LECTURE 4: SPONTANEOUS SYMMETRY BREAKING AND BANKS-CASHER RELATION

1) LOCAL NON-SINGLET AWI:

$$\boxed{\langle \partial_\mu A^a_\mu(x)\, P^b(0)\rangle = 2m \langle P^a(x) P^b(0)\rangle - \frac{\delta^{(4)}(x)}{N_f}\,\delta^{ab}\langle S^0\rangle}$$

WHERE

$$\delta^{(4)}(x) = \frac{1}{a^4}\,\delta_{x,0}$$

NOTE: $Z_S\langle S^0\rangle$ FINITE ONLY IN THE CHIRAL LIMIT

$$S^0 = \bar\psi_L \psi_R + \bar\psi_R \psi_L \quad \leftarrow \text{ UNDER } SU(N_f)_L \otimes SU(N_f)_R$$

$$(\bar f_L, f_R) + (f_L, \bar f_R)$$

SO CANNOT MIX WITH $\hat{I}$ (SINGLET) BUT CAN MIX WITH M^+ and M

$$\hat{S}^0_R = Z_S\left\{ S^0 + c_0 \frac{M}{a^2} + c_0 \frac{M^+}{a^2} + \dots \right\}$$

AND THEREFORE OUT OF THE CHIRAL LIMIT IS NOT DEFINED

2) SPONTANEOUS SYMMETRY BREAKING AND GOLSTONE BOSONS:

- LET US TAKE THE $m \to 0$ LIMIT AND $a \to 0$, ALL OPERATORS IN THIS SECTION ARE INTENDED TO BE RENORMALIZED.

$$\langle \partial_\mu A^a_\mu(x) P^b(0) \rangle = -\frac{\delta^{(4)}(x)}{N_f} \langle S^0 \rangle \qquad (2.1)$$

FOR $x \neq 0$ LORENTZ INVARIANCE IMPLIES

$$\langle A^a_\mu(x) P^b(0) \rangle = \delta^{ab} x_\mu f(x^2)$$

AND BY INSERTING IN PREVIOUS EQUATION

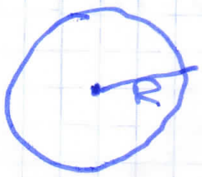
$$\sum_\mu \partial_\mu \left\{ x_\mu f(x^2) \right\} = 0 \Rightarrow 4 f(x^2) + x_\mu \partial_\mu f(x^2) = 0 \Rightarrow$$

$$\Rightarrow f(x^2) = \frac{1}{(x^2)^2}$$

$$\Downarrow$$

$$\langle A^a_\mu(x) P^b(0) \rangle = \delta^{ab} K \frac{x_\mu}{(x^2)^2} \qquad (2.2)$$

- WE CAN THEN INTEGRATE (2.1) ON A 4-SPHERE INCLUDING 0



$$\int_R \partial_\mu \langle A^a_\mu(x) P^b(0) \rangle d^4x = -\frac{1}{N_f} \delta^{ab} \langle S^0 \rangle$$

$$\Downarrow$$

$$\int_{|x|=R} d\sigma_\mu(x) \langle A^a_\mu(x) P^b(0) \rangle = -\frac{\delta^{ab}}{N_f} \langle S^0 \rangle$$

WE CAN NOW USE (2.2) [WE ARE ON-SHELL]

$$\delta^{ab} K \int_{|x|=R} d\sigma_\mu(x) \frac{x_\mu}{(x^2)^2} = -\frac{\delta^{ab}}{N_F}\langle S^0\rangle$$

$$\delta^{ab} K\, \Omega_4\, |x|^3 \cancel{\frac{x_\mu}{|x|}}\, \cancel{\frac{x_\mu}{|x|^4}} = -\frac{\delta^{ab}}{N_F}\langle S^0\rangle$$

By remembering that

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad , \; \Gamma(2) = 1 \;\Rightarrow\; \Omega_4 = 2\pi^2$$

$$\Downarrow$$

$$K = -\frac{1}{N_F}\frac{\langle S^0\rangle}{2\pi^2}$$

$$\Downarrow$$

$$\boxed{\langle A^a_\mu(x) P^b(0)\rangle = -\delta^{ab}\frac{x_\mu}{(x^2)^2}\frac{\langle S^0\rangle}{2\pi^2 N_F}} \quad (2.3)$$

Note: CURRENT-DENSITY CORRELATOR IS LONG RANGE IF $\langle S^0\rangle \neq 0$, SUPPRESSED POWER-LIKE

Note: THE ENERGY SPECTRUM OF THE THEORY DOES NOT HAVE A GAP, MASSLESS PARTICLES WITH QUANTUM NUMBERS OF A_μ P^b.

LET US PROJECT ON 0 MOMENTUM. IF

$$\langle \bar{A}_0^a(x_0) P^b(0) \rangle \equiv \int d^3x \, \langle A_0^a(x) P^b(0) \rangle$$

WE GET

$$\langle \bar{A}_0^a(x_0) P^b(0) \rangle = -\delta^{ab} \frac{\langle S^0 \rangle}{2\pi^2 N_f} \, x_0 \int d^3x \, \frac{1}{(x_0^2 + |\vec{x}|^2)^2}$$

BY REMEMBERING THAT

$$\int d^3x \, \frac{1}{(x_0^2 + |\vec{x}|^2)^2} = \frac{\pi^2}{x_0}$$

$\Downarrow$

$$\boxed{\langle \bar{A}_0^a(x_0) P^b(0) \rangle = -\frac{\delta^{ab}}{2N_f} \langle S^0 \rangle \qquad x_0 \neq 0} \quad 2.4$$

<u>NOTE:</u> CONSTANT IN EUCLIDEAN TIME,

$N_f^2 - 1$ STATES WITH $E = 0$!!!

3) GOLDSTONE BOSONS (PIONS) MATRIX ELEMENTS

BY NORMALIZING THE STATES AS USUAL

$$\langle \pi^a(p) | \pi^b(p') \rangle = (2\pi)^3 \, 2p^0 \, \delta^{(3)}(\vec{p}-\vec{p}\,') \, \delta^{ab}$$

$$d\mu(p) = \frac{d^3p}{(2\pi)^3 \, 2p^0}$$

BY REMEMBERING THAT

$$O(t) = e^{tH} O(0) e^{-Ht}$$

$$\langle \pi^b(p) | \bar\psi \gamma_5 T^a \psi | 0 \rangle = -i \delta^{ab} G_\pi e^{-E_\pi(p) x_0} e^{-i \vec{p}\cdot\vec{x}}$$

$$\langle 0 | A^a_\mu(x) | \pi^b(p) \rangle = i \delta^{ab} F_\pi p_\mu e^{-E_\pi(p) x_0} e^{i \vec{p}\cdot\vec{x}}$$

THAN

$$\langle \bar{A}^a_0(x_0) P^b(0) \rangle = \sum_c \int \frac{d^3p}{(2\pi)^3 \, 2p^0} \int d^3x \, \langle 0 | A^a_0(x) | \pi^c \rangle \langle \pi^c | P^b(0) | 0 \rangle$$

$$= \sum_c \delta^{ac} \delta^{cb} \int d^3x \int \frac{d^3p}{(2\pi)^3 \, 2p^0} (+ G_\pi F_\pi p_0) \, e^{i\vec{p}\cdot\vec{x}} e^{-E_\pi(p) x_0}$$

$$= \delta^{ab} \left(+ \frac{G_\pi F_\pi}{2} \right) e^{-E_\pi(\vec{0}) x_0}$$

$$\Downarrow$$

$$\boxed{\langle \bar{A}^a_0(x_0) P^b(0) \rangle \xrightarrow[x_0 \to \infty]{} \frac{\delta^{ab}}{2} G_\pi F_\pi}$$

BY COMPARING WITH (2.4)

$$\boxed{G_\pi F_\pi = -\frac{\langle S^0 \rangle}{N_f} \qquad m = 0} \qquad (3.1)$$

NOTES:

$\langle 0 | P | \pi \rangle \neq$ ONLY IF π IS A (PSEUDO)SCALAR

$\langle 0 | A_\mu | \pi \rangle \propto P_\mu$ BECAUSE π IS A (PSEUDO)SCALAR

NOTE:

$\langle S^0 \rangle \neq 0$ IF $G_\pi \neq 0$ AND $F_\pi \neq 0$ AND π IS A GOLDSTONE BOSON

4) GELL-MANN-OAKES-RENNER (GMOR) RELATION:

- LET US MOVE TO $m \neq 0$ AND $x_0 \neq 0$

$$\partial_0 \langle \bar{A}_0^a(x_0) P^b(0) \rangle = 2m \langle \bar{P}^a(x_0) P^b(0) \rangle$$

WE SATURATE WITH PION WHICH DOMINATES FOR $x_0 \to \infty$

$$\partial_0 \left\{ \cancel{\delta^{ab}} \frac{G_\pi F_\pi}{\cancel{2}} e^{-M_\pi x_0} \right\} = 2m \left\{ -\frac{G_\pi^2}{2M_\pi} e^{-M_\pi x_0} \cancel{\delta^{ab}} \right\}$$

$$-M_\pi F_\pi \cancel{e^{-M_\pi x_0}} = -2m \frac{G_\pi}{M_\pi} \cancel{e^{-M_\pi x_0}}$$

$$\Downarrow$$

$$\boxed{M_\pi^2 F_\pi = 2m G_\pi}$$

THEREFORE IN THE CHIRAL LIMIT

$$\lim_{m\to 0} \frac{M_\pi^2 F_\pi}{2m} = \lim_{m\to 0} G_\pi$$

WE CAN USE (B.1)

$$\boxed{\lim_{m\to 0} \frac{M_\pi^2 F_\pi^2}{2m} = \lim_{m\to 0} -\frac{\langle S^0 \rangle}{N_f}}$$

GMOR RELATION

NOTE:
- NUMERICALLY WE CAN WORK AT FINITE VOLUME, WHERE WE NEED $m \neq 0$ TO HAVE A NON ZERO CONDENSATE
- BUT $m \neq 0$ THE CONDENSATE IS DIVERGENT !! VERY BAD NUMERICALLY
- WE NEED TO FIND A WAY AROUND TO EXTRACT THE CONDENSATE AT $m \neq 0$

4) BANKS - CASHER RELATION:

$$-\frac{\langle S^0 \rangle}{N_B} = \langle \mathrm{Tr}\left[\left(1 - \frac{\bar{a} D}{2}\right)\frac{1}{D_m}\right]\rangle \frac{1}{V}$$

- BY REMEMBERING THAT

$$D_m = D + m\left(1 - \bar{a}\frac{D}{2}\right) \quad ; \quad D_m^+ D_m = DD^+ + m^2\left[1 - \left(\frac{\bar{a}}{2}\right)^2 DD^+\right]$$

IT IS EASY TO SHOW THAT (WI 11.1)

$$\mathrm{Tr}\left[\left(1 - \frac{\bar{a} D}{2}\right)\frac{1}{D_m}\right] = m\,\mathrm{Tr}\left[\frac{1 - \left(\frac{\bar{a}}{2}\right)^2 D^+D}{D^+D + m^2\left\{1 - \left(\frac{\bar{a}}{2}\right)^2 D^+D\right\}}\right]$$

AND THEREFORE

$$\boxed{-\frac{\langle S^0 \rangle}{N_B} = \frac{m}{V}\langle \mathrm{Tr}\left[\frac{1 - \left(\frac{\bar{a}}{2}\right)^2 D^+D}{D^+D + m^2\left\{1 - \left(\frac{\bar{a}}{2}\right)^2 D^+D\right\}}\right]\rangle}$$

NOTE: • $\frac{2}{a}$ MODES CLEARLY DO NOT CONTRIBUTE

NOTE: ϕ MODES GIVE CONTRIBUTIONS THAT VANISH IN THE INFINITE VOLUME LIMIT

NON-ZERO MODES:

$$DD^+ \to \left[\frac{2}{\bar{a}}\sin\left(\frac{\alpha_i}{2}\right)\right]^2 \quad ; \quad D \to \frac{1}{a}\left\{1 - e^{i\alpha}\right\}$$

We can define:

$$\lambda_i \equiv \frac{2}{\bar{a}}\frac{\sin\left(\frac{\alpha_i}{2}\right)}{\cos\left(\frac{\alpha_i}{2}\right)} \qquad \alpha_i \in [-\pi, \pi]$$

THEN

$$-\frac{\langle S^0 \rangle}{N_f} = + \frac{m}{V} \sum_i \left\langle \frac{1}{\lambda_i^2 + m^2} \right\rangle$$

WE DEFINE THE SPECTRAL DENSITY AS

$$\rho(\lambda) \equiv \frac{1}{V} \sum_i \langle \delta(\lambda - \lambda_i) \rangle$$

AND THEREFORE

$$-\frac{\langle S^0 \rangle}{N_f} = + 2m \int_0^\infty \frac{\rho(\lambda)}{\lambda^2 + m^2}\, d\lambda$$

WHERE WE HAVE USED CHIRAL SYMMETRY ~~BOC~~ FOR $\rho(-\lambda) = \rho(\lambda)$.

- WE CAN AGAIN CHANGE VARIABLE

$$x = \frac{\lambda}{m} \Rightarrow -\frac{\langle S^0 \rangle}{N_f} = + 2 \int_0^\infty \frac{1}{x^2+1} \rho(m x)\, dx$$

We are interested in the limit $m \to 0$.

$$-\frac{\langle S^0 \rangle}{N_f} = + 2 \int_0^\Lambda \frac{1}{x^2+1} \rho(m x)\, dx + \int_\Lambda^\infty \frac{1}{x^2+1} \rho(m x)\, dx$$

$$\Downarrow$$

$$\lim_{m \to 0} -\frac{\langle S^0 \rangle}{N_f} = + 2 \int_0^\Lambda \frac{1}{x^2+1} \lim_{m \to 0} \rho(m x)\, dx + \lim_{m \to 0} \int_\Lambda^\infty \frac{1}{x^2+1} \rho(m x)\, dx$$

$$\Downarrow$$

$$\boxed{-\frac{\langle S^0 \rangle}{N_f}\bigg|_{m=0} = + 2 \rho(0) \int_0^\Lambda \frac{1}{x^2+1}\, dx + \lim_{m \to 0} \int_\Lambda^\infty \frac{1}{x^2+1} \rho(m x)\, dx}$$

SINCE WE CAN TAKE Λ AS LARGE AS WE WISH BUT FINITE

$$\rho(m x) = c\, m^3 x^3 + \cdots$$

$$\Downarrow$$

$$\lim_{m\to 0} -\frac{\langle S^0 \rangle}{N_f} = +2\rho(0)\int_0^{\Lambda} \frac{dx}{x^2+1} + \underbrace{\lim_{m\to 0} c\, m^3 \int_{\Lambda}^{\infty} \frac{x^3\, dx}{x^2+1}}_{\text{divergent but goes to 0 in chiral limit}}$$

$$\Downarrow$$

Now I can take back $\Lambda \to \infty$

$$-\frac{\langle S^0 \rangle}{N_f}\bigg|_{m=0} = +2\rho(0)\, \mathrm{Arctg}(x)\Big|_0^{\infty}$$

$$\Downarrow$$

$$\boxed{-\frac{\langle S^0 \rangle}{N_f}\bigg|_{m=0} = \pi \rho(0)}$$

BANKS-CASHER RELATION

NOTE: CAN BE READ IN EITHER DIRECTIONS! A NON-ZERO SPECTRAL DENSITY AT THE ORIGIN IMPLIES THAT $\langle S^0 \rangle \neq 0$ AND VICEVERSA

5) RENORMALIZATION OF SPECTRAL DENSITY:

QUESTION: IS THE SPECTRAL DENSITY A RENORMALIZ... QUANTITY WITH A UNIVERSAL CONTINUUM LIMIT?

QUESTION: IF YES, THIS IS ALSO TRUE FOR $m \neq 0$?

TO THIS AIM LET US DEFINE THE SPECTRAL SUMS

$$\sigma_k(m, \mu) = 2V \int_0^{\infty} \frac{1}{(\lambda^2 + \mu^2)^k} \rho(\lambda, m)\, d\lambda$$

BY INTRODUCING VALENCE QUARKS AS FOR TOP. SUSC., IT IS STRAIGHTFORWARD TO SHOW THAT

$$\sigma_k(m, \mu) = - a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k\,1}(x_{2k}) \rangle$$

AGAIN BY NOTICING THAT

$$P_{12}(x) P_{23}(0) \sim \frac{1}{|x|^3} S_{13}(0)$$

IT IS POSSIBLE TO PROVE THAT ALL SHORT DISTANCE SINGULARITIES ARE INTEGRABLE IF $k \geq 3$

$\Downarrow$

$$\boxed{\sigma_k^R(m, \mu) = Z_P^{2k} \sigma_k(m, \mu) \qquad k \geq 3}$$

6) FROM SPECTRAL SUMS TO SPECTRAL DENSITY:

By noticing that

$$\int_{\mu_0^2}^{\mu_1^2}\frac{1}{(\lambda^2+\mu^2)^k}\,d\mu^2 = \frac{1}{1-k}\left[\frac{1}{(\lambda^2+\mu_1^2)^{k-1}}-\frac{1}{(\lambda^2+\mu_0^2)^{k-1}}\right]$$

By iterating a second time

$$\int_{\mu_0^2}^{\mu_2^2} d\mu_1^2\int_{\mu_0^2}^{\mu_1^2}\frac{1}{(\lambda^2+\mu^2)^k}\,d\mu^2 = \frac{1}{(1-k)(2-k)}\left[\frac{1}{(\lambda^2+\mu_2^2)^{k-2}}-\frac{1}{(\lambda^2+\mu_0^2)^{k-2}}\right]$$

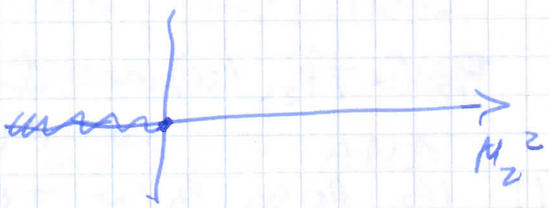
$$-\frac{(\mu_2^2-\mu_0^2)}{1-k}\,\frac{1}{(\lambda^2+\mu_0^2)^{k-1}}$$

So for $k=3$

$$\int_{\mu_0^2}^{\mu_2^2} d\mu_1^2\int_{\mu_0^2}^{\mu_1^2}\sigma_3(m,\mu)\,d\mu^2 = \frac{2V}{2}\int_0^\infty \frac{\rho(\lambda,m)}{(\lambda^2+\mu_2^2)}\,d\lambda \,+$$

$$+\,\beta(\mu_0^2,\mu_2^2)$$

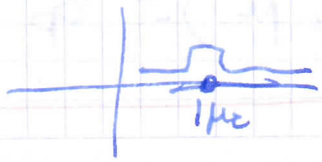
The R.H.S. as a function of μ_2^2 has a cut for $\mu_2^2<0$.

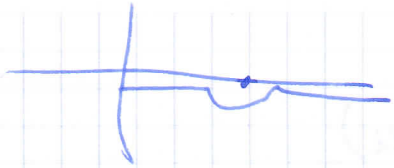


In this cas

$$\int_0^\infty \frac{\rho(\lambda,m)}{(\lambda^2+\mu_2^2)^2} = \int_0^\infty \frac{\rho(\lambda,m)}{(\lambda-|\mu_2|)(\lambda+|\mu_2|)}\,d\lambda$$

We can do the integral



OR 

THE DIFFERENCE OF THE TWO GLUES S –

7) RENORMALIZATION OF SPECTRAL DENSITY.

$$\sigma_n^R = z_p^{2k}\, 2V \int_0^{\delta} \frac{\rho(\lambda, m)}{(\lambda^2+\mu^2)^k}\, d\lambda \qquad z_p = \frac{1}{z_m}$$

$$= 2V \int_0^{\delta} \frac{\rho(\lambda, m_R \cdot z_p)}{\left[(\lambda/z_p)^2 + \mu_R^2\right]^k}\, d\lambda$$

BY DEFINING $\lambda_R = \frac{\lambda}{z_p}$

$$\sigma_n^R = 2V \int_0^{\delta} \frac{z_p\, \rho(\lambda_R z_p, m_R z_p)}{(\lambda_R^2 + \mu_R^2)^n}\, d\lambda_R$$

BY USING PREVIOUS FORMULAE WITH μ_R

$$\boxed{\rho_R(\lambda_R, m_R) = z_p\, \rho(\lambda, m) \qquad \lambda_R = \frac{\lambda}{z_p} \qquad m_R = \frac{m}{z_p}}$$

- THE SPECTRAL DENSITY IS RENORMALIZABLE AND HAS A UNIVERSAL VALUE IN CONTINUUM LIMIT FOR $m \geq 0$ –

8) MODE NUMBER

$$\nu(m,\Lambda) \equiv V \int_{-\Lambda}^{\Lambda} d\lambda \; \rho(\lambda, m)$$

<u>NOTE:</u> IT IS THE AVERAGE NUMBER OF MODES IN THE INTERVAL $[-\Lambda, \Lambda]$.

$$\nu(m,\Lambda) = V \int_{-\Lambda}^{\Lambda} d\lambda \; \frac{1}{Z_P} \cdot \rho_R(\lambda_R, m_R)$$

$$\parallel$$

$$\nu(m,\Lambda) = V \int_{-\Lambda^R}^{\Lambda^R} d\lambda_R \; \rho_R(\lambda_R, m_R) \qquad \Lambda^R = \frac{\Lambda}{Z_P}$$

$$\parallel$$

$$\nu_R(m_R, \Lambda_R)$$

$$\parallel$$

$$\boxed{\nu(m,\Lambda) = \nu_R(m_R, \Lambda_R)}$$

- THE MODE NUMBER IS A RENORMALIZATION GROUP INV. QUANTITY FOR $m \neq 0$
- IT IS ULTRAVIOLET FINITE AS IT STANDS AND HAS A UNIVERSAL MEANING.