Broken symmetries and lattice gauge theory (V): energy-momentum tensor and trace anomaly

Leonardo Giusti

University of Milano-Bicocca

Non-perturbative renormalization of $T_{\mu\nu}$

- On the lattice the Poincaré group is broken down to a discrete group and standard discretizations of $T_{\mu\nu}$ acquire finite ultraviolet renormalizations

- We focus on the SU(3) Yang–Mills. The analysis applies to other theories as well

$$T^{\rm R}_{\mu\nu} = Z_T \left\{ T^{[1]}_{\mu\nu} + z_T\, T^{[3]}_{\mu\nu} + z_S \left[T^{[2]}_{\mu\nu} - \langle T^{[2]}_{\mu\nu}\rangle_0 \right] \right\} .$$

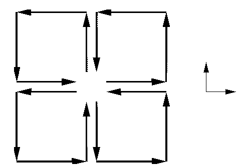
$$T^{[1]}_{\mu\nu} = (1-\delta_{\mu\nu})\frac{1}{g_0^2}\left\{ F^a_{\mu\alpha} F^a_{\nu\alpha} \right\}$$

$$T^{[2]}_{\mu\nu} = \delta_{\mu\nu}\, \frac{1}{4 g_0^2}\, F^a_{\alpha\beta} F^a_{\alpha\beta}$$

$$T^{[3]}_{\mu\nu} = \delta_{\mu\nu}\frac{1}{g_0^2}\left\{ F^a_{\mu\alpha} F^a_{\mu\alpha} - \frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} \right\}$$

where

$$F^a_{\mu\nu}(x) = -\frac{i}{4a^2}{\rm Tr}\Big\{\Big[Q_{\mu\nu}(x) - Q_{\nu\mu}(x)\Big] T^a\Big\}\, , \qquad Q_{\mu\nu}(x) = \sum$$

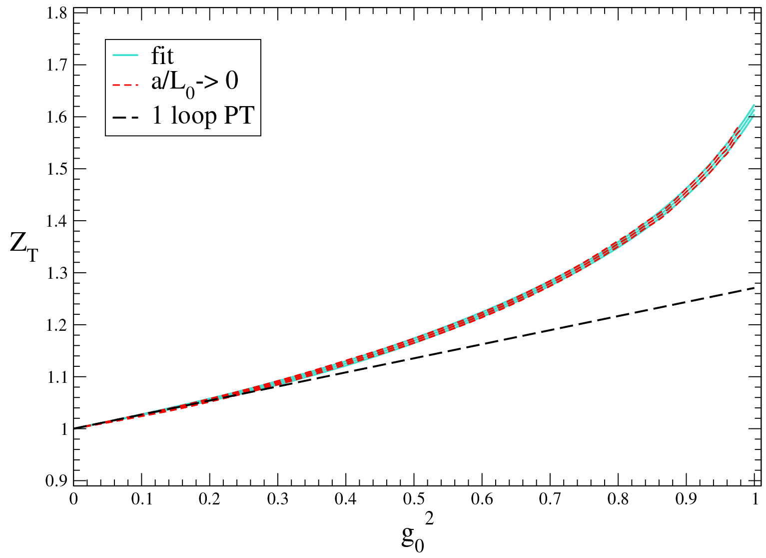


- The continuum relation

$$\langle T_{0k}\rangle_{\boldsymbol{\xi}} = \frac{1}{L_0}\lim_{V\to\infty}\frac{1}{V}\frac{\partial}{\partial\xi_k}\ln Z(L_0,\boldsymbol{\xi})$$

can be imposed on the lattice to fix Z_T

$$Z_T(g_0^2) = -\frac{\Delta f}{\Delta\xi_k}\,\frac{1}{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}$$



where the derivative in the shift is discretized by the symmetric finite difference

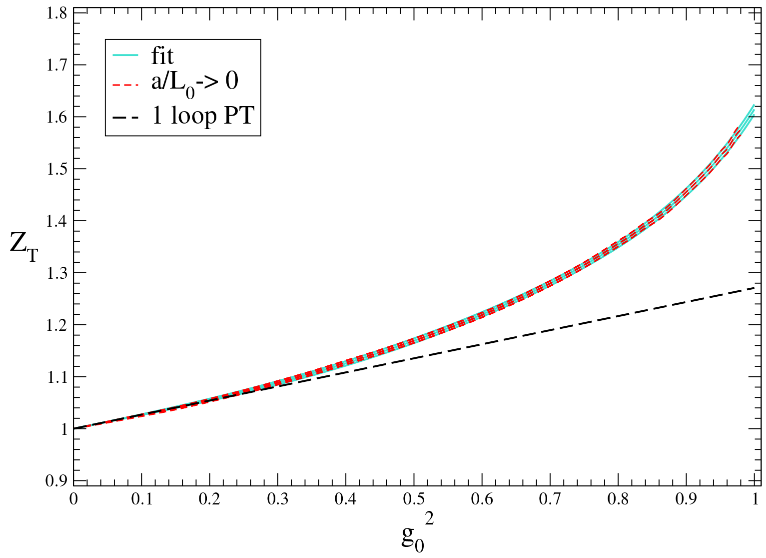
$$\frac{\Delta f}{\Delta\xi_k} = \frac{1}{2aV}\ln\Big[\frac{Z(L_0,\boldsymbol{\xi}-a\hat{k}/L_0)}{Z(L_0,\boldsymbol{\xi}+a\hat{k}/L_0)}\Big]$$

- The final results for $Z_T(g_0^2)$ are well represented by

$$Z_T(g_0^2) = \frac{1-0.4457\,g_0^2}{1-0.7165\,g_0^2} - 0.2543\,g_0^4 + 0.4357\,g_0^6 - 0.5221\,g_0^8$$

with the error that varies from 0.4% up 0.7% in the range $0 \leq g_0^2 \leq 1$

- The continuum relation



$$\langle T_{0k}\rangle_{\boldsymbol{\xi}} = \frac{1}{L_0}\lim_{V\to\infty}\frac{1}{V}\frac{\partial}{\partial\xi_k}\ln Z(L_0,\boldsymbol{\xi})$$

can be imposed on the lattice to fix Z_T

$$Z_T(g_0^2) = -\frac{\Delta f}{\Delta\xi_k}\,\frac{1}{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}$$

where the derivative in the shift is discretized by the symmetric finite difference

$$\frac{\Delta f}{\Delta\xi_k} = \frac{1}{2aV}\ln\left[\frac{Z(L_0,\boldsymbol{\xi}-a\hat{k}/L_0)}{Z(L_0,\boldsymbol{\xi}+a\hat{k}/L_0)}\right]$$

- Within statistical errors, the non-perturbative determination starts to deviate significantly from the one-loop result [Caracciolo et al. 88, 90]

$$Z_T(g_0^2) = 1 + 0.27076\,g_0^2$$

already at $g_0^2 \sim 0.25$

The triplet renormalization constant z_T

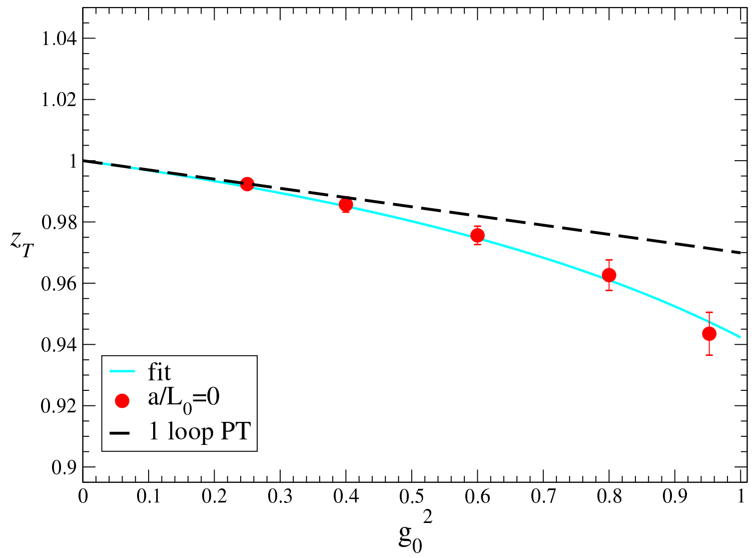
- The continuum relation

$$\langle T_{0k}\rangle_{\boldsymbol{\xi}} = \frac{\xi_k}{1-\xi_k^2}\left\{\langle T_{00}\rangle_{\boldsymbol{\xi}} - \langle T_{kk}\rangle_{\boldsymbol{\xi}}\right\}$$

is enforced on the lattice to determine z_T

$$z_T(g_0^2) = \frac{1-\xi_k^2}{\xi_k}\frac{\langle T_{0k}^{[1]}\rangle_{\boldsymbol{\xi}}}{\langle T_{00}^{[3]}\rangle_{\boldsymbol{\xi}} - \langle T_{kk}^{[3]}\rangle_{\boldsymbol{\xi}}}$$

with the condition $\frac{L\,\xi_k}{L_0(1+\xi_k^2)} = q \in \mathbb{Z}$



- The results for $z_T(g_0^2)$ are well represented by

$$z_T(g_0^2) = \frac{1-0.5090\,g_0^2}{1-0.4789\,g_0^2}$$

where the error grows linearly from 0.15% to 0.75% in the interval $0 \leq g_0^2 \leq 1$

The triplet renormalization constant z_T

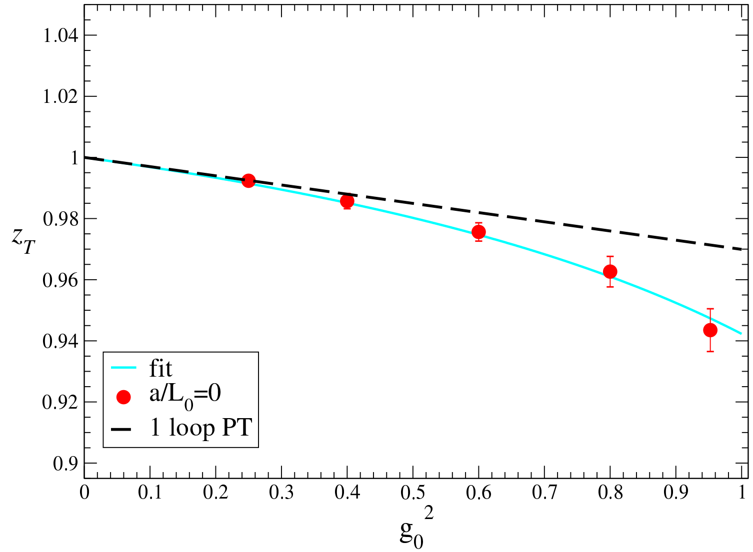
- The continuum relation

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{\xi_k}{1-\xi_k^2} \left\{ \langle T_{00} \rangle_{\boldsymbol{\xi}} - \langle T_{kk} \rangle_{\boldsymbol{\xi}} \right\}$$

is enforced on the lattice to determine z_T

$$z_T(g_0^2) = \frac{1-\xi_k^2}{\xi_k} \frac{\langle T_{0k}^{[1]} \rangle_{\boldsymbol{\xi}}}{\langle T_{00}^{[3]} \rangle_{\boldsymbol{\xi}} - \langle T_{kk}^{[3]} \rangle_{\boldsymbol{\xi}}}$$

with the condition $\frac{L\,\xi_k}{L_0(1+\xi_k^2)} = q \in \mathbb{Z}$



- Within statistical errors, the non-perturbative determination starts to deviate significantly from the one-loop result [Caracciolo et al. 88, 90]

$$z_T(g_0^2) = 1 - 0.03008\, g_0^2$$

already at $g_0^2 \sim 0.4$

The singlet renormalization constant z_S

- The continuum relation

$$\frac{\partial}{\partial \xi_k} \langle T_{\mu\mu} \rangle_{\boldsymbol{\xi}} = \frac{1}{(1+\xi^2)^2} \frac{\partial}{\partial \xi_k} \left[\frac{(1+\xi^2)^3}{\xi_k} \langle T_{0k} \rangle_{\boldsymbol{\xi}} \right]$$

  is enforced on the lattice to determine z_S
  by discretizing the derivative in ξ_k

- Only the the one-loop result is known so far [Caracciolo et al. 88, 90]

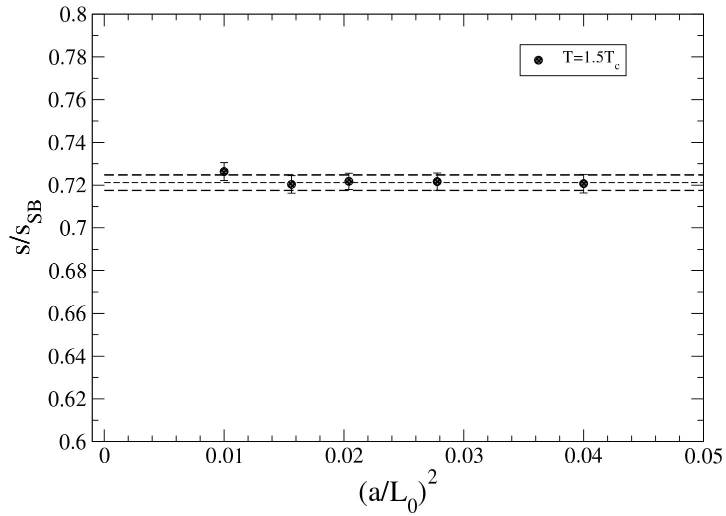
$$z_S(g_0^2) = \frac{1}{(4\pi^2)} \frac{11}{6} N_c \, g_0^2$$

Entropy density in the continuum

- Entropy density obtained by extrapolating

$$\frac{s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\frac{(1+\xi^2)}{\xi_k}\frac{Z_T\,\langle T^{[1]}_{0k}\rangle_{\boldsymbol{\xi}}}{T^4}$$

  to the continuum limit

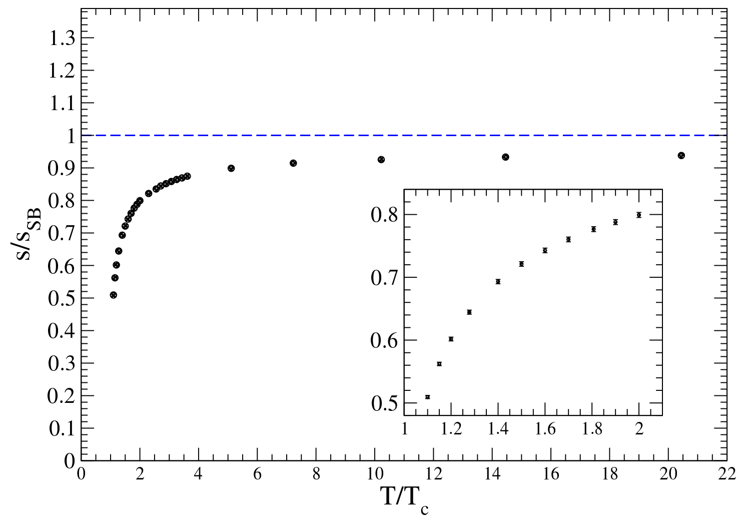


Entropy density in the continuum

- Entropy density obtained by extrapolating

$$\frac{s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\frac{(1+\xi^2)}{\xi_k}\frac{Z_T\,\langle T^{[1]}_{0k}\rangle_{\boldsymbol{\xi}}}{T^4}$$

to the continuum limit



- Precision of $\sim 0.5\%$ for all points

- The computation at temperatures up to $T \sim 200\,T_c$ is completed. The analysis of the data is in progress

- At $T \sim 20\,T_c$ the entropy still differs from the Stefan-Boltzmann value by roughly 5%

- When matching with perturbation theory, the series has oscillating coeffs.
  At $T \sim 20\,T_c$, the $O(g^6)$ is roughly 40% of total correction with respect to SB

Summary of today lecture

- Lorentz invariance implies a great degree of redundancy in defining a relativistic thermal theory in the Euclidean path-integral formalism

- In the thermodynamic limit, the orientation of the compact periodic direction with respect to the coordinate axes can be chosen at will and only its length is relevant

$$f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right) = -\lim_{V\to\infty} \frac{1}{L_0 V} \ln Z(L_0, \boldsymbol{\xi})$$

- The redundancy in the description implies that the total energy and momentum distributions in the canonical ensemble are related

- For a finite-size system, the lengths of the box dimensions break this invariance. Being a soft breaking, however, interesting exact Ward Identities survive

- As in the standard case, if the lightest screening mass $M \neq 0$, leading finite-size corrections are exponentially small in (ML)

- When the theory is regularized on a lattice, the overall orientation of the periodic directions with respect to the lattice coordinate system affects renormalized observables at the level of lattice artifacts

- As the cutoff is removed, the artifacts are suppressed by a power of the spacing

- The flexibility in the lattice formulation added by the introduction of a triplet $\boldsymbol{\xi}$ of (renormalized) parameters has interesting consequences:

  * WIs to renormalize non-perturbatively $T_{\mu\nu}$ (including the trace anomaly)

  * Simpler ways to compute thermodynamic potentials

$$s = -\frac{Z_T L_0 (1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k} \langle T_{0k} \rangle_{\boldsymbol{\xi}}$$

Final considerations

- Lattice gauge theory is a theoretical femtoscope to explore strong dynamics non-perturbatively. Its lenses are made of quantum field theory, numerical techniques and computers

- It allows us to look at quantities not accessible to experiments that may unveil the underlying mechanisms of non-perturbative strong dynamics

- A large variety of physics applications: QCD, flavour physics, beyond Standard Model physics, etc.

- Thanks to the recent extraordinary conceptual, technical and algorithmic advances the chiral regime of the theories is becoming accessible

- The femtoscope, however, is still rather crude. There is continuous conceptual and technical progress to empower it