LECTURE 5: ENERGY-MOMENTUM TENSOR AND TRACE ANOMALY

1) ENERGY MOMENTUM TENSOR

BY DEFINING TRANSLATIONS AS

$$x'_\mu = x_\mu - \varepsilon_\mu \;;\; \phi'(x') = \phi(x) \;;\; \delta\phi = \phi - \phi'$$

IF THE ACTION IS TRANS. INVARIANT

$$\delta S = 0 \Rightarrow \boxed{\langle \partial_\mu T_{\mu\nu}(x)\, \mathcal{O}(0)\rangle = 0 \quad x \neq 0}$$

$T_{\mu\nu} \equiv$ ENERGY-MOMENTUM TENSOR

IF WE DEFINE

$$\overline{T}_{\mu\nu}(x_0) = \int d^3x\, T_{\mu\nu}(x)$$

THEN BY TAKING INTO ACCOUNT MINKOWSKI-EUCLIDEAN ROTATION

$$\langle \overline{T}_{00}\rangle = -E \;;\; \langle \overline{T}_{11}\rangle = pV \;;\; \hat{P}_i \longrightarrow -i\,\overline{T}_{0i}$$

- IF THE THEORY IS SCALE INVARIANT

$$x' = \lambda x \;;\; \phi'(x') = \lambda^D \phi(x')$$ D = CANONICAL DIMENSION OF FIELD

THEN THE DILATATION CURRENT IS CONSERVED

$$D_\mu = T_{\mu\nu} x_\nu \Rightarrow \partial_\mu D_\mu = T_{\mu\mu} = 0$$

SO SCALE INVARIANCE IMPLIES $\boxed{T_{\mu\mu} = 0}$

NOTE:

NOT EASY TO DEFINE $T_{\mu\nu}$ NON-PERTURBATIVELY (AND THEREFORE THE TRACE ANOMALY) ONLY RECENTLY SOLUTION WAS FOUND

2) FAST INTRODUCTION TO THERMAL FIELD THEORY

$$Z(L_0) = \mathrm{Tr}\left[e^{-L_0 \hat{H}} \right] , \quad T = \frac{1}{L_0} , \quad V = L^3$$

$$\hookrightarrow T\frac{\partial}{\partial T} = -L_0 \frac{\partial}{\partial L_0}$$

WE WILL BE INTERESTED IN

$$f \equiv -\frac{1}{L_0 V} \ln Z(L_0) \qquad \text{FREE ENERGY}$$

$$\boxed{e \equiv -\frac{1}{V} \frac{\partial}{\partial L_0} \ln Z(L_0)} \qquad \text{ENERGY DENSITY}$$

$$s = -\frac{L_0^2}{V} \frac{\partial}{\partial L_0} \left\{ \frac{1}{L_0} \ln Z(L_0) \right\} \qquad \text{ENTROPY DENSITY}$$

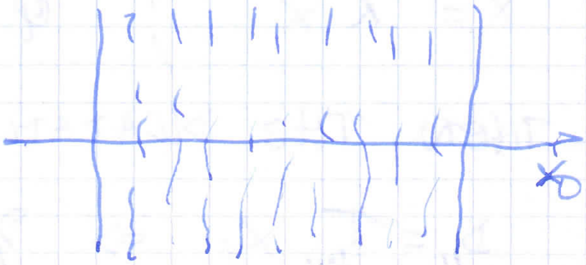
IN THE THERMODYNAMIC LIMIT ($p \equiv$ PRESSURE)

$$p = -f , \quad s = L_0 (e + p) , \quad s = -L_0^2 \frac{\partial}{\partial L_0} p$$

IN PATH INTEGRAL FORMALISM THERMAL THEORY IS DEFINED WITH PERIODIC (ANTIPERIODIC) BOUNDARY CONDITIONS IN TIME

$$\phi(x) = \phi(x + V_{pbc}\, m) \qquad m \in \mathbb{Z}^4$$

$$V_{pbc} = \begin{pmatrix} L_0 & & & 0 \\ & L_1 & & \\ & & L_2 & \\ 0 & & & L_3 \end{pmatrix}$$



$V \equiv$ PERIODICITY MATRIX

IF THEORY IS SCALE INVARIANT, T IS THE ONLY SCALE IN THE PROBLEM

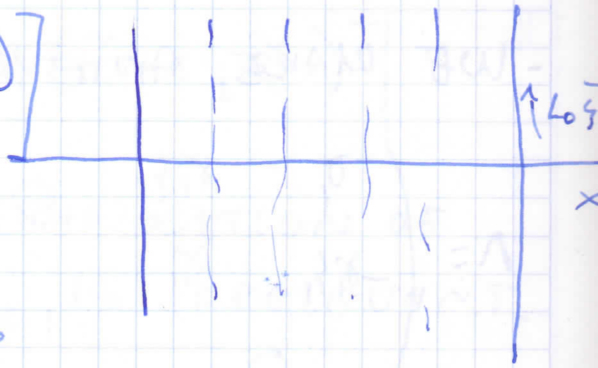
$p = c \frac{1}{L_0^4}$, $\quad e = -3c \frac{1}{L_0^4}$

AND THEREFORE

$\langle T_{\mu\mu} \rangle = -e + 3p = 0 \Rightarrow \boxed{\langle T_{\mu\nu} \rangle = 0}$ C AS EXPE FROM WI

3) PARTITION FUNCTION WITH SHIFTED BOUNDARY CONDITIONS

$$Z(L_0, \vec{\xi}) = \mathrm{Tr}\left[e^{-L_0(\hat{H} - i \vec{\xi} \hat{\vec{p}})} \right]$$



WHICH IN THE PATH INTEGRAL FORMALISM IS WRITTEN AS

$$\phi(x) = \phi(x + V_{abc}\, n) \qquad n \in Z^4$$

$$V_{abc} = \begin{pmatrix} L_0 & & & 0 \\ L_0\xi_1 & L_1 & & \\ L_0\xi_2 & 0 & L_2 & \\ L_0\xi_3 & 0 & 0 & L_3 \end{pmatrix}$$

$$f(L_0, \vec{\xi}) = -\frac{1}{L_0 V} \ln\left(Z(L_0, \vec{\xi})\right)$$

IN INFINITE VOLUME IT HOLDS

$$\boxed{f(L_0,\vec{\xi}) = f(L_0\sqrt{1+\vec{\xi}^2},0)}$$

COMMENT: A THERMAL THEORY CAN BE IMPLEMENTED IN THE PATH INTEGRAL WITH SHIFTED BOUNDARY CONDITIONS. TEMPERATURE IS

$$T=\frac{1}{L_0\sqrt{1+\vec{\xi}^2}}$$

<u>DIM</u>:

FOR SIMPLICITY LET US CHOOSE $\xi_1 \neq 0$. WE WILL THEN USE ROTATIONAL SIMMETRY TO GENERALIZE THE RESULT

- WE MAKE AN EUCLIDEAN BOOST

$$\Lambda = \begin{pmatrix} \gamma_1 & \gamma_1\xi_1 & & 0 \\ -\gamma_1\xi_1 & \gamma_1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad \gamma_1 = \frac{1}{\sqrt{1+\xi_1^2}}$$

$x'=\Lambda x$; $\phi^\Lambda(x') = U(\Lambda)\phi(x)$; $D\phi^\Lambda = D\phi$

WHERE MEASURE IS INVARIANT UNDER LORENTZ SYMMETRY

<u>NOTE</u>: TEMPERATURE BREAKS EUCLIDEAN LORENTZ SYMMETRY (SO(4))

THE FORCE OF ACTION IS INVARIANT BUT

$$V_{sbc} \to V'_{sbc} = \alpha V_{sbcc}$$

(exactly as for spurious breaking of planar mass

$$V'_{sbc} = \begin{pmatrix} L_0/\gamma_1 & L_1 \gamma_1 \xi_1 & 0 & 0 \\ 0 & L_1 \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- WE CAN INTERPRET DIRECTION 1 AS FICTITIOUS TEMPERATURE

$$Z(L_0, \vec{\xi}) = \mathrm{Tr}\left[e^{-L_1 \gamma_1 (\tilde{H} + i \xi_1 \tilde{P}_0)} \right]$$

FOR $L_1 \to \infty$ THE INV. UNDER TRANSLATION OF VACUUM $\Rightarrow$ 21 TERM IN V'_{sbc} IS IRRELEVANT.

SO FOR $V \to \infty$

$$Z(V) = Z(V'') \qquad Z(V'') = \begin{pmatrix} L_0/\gamma_1 & & & 0 \\ & L_1 \gamma_1 & & \\ & & L_2 & \\ 0 & & & L_3 \end{pmatrix}$$

C.V.D.

NOTE: f DEPENDS ON L_0 (SOFT BREAKING) AND $\vec{\xi}$ ONLY THROUGH THE INVARIANT COMBINATION $L_0 \sqrt{1+\xi^2}$

NOTE: NOTHING UNEXPECTED. IF WE ~~TAKE~~ DESCRIBE THE THERMAL SYSTEM IN A MOVING FRAME

$$Z = \mathrm{TR}\left[e^{-L_0 (\hat{H} - \vec{v}\,\vec{P})} \right]$$

IF WE DEFINE $v = i\xi$ WE GET OUR PARTITION FUNCTION.

- MOREOVER IN THE MOVING FRAME

$$T' = T_0 \gamma$$

4) WI_1 AT $\xi = 0$

BY REMEMBERING THAT

$$e^{-L_0 V f} = \mathrm{TR}\left[e^{-L_0 (\hat{H} - i\vec{\xi}\,\vec{P})} \right]$$

$$\Downarrow$$

$$\frac{df}{d\xi_i} = -\frac{i}{V} \langle \hat{P}_i \rangle \quad ; \quad \frac{d^2 f}{d\xi_i d\xi_j} = \frac{L_0}{V} \langle \hat{P}_i \hat{P}_j \rangle$$

IF WE NOW USE THE INVARIANCE

$$f(L_0 \sqrt{1+\vec{\xi}^2}, 0) = f(L_0, \vec{\xi}) \quad ; \quad \beta = L_0 \sqrt{1+\vec{\xi}^2}$$

$$\Downarrow$$

$$\frac{\partial}{\partial \xi_i} f(L_0 \sqrt{1+\vec{\xi}^2}) = \frac{d}{d\beta} f \, \frac{d\beta}{d\xi_i} \quad ; \quad \frac{d\beta}{d\xi_i} = \frac{L_0 \xi_i}{\sqrt{1+\vec{\xi}^2}}$$

BY ITERATING IT IS EASY TO SEE THAT

$$\frac{\partial^2}{\partial \xi_i^2} f\left(L_0\sqrt{1+\vec{\xi}^2}\right)\Big|_{\vec{\xi}=0} = \frac{df}{d\beta}\cdot L_0$$

$$\boxed{\frac{1}{V}\langle \hat{P}_i \hat{P}_i\rangle\Big|_{\xi=0} = \frac{df}{d\beta}\Big|_{\xi=0}}$$

or

BUT

$$\frac{\partial f}{\partial L_0}\Big|_{\xi=0} = \frac{1}{L_0 V}\langle\hat{H}\rangle\Big|_{\xi=0} - \frac{1}{L_0} f\Big|_{\xi=0}$$

AND THE INVARIANCE SAYS

$$\frac{\partial f}{\partial L_0}\Big|_{\xi=0} = \frac{df}{d\beta}\Big|_{\xi=0}$$

SO:

$$\boxed{\frac{1}{V}\langle \hat{P}_i \hat{P}_i\rangle\Big|_{\xi=0} = \frac{1}{L_0}\left\{\frac{\langle\hat{H}\rangle}{V} - f\right\}_{\xi=0}}$$

BY REPLACING THE FIELD EXPRESSIONS

$$\boxed{L_0\langle \overline{T}_{01}(x_0)\, T_{01}(0)\rangle = \langle T_{00}\rangle - \langle T_{11}\rangle}$$

NOTE: SINCE T_{0i} AND $T_{00}-T_{ii}$ BELONG TO THE SAME MONOT (TWO INDEX SIMMETRIC TRACELESS OPERATOR) WHICH MIX UNDER RENORMALIZATION ONLY WITH ITSELF

$$Z_T = 1$$

5) EXPRESSION FOR ENTROPY DENSITY AT $\xi \neq 0$

$$\frac{\partial f}{\partial \xi_i} = -\langle T_{0i} \rangle_\xi \quad ; \quad \beta = L_0 \sqrt{1+\vec{\xi}^2}$$

$$\frac{\partial f}{\partial \xi_i} = \frac{df}{d\beta} \cdot \frac{d\beta}{d\xi_i} \quad ; \quad \frac{d\beta}{d\xi_i} = \frac{L_0\, \xi_i}{\sqrt{1+\vec{\xi}^2}}$$

BUT FROM THE VERY BEGINNING (point 2)

$$s = \beta^2 \frac{d}{d\beta} f$$

SO:

$$s = L_0^2 (1+\vec{\xi}^2) \left(-\langle T_{0i} \rangle_\xi \right) \frac{\sqrt{1+\vec{\xi}^2}}{L_0\, \xi_i}$$

$$\boxed{s = -\frac{L_0}{\gamma^3} \frac{\langle T_{0i} \rangle_\xi}{\xi_i}}$$

ok

6) WIs FOR $\xi \neq 0$ FOR NON-SINGLET

WE START FROM POINT 2 BUT WITH A MODIFIED PERIODICITY MATRIX

$$V = \begin{pmatrix} L_0 \gamma_1 & & & \\ L_0 \xi_1 \gamma_1 & L_1 & 0 & \\ L_0 \xi_2 & & L_2 & \\ L_0 \xi_3 & 0 & & L_3 \end{pmatrix} \quad ; \quad \gamma_1 = \frac{1}{\sqrt{1+\xi_1^2}}$$

IF WE MAKE THE USUAL ROTATION

$$\Lambda = \begin{pmatrix} \gamma_1 & \gamma_1 \xi_1 & 0 & \\ -\gamma_1 \xi_1 & \gamma_1 & & \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}$$

WE ARRIVE TO

$$V' = \Lambda V = \begin{pmatrix} L_0 & \gamma_1 L_1 \xi_1 & & 0 \\ 0 & \gamma_1 L_1 & & \\ L_0 \xi_2 & & L_2 & \\ L_0 \xi_3 & 0 & & L_3 \end{pmatrix}$$

AND

$$Z(V) = Z(V') \Rightarrow \frac{\partial}{\partial \xi_1} Z(V) = \frac{\partial}{\partial \xi_1} Z(V')$$

BY PERFORMING THE DERIVATIVES

$$\boxed{\frac{1}{Z(V)} \frac{\partial}{\partial \xi_1} Z(V) = L_0 L_1 L_2 L_3 \, \gamma_1^3 \left\{ \langle T_{01} \rangle_V - \xi_1 \langle T_{00} \rangle_V \right\}}$$

IF WE DO THE SAME FOR V'

$$\frac{1}{Z(V')}\frac{\partial}{\partial \xi_1} Z(V') = L_0 L_1 L_2 L_3 \gamma_1^3 \left\{ \langle T_{01} \rangle_{V'} - \xi_1 \langle T_{11} \rangle_{V'} \right\}$$

$$\Downarrow$$

$$\boxed{\langle T_{01} \rangle_V - \xi_1 \langle T_{00} \rangle_V = \langle T_{01} \rangle_{V'} - \xi_1 \langle T_{11} \rangle_{V'}} \quad \text{OK}$$

(7.1)

BUT:

$$\langle T_{\mu\nu}(x) \rangle_{V'} = \Lambda_{\mu\rho} \Lambda_{\nu\sigma} \langle T_{\rho\sigma}(x) \rangle_V$$

BY DOING THE ALGEBRA FOR $\langle T_{11} \rangle_{V'}$ AND $\langle T_{01} \rangle_{V'}$ WE GET

$$\langle T_{11} \rangle_{V'} = \gamma_1^2 \xi_1^2 \langle T_{00} \rangle_V + \gamma_1^2 \langle T_{11} \rangle_V - 2\gamma_1^2 \xi_1 \langle T_{01} \rangle_V$$

BY INSERTING IN (7.1) THIS EQUATION

$$\langle T_{01} \rangle_V - \frac{\{1+\xi_1^2\}}{1-\xi_1^2} \langle T_{01} \rangle_{V'} = \frac{\xi_1}{1-\xi_1^2} \left\{ \langle T_{00} \rangle_V - \langle T_{11} \rangle_V \right\}$$

BY TAKING $L_1, L_2, L_3 \to \infty \Rightarrow \langle T_{01} \rangle_{V'} \to 0$

$$\boxed{\langle T_{0k} \rangle_\xi = \frac{\xi_k}{1-\xi_k^2} \left\{ \langle T_{00} \rangle_\xi - \langle T_{kk} \rangle_\xi \right\}}$$

$\hookrightarrow$ NO SUMMATION OVER k

RELATES OFF-DIAGONAL AN DIAGONAL COMPONENTS OF THE MOMENT

7) WI FOR $\xi \neq 0$ FOR SINGLET

BY USING THE DEFINITIONS IN POINT 2

$$\langle T_{\mu\mu} \rangle = -\frac{\Delta}{\beta} + 4P$$

$$\Downarrow$$

$$\beta \frac{\partial}{\partial \beta} \langle T_{\mu\mu} \rangle = 4\beta \frac{\partial}{\partial \beta} P - \beta \frac{\partial}{\partial \beta} \left\{ \frac{\Delta}{\beta} \right\}$$

$$\Downarrow$$

$$\boxed{\beta \frac{\partial}{\partial \beta} \langle T_{\mu\mu} \rangle = -\frac{1}{\beta^3} \frac{\partial}{\partial \beta} \left\{ \beta^3 \Delta \right\}}$$

SINCE $T_{\mu\mu}$ IS INVARIANT UNDER $SO(4)$

$$\langle T_{\mu\mu} \rangle = g\left(L_0 \sqrt{1+\xi^2}\right) \Rightarrow \frac{\partial}{\partial \beta} \langle T_{\mu\mu} \rangle = \frac{\partial g}{\partial \beta} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial \beta} \langle T_{\mu\mu} \rangle_\xi = \frac{\sqrt{1+\xi^2}}{L_0\, \xi_R} \frac{\partial}{\partial \xi_R} \langle T_{\mu\mu} \rangle_\xi$$

BY USING ALSO FOR R.H.S.

$$\boxed{\frac{\partial}{\partial \xi_R} \langle T_{\mu\mu} \rangle_\xi = \frac{1}{(1+\xi^2)^2} \frac{\partial}{\partial \xi_R} \left\{ (1+\xi^2)^3 \frac{\langle T_{0R} \rangle_\xi}{\xi_R} \right\}}$$

Note: $\langle T_{0R} \rangle_\xi$ DOES NOT RENORMALIZE $\Rightarrow$

$\langle T_{\mu\mu} \rangle_\xi$ DOES NOT RENORMALIZE EITHER!!

(RENORMALIZATION CAN BE CHOSEN TO BE INDEPENDENT ON ξ)

8) TRACE ANOMALY IN DIMENSIONAL REGULARIZATION (SU(3) GAUGE THEORY)

$$T_{\mu\mu} = \frac{2\epsilon}{D g_0^2} F^a_{\alpha\beta} F^a_{\alpha\beta} \qquad 8.1$$

WI IN PREVIOUS SECTION IMPLIES

$$T^R_{\mu\mu} = T_{\mu\mu} - \langle T_{\mu\mu} \rangle_0 \qquad 8.2$$

RENORMALIZED COUPLING AND OP. ARE DEFINED AS

$$\{ F^a_{\alpha\beta} F^a_{\alpha\beta} \}^R = Z_E^{-1} \{ F^a_{\alpha\beta} F^a_{\alpha\beta} - \langle F^a_{\alpha\beta} F^a_{\alpha\beta} \rangle_0 \} \qquad 8.3$$

$$g_0^2 = \mu^{2\epsilon} g^2 Z_g^{-1} \qquad 8.4$$

BY USING SHIFTED BOUNDARY CONDITIONS AND DERIVATIVE WITH RESPECT TO g_0 IT IS POSSIBLE TO SHOW THAT

$$\boxed{\epsilon Z_E Z_g = \epsilon - \frac{\beta}{g}} \qquad 8.5$$

THEREFORE 8.1 + 8.3 + 8.4 IMPLIES

$$T_{\mu\mu} - \langle T_{\mu\mu} \rangle_0 = \frac{2\epsilon Z_g Z_E}{D \mu^{2\epsilon} g^2} \{ F^a_{\alpha\beta} F^a_{\alpha\beta} \}^R$$

BY USING 8.5 AND 8.2

$$\boxed{T^R_{\mu\mu} = -\frac{1}{2g^3} \beta \{ F^a_{\alpha\beta} F^a_{\alpha\beta} \}^R} \quad \leftarrow \text{TRACE ANOMALY}$$