LECTURE 3: WITTEN-VENEZIANO RELATION

1) DENSITY CHAINS

- WE ASSUME AGAIN

$$M = M^{+} = m \mathbb{1}$$

AND WE DO A NON-SINGLET ~~AXIAL TRANSF.~~

$$\delta S_F = -i \, \varepsilon_A^a \, 2 m a^4 \sum_x P^a(x)$$

- LET US TAKE THE OPERATOR

$$S^b \equiv \bar{\psi}_L T^b \psi_R + \bar{\psi}_R T^b \psi_L$$

$$\Downarrow$$

$$\delta S^b = -i \, \varepsilon_A^a \left[\bar{\psi}_L \{T^a, T^b\} \psi_R - \bar{\psi}_R \{T^a, T^b\} \psi_L \right]$$

AND THEN THE OPERATOR

$$O = S^b(x_2) \, P_{33}(x_4) \qquad b = 1, 2, 3$$

and we choose $\varepsilon_A^a \neq 0$ for $a = 1, 2, 3$

$$\{T^a, T^b\} = \frac{\delta^{ab}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

THEREFORE IN THIS CASE

$$\delta P_{33} = 0 \quad , \qquad \delta S^b = -i \frac{\varepsilon_A^a}{2} \left[P_{11} + P_{22} \right]$$

BY USING THE GENERAL RELATION

$$\langle \delta \varsigma_F \, O \rangle = \langle \delta O \rangle$$

$$i \, \varepsilon_A^a \, 2m \, a^4 \sum_{x_1} \langle P^a(x_1) \varsigma^b(x_2) \, P_{33}(x_4) \rangle =$$

$$= -i \frac{\varepsilon_A^a}{2} \langle \left\{ P_{11}(x_2) + P_{22}(x_2) \right\} P_{33}(x_4) \rangle$$

Now, BY WICK CONTRACTING $P^a \varsigma^b$ ON R.H.S.

$$P^a(x_1) \varsigma^b(x_2) = - \underbrace{T_R \left[T^a T^b \right]}_{\frac{1}{2}} \, a$$

$$\Downarrow$$

$$\boxed{m \, a^4 \sum_{x_1} \langle P_{12}(x_1) \varsigma_{21}(x_2) \, P_{33}(x_4) \rangle = \langle P_{11}(x_2) \, P_{33}(x_4) \rangle} \quad (1.1)$$

BY INSERTING IN (3.1) OF LECTURE 2

$$\boxed{a^4 \sum_x \langle q(x) q(0) \rangle = m^3 a^8 \sum_{x_1, x_2} \langle P_{12}(x_1) \varsigma_{21}(x_2) P_{33}(x_4) \rangle}$$

__Note__: THIS STARTS TO BE INTERESTING DUE TO THE FLAVOUR STRUCTURE

- IF $N_f = 5$ WE CAN ITERATE THE ARGUMENT IN A STRAIGHTFORWARD WAY, AND GET

$$\boxed{a^4 \sum_x \langle q(x)\, q(0)\rangle = m^5 a^{16} \sum_{x_1,x_2,x_3,x_4} \langle P_{12}(x)\, S_{23}(x_2)\, S_{31}(x_3) \times P_{45}(x_4)\, S_{54}(0)\rangle} \qquad (1.2)$$

THIS IS THE FORMULA WE ARE INTERESTED IN!!!

2) ~~DERIVATION~~ INTERPRETATION OF FORMULA FROM SPECTRAL PROPERTIES OF D:

- WE START FROM

$$a^4 \sum_x \langle q(x)\, q(0)\rangle = m^2 a^4 \sum_x \langle P_{11}(x)\, P_{22}(0)\rangle$$

- BY REMEMBERING THAT

$$P_{11} = \bar\psi_1 \gamma_5 \left(1 - \frac{\bar a D}{2}\right)\psi_1$$

$$\Downarrow$$

$$P_{11}(x) = -\,\mathrm{tr}\left[\gamma_5 \left\{\left(1 - \frac{\bar a D}{2}\right) D_m^{-1}\right\}(x,x)\right]$$

$$D_m \equiv D + m\left(1 - \frac{\bar a D}{2}\right)$$

D and D_m HAVE SAME EIGENVECTORS

$\lambda \neq 0$ DO NOT CONTRIBUTE to $\sum_x \hat{P}_{11}(x) \; \langle u_\lambda | \gamma_5 | u_\lambda \rangle = 0$

$\lambda = \frac{2}{a} \Rightarrow \lambda_m = \frac{2}{a} \Rightarrow$ DO NOT CONTRIBUTE

$\lambda = 0 \Rightarrow \lambda_m = m$ contribute

$$a^4 \sum_x \hat{P}_{11}(x) = -\frac{1}{m}(n_+ - n_-)$$

THIS IS THE FORMULA WE ARE INTERESTED IN !!!

$$\boxed{m^2 a^4 \sum_x \langle P_{11}(x) P_{22}(0) \rangle = m^2 \frac{1}{m^2} \frac{\langle (n_+ - n_-)^2 \rangle}{V}}$$

c.v.d.

IF WE HAVE MORE DENSITIES

$$a^8 \sum_{x_1, x_2} \langle P_{12}(x_1) S_{21}(x_2) \rangle = -a^8 \sum_{x_1 x_2} \mathrm{Tr}\left[\gamma_5 \left(1 - \frac{aD}{2}\right) D_m^{-1} \times \times \left(1 - \frac{aD}{2}\right) D_m^{-1}\right]$$

SINCE IT HOLDS

$$\mathrm{Tr}\left[\gamma_5 f(D)\right] = \left\{ f(0) - f\left(\frac{2}{a}\right) \right\} (n_+ - n_-)$$

then

$$a^4 \sum_{x_1 x_2} P_{12}(x_1) S_{21}(x_2) = -\left\{ \frac{1}{m^2} - 0 \right\} (n_+ - n_-)$$

$$\boxed{m^3 a^8 \sum_{x_1 x_2} \langle P_{12}(x_1) S_{21}(x_2) P_{33}(x_1) \rangle = \cancel{m}^3 \frac{1}{\cancel{m}^3} \frac{(n_+ - n_-)^2}{V} = Q^2}$$

end

3) OPE AND FINITENESS OF χ:

BY TAKING (1.2)

$$a^4 \sum_x \langle q(x) q(0) \rangle = m^5 a^{16} \sum_{x_1 - x_4} \langle P_{12}(x_1)\, S_{23}(x_2)\, S_{31}(x_3) \times$$

$$\times P_{45}(x_4)\, S_{54}(0) \rangle$$

ON R.H.S. THE PRODUCTS

$$m \cdot P_{15} \quad , \quad m \cdot S_{ij}$$

- DO NOT NEED RENORMALIZATION WHEN INSERTED AT PHYSICAL DISTANCE SINCE

$$\hat{m}_R = Z_m\, m \quad , \quad \hat{P}_{ij,R} = Z_P\, P_{ij} \; ; \quad \hat{S}_{ij,R} = Z_S\, S_{ij}$$

AND NON-SINGLET WIS GUARANTEE

$$Z_P = \frac{1}{Z_m} \quad , \quad Z_P = Z_S$$

- SINCE IS AN INTEGRATED CORRELATOR, WE NEED TO SEE IF SHORT DISTANCE SINGULARITIES ARE INTEGRABLE. IN THE CONTINUUM

$$\hat{P}_{12R}(x)\, \hat{S}_{23,R}(0) \xrightarrow[|x| \to 0]{} \frac{1}{|x|^3}\, \hat{P}_{13,R}(0)$$

AND THEREFORE

$$\int d^4x\, \hat{P}_{12R}(x)\, \hat{S}_{23R}(0) = \text{FINITE}$$

SINCE AT $|x| \to 0$ THE SHORT DISTANCE BEHAVIOUR IS INTEGRABLE.

- SINCE ALL P ~~AN~~ AND S HAVE NON TRIVIAL FLAVOUR NUMBERS, THIS CAN BE USED FOR ALL COUPLE OF THEM, AND ALSO WHEN MORE THAN 2 CONVERGE TO THE SAME POINT. ALL SINGULARITIES ARE INTEGRABLE

- FOR INSTANCE WHEN ALL OF THEM CONVERGES TO THE SAME POINT

$$PSS\,PS \longrightarrow \frac{1}{|x|^{15}}\,I$$

WHICH INTEGRATED OVER $d^4x_1 - d^4x_4$ GIVES A FINITE CONTRIBUTION.

CONCLUSION!

$$\boxed{\begin{aligned}\chi = m^5 a^{16} \sum_{x_1 - x_4} \langle P_{12}(x_1)\, S_{23}(x_2)\, S_{31}(x_3) \;\times \\ \times\, P_{45}(x_4)\, S_{54}(0) \rangle \qquad (3.1)\end{aligned}}$$

IS FINITE AS IT STANDS, AND HAS A FINITE CONTINUUM LIMIT. WITH GW FERMION IT COINCIDE WITH

$$\chi = \frac{1}{V}\langle (n_+ - n_-)^2 \rangle$$

4) UNIVERSALITY OF χ:

- ONCE DEFINED AS IN (3.1), THE FINITENESS IS SHOWN ONLY BY USING NON-SINGLET WI1 AND FLAVOUR QUANTUM NUMBERS.

- WE CAN THEREFORE WRITE ~~THE~~ VERY SAME FORMULA, BUT WITH WILSON-FERMIONS

- BY JUST REPLACING $\frac{1}{Z_W} \to Z_P$ WE CAN PROVE THAT (3.1) WITH WILSON FERMIONS HAS NO SHORT-DISTANCE SINGULARITIES

$$\Downarrow$$

$$\chi = m^5 a^{16} \sum_{x_1 \ldots x_4} \langle PSS \quad PS \rangle$$

IS A UNIVERSAL DEFINITION. ALL CONTINUUM LIMITS TEND TO THE SAME VALUES.

5) FINITINESS FOR $N_f < 5$:

- WE CAN INTRODUCE A MULTIPLET OF VALENCE QUARKS, WITH FERMION ACTION GIVEN BY

$$S_F = a^4 \sum_x \left\{ \sum_{i=1}^{2N_v} \bar{\psi}_i D_m \psi_i + \sum_{k=1}^{N_v} |D_m \phi_k|^2 \right\}$$

$\psi_i, \bar{\psi}_i$ ARE THE VALENCE QUARK, ϕ_k ARE THE ASSOCIATED PSEUDOFERMION FIELDS

- WITH SMALL MODIFICATIOS ALL GOES AS BEFORE

6) WITTEN - VENEZIANO RELATION:

- FROM PREVIOUS POINTS WE KNOW

$$\chi \begin{cases} \to \text{FINITE} \\ \to \text{UNAMBIGOUSLY DEFINED} \end{cases}$$

- POINT OF WITTEN:

(6.1) $\boxed{\chi^{YM} \neq 0}$ NO REASON WHY $= 0$

- WITH FERMIONS IF χ HAS A REGULAR EXPANSION IN N_f/N_c THEN

(6.2) $$\boxed{\chi = \chi^{YM} + O\left(\frac{N_f}{N_c}\right)}$$

IN THE CHIRAL LIMIT THE LOCAL WI SAYS

$$\langle \partial_\mu A^0_\mu(x)\, q(0) \rangle = 2 N_f \langle q(x)\, q(0) \rangle$$

~~POSTULATING~~ ASSUMING THE ABSENCE OF MASSLESS PARTICLES IN THIS CHANNEL

$$\int d^4x\, \langle \partial_\mu A^0_\mu(x)\, q(0) \rangle = 0 \implies \boxed{\chi\big|_{m=0} = 0 \quad \forall \frac{N_f}{N_c}} \quad (6.3)$$

~~NOTE: BY USING THE STANDARD DEFINITIONS~~

WITTEN QUESTION: HOW IT IS POSSIBLE TO RECONCILE (6.1), (6.2) AND (6.3)?

SOLUTION: (6.2) MUST BE WRONG.

BY USING THE KÄLLEN-LEHMANN REPRESENTATION

$$\chi(p) \equiv \int d^4x\, e^{ipx} \langle q(x)\, q(0) \rangle$$

All contribu[...] except the [...] of η'

$$\boxed{\chi(p) = C_0 \Lambda^4 + C_1 \Lambda^2 p^2 + C_2 (p^2)^2 - \frac{R^2}{p^2 + m^2_{\eta'}} + (p^2)^3 \int_{M^2}^{\infty} \frac{\rho(t)}{(t+p^2)\, t^3}\, dt} \quad (6.4)$$

THEN $\forall m$

$$\chi \equiv \lim_{p\to 0} \chi(p) = C_0 \Lambda^4 - \frac{R^2}{m^2_{\eta'}} \qquad (6.5)$$

IS UNAMBIGUOUS THANKS TO ALL SHOWN BEFORE

BY USING (6.3)

$$\lim_{m\to 0} \chi = 0 \implies \boxed{\frac{R^2}{m^2_{\eta'}}\bigg|_{m=0} = C_0 \Lambda^4\Big|_{m=0}} \quad (6.6)$$

BY ASSUMING THE SMOOTHNESS (IN $\frac{N_f}{N_c}$) OF $\chi(p)$ at $p \neq 0$

$$\lim_{\frac{N_f}{N_c}\to 0} \chi(p) = \chi^{YM}(p) \qquad \forall m$$

BY USING 6.4

$$\chi^{YM} = \lim_{p\to 0} \chi^{YM}p = \lim_{p\to 0} \lim_{\frac{N_f}{N_c}\to 0} \chi(p) = \lim_{\frac{N_f}{N_c}\to 0} c_0 \Lambda^4$$

$$\Downarrow$$

$$\boxed{\chi^{YM} = \lim_{\frac{N_f}{N_c}\to 0} c_0 \Lambda^4} \qquad \forall m \quad \text{ou. 2.4 n.} \infty$$

SINCE R.H.S. IS VALID $\forall m$ AND IN PARTICULAR FOR $m=0$

$$\boxed{\lim_{\frac{N_f}{N_c}=0} \lim_{m\to 0} \frac{R^2}{m_{\eta'}^2} = \chi^{YM}}$$

WITTEN-VENEZIANO FORMULA FOR THE MASS

COMPUTATION OF R:

BY USING THE STANDARD CONVENTION FOR $F_{\eta'}$

$$\langle 0|A_0(x)|\eta'\rangle = i\sqrt{2N_f}\, F_{\eta'}\, M_{\eta'}\, e^{-M_{\eta'}x_0}$$

$$\partial_0 \bar{A}_0(x_0) = 2N_f\, \bar{q}(x_0)$$

$$\Downarrow$$

$$\langle 0|q(0)|\eta'\rangle = -i \frac{F_{\eta'} M_{\eta'}^2}{\sqrt{2N_f}} \Rightarrow R^2 = \frac{F_{\eta'}^2 m_{\eta'}^4}{2N_f}$$

AND THEREFORE

$$\boxed{\lim_{\frac{N_f}{N_c}\to 0} \lim_{m\to 0} \frac{F_{\eta'}^2 m_{\eta'}^2}{2N_f} = \chi^{YM}}$$

NOTE:

1) χ^{YM} IS $O(1)$ IN $\frac{1}{N_c}$, $F_{\eta'}^2 \propto N_c$ =>

=> $m_{\eta'}^2 = O\left(\frac{N_F}{N_C}\right)$ this is the answer to Witten question, there is 1 particle with a mass $\frac{N_F}{N_C}$, so we need to rescue all series of diagrams and we get $O(1)$

2) when $N_c \to \infty$, or $\frac{N_F}{N_C} \to 0$, $m_{\eta'}^2$ BECOMES A GOLDSTONE BOSON

3) IN THIS LIMIT $U(1)_A$ IS RESTORED

WITTEN 79: "WE CANNOT ASK WHETHER THE FORMULA IS CORRECT, BECAUSE IT INVOLVES χ^{YM}, WHICH WE CAN NEITHER MEASURE NOR CALCULATE.