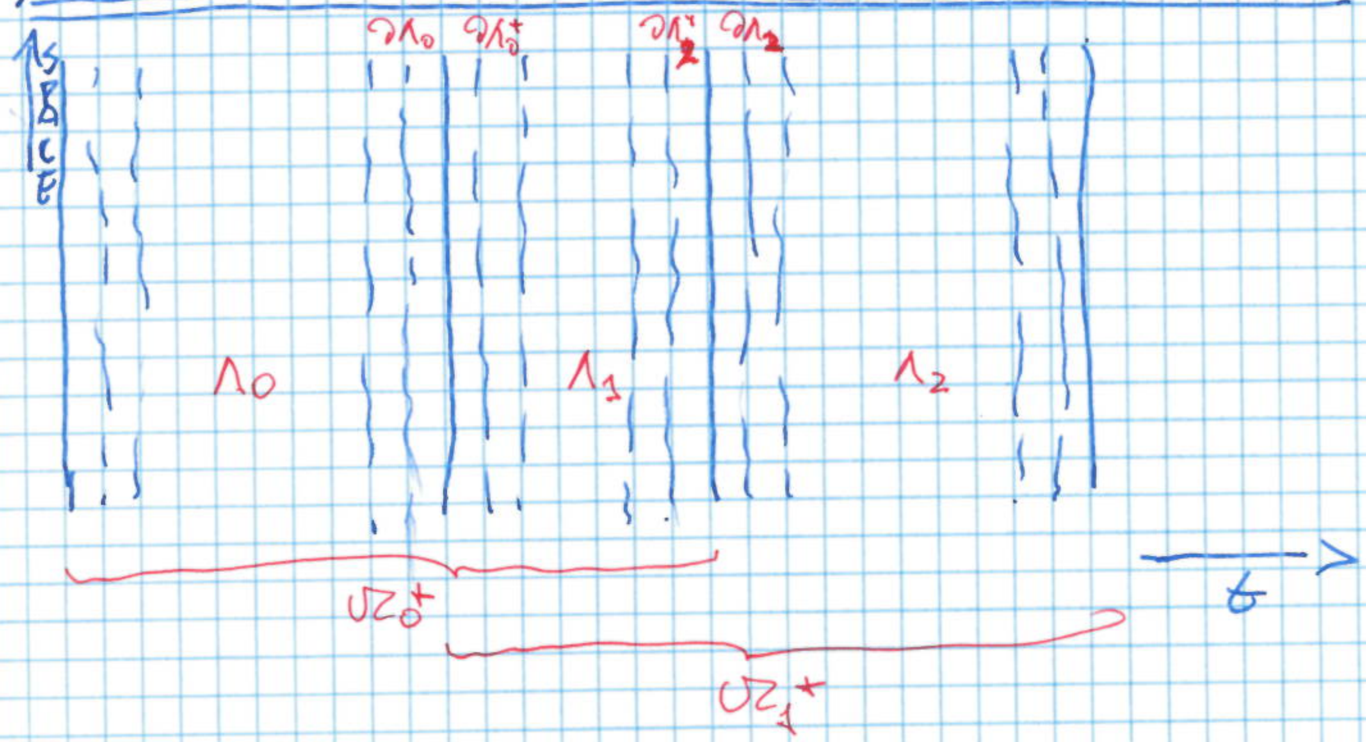
$$\det[Q] = \frac{1}{\det[Q^{-1}]} = c' \int D\phi \, D\phi^+ \exp\{-\phi^+ Q^{-1} \phi\}$$

NOTE: Q^{-1} IS A NON LOCAL FUNCTIONAL OF THE GAUGE FIELD. THE ACTION OF SCALAR FIELDS THUS DEPENDS NON-LOCALLY ON $U_\mu(x)$, AND SO DO THE WICK CONTRACTIONS

CONSEQUENCE = MULTI-LEVEL INTEGRATION NOT APPLICABLE STRAIGHTFORWARDLY

GOAL OF THESE LECTURES = REWRITE THE DETERMINANT
- AS INTEGRAL OVER AUXILIARY SCALAR FIELDS WITH AN ACTION WHICH DEPENDS "LOCALLY" (BLOCK LOCAL) ON THE GAUGE FIELD
- REACH A SIMILAR RESULT FOR WICK CONTRACTIONS TOO.

2) DOMAIN DECOMPOSITION OF THE LATTICE



LATTICE CAN BE DECOMPOSED IN SEVERAL DOMAINS (SUBLATTICES)

- HERE WE WILL CONSIDER ONLY DECOMPOSITIONS ALONG THE TEMPORAL DIRECTION

- NON-OVERLAPPING DOMAIN DECOMPOSITIONS:

  - $\Lambda_0, \Lambda_1, \Lambda_2$
  - $\Gamma = \Lambda_0 \cup \Lambda_2$, $\Gamma^+ = \Lambda_1$ Γ = DISCONNECTED

- OVERLAPPING DOMAIN DECOMPOSITION:

$$\Omega_0^* = \Lambda_0 \cup \Lambda_1 \quad , \quad \Omega_1^* = \Lambda_1 \cup \Lambda_2$$

- PROJECTORS ON QUARK FIELDS SUPPORTED ON Λ_i:

$$[P_{\Lambda_i} \psi](x) = \begin{cases} \psi(x) & x \in \Lambda_i \\ 0 & \text{ELSEWHERE} \end{cases}$$

ANALOGOUS PROJECTORS CAN BE DEFINED FOR INTERNAL AND EXTERNAL BOUNDARIES $P_{\partial\Lambda}, P_{\partial\Lambda^*}, \ldots$

- Q CAN THUS BE WRITTEN IN BLOCK FORM:

$$Q = \begin{pmatrix} Q_{\Lambda_{0,0}} & Q_{\Lambda_{0,1}} & 0 \\ Q_{\Lambda_{1,0}} & Q_{\Lambda_{1,1}} & Q_{\Lambda_{1,2}} \\ 0 & Q_{\Lambda_{2,1}} & Q_{\Lambda_{2,2}} \end{pmatrix}$$

AND ANALOGOUSLY

$$Q_{\Omega_0^+} = \begin{pmatrix} Q_{\Lambda_{0,0}} & Q_{\Lambda_{0,1}} \\ Q_{\Lambda_{1,0}} & Q_{\Lambda_{1,1}} \end{pmatrix}, \quad Q_{\Omega_1^+} = \begin{pmatrix} Q_{\Lambda_{1,1}} & Q_{\Lambda_{1,2}} \\ Q_{\Lambda_{2,1}} & Q_{\Lambda_{2,2}} \end{pmatrix}$$

NOTE: $Q_{\Lambda_{0,0}} \equiv P_{\Lambda_0} Q P_{\Lambda_0}$ SO INSIDE Λ_0 IS DEFINED AS Q, BUT WITH DIRICHLET BOUNDARY CONDITIONS (ON ψ) ON THE EXTERIOR BOUNDARIES OF THE BLOCK.

ANALOGOUSLY FOR $Q_{\Omega_0^+}, \ldots$

$Q_{\Lambda_{0,1}} = P_{\Lambda_0} Q P_{\Lambda_1}, \ldots$

NOTE: $Q_{\Lambda_{0,0}}$ DEPENDS ON THE GAUGE FIELD IN Λ_0 ONLY!! CRUCIAL!

NOTE: FROM NOW ON WILL DROP THE SUBSCRIPT $\Lambda_{i,j}$, I.E. $Q_{\Lambda_{i,j}} \to Q_{ij}$ AND WILL BE CLEAR FROM THE CONTEXT WHICH OPERATOR WE ARE CONSIDERING.

3) INVERSE AND DETERMINANT OF A 2×2 BLOCK MATRIX:

LU DECOMPOSITION

$$Q = \begin{pmatrix} Q_\Gamma & Q_{\partial\Gamma} \\ Q_{\partial\Gamma^+} & Q_{\Gamma^+} \end{pmatrix} = \begin{pmatrix} I & Q_{\partial\Gamma} Q_{\Gamma^+}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_\Gamma & 0 \\ Q_{\partial\Gamma^+} & Q_{\Gamma^+} \end{pmatrix}$$

WHERE THE SCHUR COMPLEMENT IS

$$S_\Gamma \equiv Q_\Gamma - Q_{\partial\Gamma} Q_{\Gamma^+}^{-1} Q_{\partial\Gamma^+}$$

DETERMINANT

$$\det Q = \det S_\Gamma \det Q_{\Gamma^+}$$

INVERSE

$$Q^{-1} = \left(\begin{array}{c|c} S_\Gamma^{-1} & -S_\Gamma^{-1} Q_{\partial\Gamma} Q_{\Gamma^+}^{-1} \\ \hline -Q_{\Gamma^+}^{-1} Q_{\partial\Gamma^+} S_\Gamma^{-1} & Q_{\Gamma^+}^{-1} + Q_{\Gamma^+}^{-1} Q_{\partial\Gamma^+} S_\Gamma^{-1} Q_{\partial\Gamma} Q_{\Gamma^+}^{-1} \end{array}\right)$$

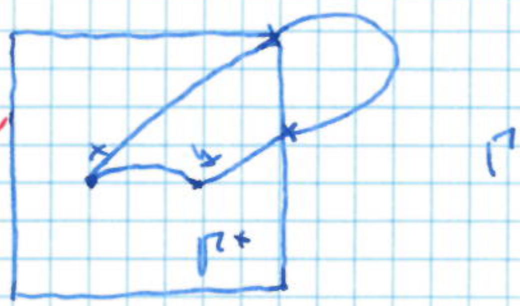
Note: S_Γ^{-1} IS THE EXACT INVERSE OF Q IN Γ

$$P_\Gamma Q^{-1} P_\Gamma = S_\Gamma^{-1}$$

Note: THE 22 ELEMENT IS

$$Q^{-1}(y,x) = Q_{\Gamma^+}^{-1}(y,x) + Q_{\Gamma^+}^{-1}(y,\cdot)\left[Q_{\partial\Gamma^+} S_\Gamma^{-1} Q_{\partial\Gamma}\right](\cdot,\cdot)\, Q_{\Gamma^+}^{-1}(\cdot,x)$$

- FIRST TERM DEPENDS ON GAUGE FIELD IN Γ^+ ONLY
- SECOND TERM ~~CORRECTION~~ HAS AT LEAST EXTRA TWO PROPAGATORS FROM x OR y TO THE BOUNDARY



- THIS FORMALIZES OUR INTUITION THAT THE PROPAGATOR BETWEEN x AND y DEPENDS WEAKLY ON THE GAUGE FIELD ON THE MOON

4) DYNAMICAL INFORMATION FROM NUMERICAL LATTICE QCD:

CONFIGURATION BY CONFIGURATION IN THE RELEVANT ENSEMBLE

$$Q^{-1}(y,x) \propto e^{-\frac{M_\pi}{2}|y-x|}$$

FOR LARGE DISTANCES $|y-x|$

<u>NOTE:</u> IN THE FREE CASE WOULD BE $e^{-m_0|y-x|}$.

THIS OBSERVATION COMBINED WITH THE PREVIOUS BLOCK FORMS OF THE DETERMINANT AND THE QUARK PROPAGATOR LEADS TO OUR GOAL, I.E. TO A FACTORIZATION OF THE GAUGE DEPENDENCE OF THE QUARK DETERMINANT AND WICK CONTRACTIONS, of the type that ~~WHICH~~ CAN BE MANAGED IN NUMERICAL SIMULATIONS (MB-DD-HMC)

IN PARTICULAR IN THE 22 ELEMENT OF PREVIOUS PAGE, THE DEPENDENCE ON THE GAUGE FIELD IN Γ IS SUPPRESSED EXPONENTIALLY WITH THE DISTANCE OF x AND y FROM THE BOUNDARY (MOON), ~~THE~~ (SECOND TERM).
THE FIRST TERM DEPENDS ON THE GAUGE FIELD IN Γ^* ONLY.

5) FACTORIZATION OF $Q^{-1}(y,x)$:

LET US CHOOSE

$$\Gamma = \Lambda_0 \cup \Lambda_2 \quad , \quad \Gamma^{*} = \Lambda_1$$

AND THE THICKNESS Δ OF Λ_1 SO THAT
$M_\pi \Delta \gg 1$ (IN PRACTICE $\Delta \simeq 0.5\,\mathrm{fm}$ MAY BE ENOUGH)

THE SCHUR COMPLEMENT READS

$$S_\Gamma = \begin{pmatrix} Q_{00} - Q_{01} Q_{11}^{-1} Q_{10} & -Q_{01} Q_{11}^{-1} Q_{12} \\ -Q_{21} Q_{11}^{-1} Q_{10} & Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} \end{pmatrix}$$

$$Q_{\Gamma^*} = Q_{11}$$

BY APPLYING THE ~~TWO~~ 2×2 DECOMPOSITION TO
Ω_0^* AND Ω_1^* INDEPENDENTLY

$$P_{\Lambda_0} Q^{-1}_{\Omega_0^*} P_{\Lambda_0} = \left[Q_{00} - Q_{01} Q_{11}^{-1} Q_{10} \right]^{-1}$$

$$P_{\Lambda_2} Q^{-1}_{\Omega_1^*} P_{\Lambda_2} = \left[Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} \right]^{-1}$$

WHICH LEADS TO

$$S_P = \begin{pmatrix} Q_{00} - Q_{01} Q_{11}^{-1} Q_{10} & 0 \\ 0 & Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & P_{\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} \\ P_{\Lambda_2} Q_{\nu z_1^+}^{-1} Q_{10} & 1 \end{pmatrix}}_{\tilde{W}}$$

AGAIN USING INVERSE OF BLOCK MATRIX AGAIN:

$$\tilde{W}^{-1} = \begin{pmatrix} 1 & -P_{\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} \\ -P_{\Lambda_2} Q_{\nu z_1^+}^{-1} Q_{10} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1 - P_{\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} Q_{\nu z_1^+}^{-1} Q_{10}} & 0 \\ 0 & \frac{1}{1 - P_{\Lambda_2} Q_{\nu z_1^+}^{-1} Q_{10} Q_{\nu z_0^+}^{-1} Q_{12}} \end{pmatrix}$$

THEREFORE

$$P_{\Lambda_0} Q^{-1} P_{\Lambda_0} = P_{\Lambda_0} S_{P}^{-1} P_{\Lambda_0} = P_{\Lambda_0} \frac{1}{1 - P_{\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} Q_{\nu z_1^+}^{-1} Q_{10}} Q_{\nu z_0^+}^{-1} P_{\Lambda_0}$$

By EXPANDING DENOMINATOR IN NEUMANN SERIES

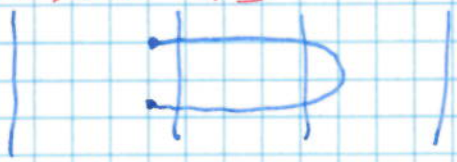
$$\boxed{P_{\Lambda_0} Q^{-1} P_{\Lambda_0}} = P_{\Lambda_0} Q_{\nu z_0^+}^{-1} P_{\Lambda_0} + P_{\Lambda_0} \sum_{m=1}^{\infty} \left[P_{\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} Q_{\nu z_1^+}^{-1} Q_{10} \right]^m Q_{\nu z_0^+}^{-1} P_{\Lambda_0}$$

$$= P_{\Lambda_0} Q_{\nu z_0^+}^{-1} P_{\Lambda_0} + P_{\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} Q_{\nu z_1^+}^{-1} Q_{10} \frac{1}{1-W} Q_{\nu z_0^+}^{-1} P_{\Lambda_0}$$

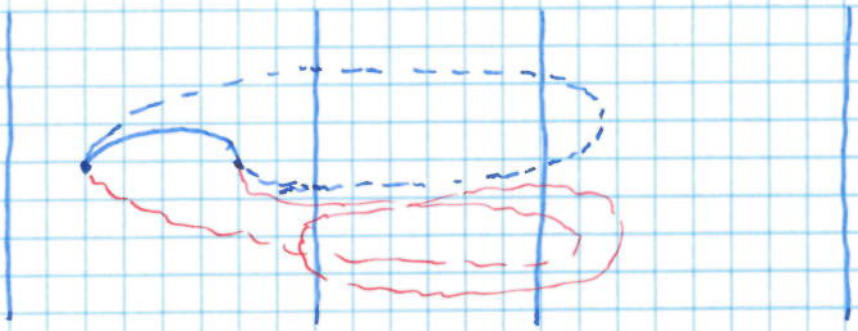
$$W = P_{\partial\Lambda_0} Q_{\nu z_0^+}^{-1} Q_{12} Q_{\nu z_1^+}^{-1} Q_{10}$$

NOTE: BOUNDARY $P_{\partial\Lambda_0}$ TO BOUNDARY $P_{\partial\Lambda_0}$ OPERATOR, SO SUPPRESSED AS

$$w \approx e^{-M_\pi \cdot \Delta}$$



LET US READ THE RESULT



$$P_{\Lambda_0} Q^{-1} P_{\Lambda_0} = P_{\Lambda_0} Q^{-1}_{\Omega_0^+} P_{\Lambda_0} +$$

$$P_{\Lambda_0} Q^{-1}_{\Omega_0^+} Q_{12} Q^{-1}_{\Omega_1^+} Q_{10} \sum_{n=0}^{\infty} \left[P_{\partial\Lambda_0} Q^{-1}_{\Omega_0^+} Q_{12} Q^{-1}_{\Omega_1^+} Q_{10} \right]^n Q^{-1}_{\Omega_0^+} P_{\Lambda_0}$$

ON THE LEFT ~~SWA~~ SCHWARZ ALTERNATING PROCEDURE (SAP) WITH OVERLAPPING DOMAINS Ω_0^+ AND Ω_1^+.

NOTE! THE PROPAGATOR IS WRITTEN AS SERIES OF TERMS, EACH HAVING A FACTORIZED GAUGE-FIELD DEPENDENCE OF INCREASING COMPLEXITY. GOAL!!!

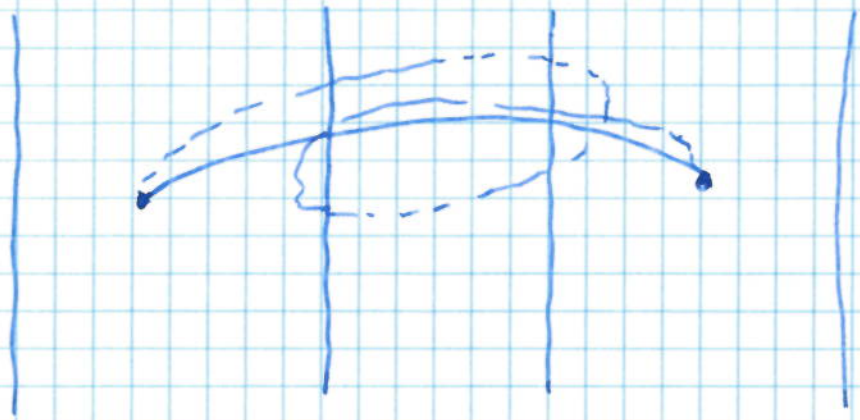
NOTE: THE INDEX $(n+1)$ COUNTS THE NUMBER OF TIMES A QUARK LOOPS FROM $\partial\Lambda_0$ TO $\partial\Lambda_2$ AND BACK TO $\partial\Lambda_0$. THE CONTRIBUTIONS OF THESE PATHS IS SUPPRESSED AS $e^{-(n+1) M_\pi \Delta}$

NOTE: IF WE NEGLECT THE LOOP CONTRIBUTIONS, THE PROPAGATOR DEPENDS ON THE GAUGE FIELD IN Ω_0^+ ONLY, I.E. INDEPEND. OF GAUGE FIELD IN Λ_2 (MOON).

LET US SEE THE OTHER COMPONENT

$$P_{\Lambda_2} Q^{-1} P_{\Lambda_0} = - P_{\Lambda_2} Q^{-1}_{\Omega_1^+} Q_{10} \frac{1}{1-\omega} Q^{-1}_{\Omega_0^+} P_{\Lambda_0}$$

SAME COMMENTS AND SAME LOOP EXPANSION



6) DETERMINANT FACTORIZATION!

AGAIN DO WITH

$$\Gamma = \Lambda_0 \cup \Lambda_2 \quad , \quad \Gamma^+ = \Lambda_1$$

AND BY USING LU

$$\det Q = \det Q_{11} \det S_\Gamma = \frac{1}{\det Q_{11}^{-1}} \; \frac{1}{\det S_\Gamma^{-1}}$$

BUT

$$\frac{1}{\det S_\Gamma^{-1}} = \frac{1}{\det\left\{P_{\Lambda_0} Q^{-1}_{\Omega_0^+} P_{\Lambda_0}\right\} \det\left[P_{\Lambda_2} Q^{-1}_{\Omega_1^+} P_{\Lambda_2}\right]} \det \tilde{w}$$

AND

$$\det \tilde{w} = \det \begin{pmatrix} 1 & P_{\Lambda_0} Q^{-1}_{\Omega_0^+} Q_{12} \\ P_{\Lambda_2} Q^{-1}_{\Omega_1^+} Q_{10} & 1 \end{pmatrix}$$

BY USING THE STANDARD RELATION

$$\det \begin{pmatrix} 1 & B \\ C & 1 \end{pmatrix} = \det (1 - BC)$$

$$\Downarrow$$

$$\boxed{\det \tilde{w} = \det \begin{pmatrix} 1 & P_{\Lambda_0} Q^{-1}_{\Omega_0^+} Q_{12} \\ P_{\Lambda_2} Q^{-1}_{\Omega_1^+} Q_{10} & 1 \end{pmatrix} = \det (1 - w)}$$

FINALLY:

$$\det Q = \frac{1}{\det Q_{11}^{-1} \det\{P_{\Lambda_0} Q^{-1}_{UZ_0^+} P_{\Lambda_0}\} \det\{P_{\Lambda_2} Q^{-1}_{UZ_1^+} P_{\Lambda_2}\} \det\{\frac{1}{1-\omega}\}}$$

DEP. ON U_μ IN Λ_1 ONLY

U_μ IN $\Lambda_0 \cup \Lambda_1$

U_μ IN $\Lambda_1 \cup \Lambda_2$

U_μ EVERYWHERE

NOTE: FOR FIRST THREE DETERMINANTS GOAL REACHED - THE CONTRIBUTION FROM QUARK PATHS LOOPING AROUND THE BOUNDARIES STILL DEPENDS ON THE GAUGE FIELD EVERYWHERE.
BUT IT IS SUPPRESSED AS $e^{-M_\pi \Delta}$!!!

LET US EXPAND IN NUMBER OF LOOPS AGAIN

$$\frac{1}{\det\{\frac{1}{1-\omega}\}} = \frac{1}{\det\{\sum_{n=0}^{\infty} \omega^n\}} = \frac{\det\{1-\omega^{N+1}\}}{\det\{\sum_{n=0}^{N} \omega^n\}}$$

WITH N EVEN. THIS IS BECAUSE

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{N} x^n + \sum_{n=N+1}^{\infty} x^n = \sum_{n=0}^{N} x^n + x^{N+1} \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{N} x^n + \frac{x^{N+1}}{1-x}$$

$$\Downarrow$$

$$\frac{1-x^{N+1}}{1-x} = \sum_{n=0}^{N} x^n$$

c.v.d.

THE ROOTS OF THE POLYNOMIAL $\sum_{n=0}^{N} w^n$ ARE

$$u_k = e^{i\frac{2\pi k}{N+1}} \qquad k = 1, \ldots, N$$

AND COME IN COMPLEX CONJUGATE PAIRS. THEREFORE

$$\det\left\{\sum_{n=0}^{N} w^n\right\} = \prod_{k=1}^{N/2}\left\{\det\left[(u_k^+ - w)(u_k - w)\right]\right\}$$

SINCE w IS SIMILAR TO w^+ (SEE EXERCISES)

$$w = S w^+ S^{-1}$$

AND THEREFORE

$$\det\left\{\sum_{n=0}^{N} w^n\right\} = \prod_{k=1}^{N/2}\left\{\det\left[(u_k - w)^+(u_k - w)\right]\right.$$

WE CAN NOW PERFORM THE REVERSE SUBSTITUTION!

$$W_z \equiv \begin{pmatrix} z\, P_{\theta n_0} & P_{\theta n_0} Q^{-1}_{vz_0^+} Q_{12} \\ P_{\theta n_2} Q^{-1}_{vz_1^+} Q_{10} & z\, P_{\theta n_2} \end{pmatrix}$$

TO FINALLY OBTAIN

$$\prod_{k=1}^{N/2}\left\{\det\left[(u_k - w)^+(u_k - w)\right]\right. = \prod_{k=1}^{N/2} \det\left[W^+_{\sqrt{u_k}} W_{\sqrt{u_k}}\right]$$

THE FINAL RESULT READS:

$$\det Q = \frac{1}{\det Q_{11}' \det\{P_{\Lambda_0} Q^{-1}_{\nu z_0^+} P_{\Lambda_0}\} \det\{P_{\Lambda_2} Q^{-1}_{\nu z_1^+} P_{\Lambda_2}\}} \times \frac{\det(1 - W^{N+1})}{\prod_{k=1}^{N/2} \det[W^+_{\sqrt{u_k}} W_{\sqrt{u_k}}]} \qquad (\alpha)$$

NOTE: W SMALL, VERY GOOD APPROXIMATION ALREADY WITH $N \simeq 5-10$

7) REPRESENTATION WITH AUXILIARY SCALAR FIELDS

FIRST THREE DETS IN (α) REPRESENTED AS INTEGRAL ON PSEUDOFERMIONS AS USUAL (SIMPLEST CASE ONE PSEUDOFERMION), E.G. FOR $N_f = 2$

$$\frac{1}{\det\{P_{\Lambda_0} Q^{-1}_{\nu z_0^+} P_{\Lambda_0}\}^2} = c' \int D\phi_0 D\phi_0^+ e^{-|P_{\Lambda_0} Q^{-1}_{\nu z_0^+} \phi_0|^2}$$

- LOOP CORRECTION WITH $N/2$ MULTIBOSONS

$$\frac{1}{\prod_{k=1}^{N} \det[W^+_{\sqrt{u_k}} W_{\sqrt{u_k}}]} = c' \prod_{k=1}^{N} \left\{ \int D\chi_k D\chi_k^+ e^{-|W_{\sqrt{u_k}} \chi_k|^2} \right\}$$

$$\chi_k = P_{\partial\Lambda_0} \eta_k + P_{\partial\Lambda_2} \xi_k$$

SO η_k AND ζ_k ARE SCALAR FIELDS LIVING ON THE THICK TIME-SLICES $\partial\Lambda_0$ AND $\partial\Lambda_2$ RESPECT.

8) MULTILEVEL INTEGRATION WITH FERMIONS:

$$\langle O \rangle = \frac{\langle O\, \tilde{W}_N \rangle_N}{\langle \tilde{W}_N \rangle_N} = \frac{\langle O_{FACT} \rangle_N}{\langle W_N \rangle_N} + \frac{\langle O\tilde{W}_N - O_{FACT} \rangle_N}{\langle W_N \rangle_N}$$

$$W_N = \det\left(1 - w^{N+1}\right) = \frac{\int d\zeta\, d\zeta^+ \, e^{-\left|(1-w^{N+1})^{-1}\zeta\right|^2}}{\int d\zeta\, d\zeta^+ \, e^{-\zeta^+\zeta}}$$

SUMMARY OF VARIOUS CONTRIBUTIONS!

1) PATHS WITHOUT QUARK LOOPS AROUND $\partial\Lambda_0$ AND $\partial\Lambda_2$ HAVE WEIGHTS WITH FACTORIZED GAUGE-FIELD DEPENDENCE $\Rightarrow$ STANDAR PSEUDOFERMIONS IN Ω_0^* AND Ω_1^*

2) PATHS WITH 1 UP TO N LOOPS AROUND $\partial\Lambda_0$ AND $\partial\Lambda_2$ HAVE WEIGHTS REPRESENTED BY MULTIBOSON FIELDS

3) PATHS WITH HIGHER LOOPS AROUND $\partial\Lambda_0$ AND $\partial\Lambda_2$ HAVE WEIGHTS REPRESENTED BY THE REWEIGHTING FACTOR (NEGLIGIBLE IN PRESENT DAY SIMULATIONS)