Lecture 2: Ginsparg-Wilson fermions and chiral anomaly

1) Ginsparg-Wilson relation and Lüscher symmetry

$$\gamma_5 D + D\gamma_5 = \bar{a}\, D\gamma_5 D \quad , \quad \bar{a} = \frac{a}{1+1}$$

$$\Updownarrow$$

$$\gamma_5 D + D\hat{\gamma}_5 = 0 \quad , \quad \hat{\gamma}_5 \equiv \gamma_5 (1 - \bar{a} D)$$

Avoid NN theorem due to presence of $\hat{\gamma}_5$

- The massless fermion action

$$S = a^4 \sum_x \bar{\psi}(x) [D\psi](x)$$

under the Lüscher symmetry

$$\psi' = \psi + i\, \varepsilon_A^0\, \hat{\gamma}_5 \psi \Rightarrow \delta\psi = \psi - \psi' = -i\, \varepsilon_A^0\, \hat{\gamma}_5 \psi$$

$$\bar{\psi}' = \bar{\psi} + i\, \varepsilon_A^0\, \bar{\psi}\gamma_5 \Rightarrow \delta\bar{\psi} = \bar{\psi} - \bar{\psi}' = -i\, \varepsilon_A^0\, \bar{\psi}\gamma_5$$

is invariant

$$\delta S = -a^4\, i\, \varepsilon_A^0 \sum_x \bar{\psi}(x) \left\{ [\gamma_5 D + D\hat{\gamma}_5] \psi \right\}(x) = 0$$

<u>Note</u>: $\hat{\gamma}_5$ depends on background gauge field

2) TRANSFORMATION OF MEASURE

$$\psi' = e^{i \overset{\circ}{\varepsilon}_A \hat{\gamma}_5} \psi \quad , \quad \bar{\psi}' = \bar{\psi} e^{i \overset{\circ}{\varepsilon}_A \gamma_5}$$

$$d\bar{\psi}' d\psi' = \frac{1}{\det\left(e^{i \varepsilon_A \gamma_5}\right)} \frac{1}{\det\left(e^{i \varepsilon_A \hat{\gamma}_5}\right)} d\bar{\psi} d\psi$$

trace over space, spin, color

$$\Downarrow$$

$$d\bar{\psi}' d\psi' = e^{-i \overset{\circ}{\varepsilon}_A \mathrm{TR}\left[\hat{\gamma}_5\right]} d\bar{\psi} d\psi$$

- IF WE INTRODUCE THE TOP. CHARGE DENSITY

$$a^4 q(x) \equiv - \frac{\bar{a}}{2} \mathrm{tr}\left[\gamma_5 D(x,x)\right] \quad ; \quad Q \equiv a^4 \sum_x q(x)$$

$$\Downarrow$$

$$Q = \frac{1}{2} \mathrm{TR}\left[\hat{\gamma}_5\right]$$

NOTE: IN CLASSICAL CONTIN. LIMIT

$$q(x) \longrightarrow \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \mathrm{tr}\left[F_{\mu\nu}(x) F_{\rho\sigma}(x)\right]$$

3) SPECTRAL PROPERTIES OF GW OPERATORS:

$\bar{a} D = 1 - \gamma_5 \hat{\gamma}_5$, and $(\gamma_5 \hat{\gamma}_5)(\gamma_5 \hat{\gamma}_5)^+ = 1$

so

$$\bar{a} \bar{\lambda} = 1 - e^{i\alpha} \Rightarrow \begin{cases} \bar{\lambda} = 0 \\ \bar{\lambda} \in \text{complex} \\ \bar{\lambda} = \frac{2}{\bar{a}} \end{cases}$$

$\bar{\lambda} = 0$:

$\bar{\lambda} = 0 \Rightarrow (\gamma_5 D + D\gamma_5)|u_\lambda\rangle = \bar{a} D \gamma_5 D |u_\lambda\rangle \Rightarrow D\gamma_5 |u_\lambda\rangle = 0$

$\Rightarrow$ we can choose

$$\gamma_5 |u_\lambda\rangle = \pm |u_\lambda\rangle \qquad \lambda = 0$$

$n_\pm$ = number of ϕ modes with chirality $\pm$

$\bar{\lambda} = \frac{2}{\bar{a}}$:

$\bar{\lambda} = \frac{2}{\bar{a}} \Rightarrow (\gamma_5 D + D\gamma_5)|u_\lambda\rangle = \bar{a} D\gamma_5 D |u_\lambda\rangle \Rightarrow D\gamma_5 |u_\lambda\rangle = \frac{2}{\bar{a}} \gamma_5 |u_\lambda\rangle$

again we can choose

$$\gamma_5 |u_\lambda\rangle = \pm |u_\lambda\rangle \qquad \bar{\lambda} = \frac{2}{\bar{a}}$$

$n'_\pm$ = number of $\frac{2}{\bar{a}}$ modes with chirality $\pm$

$\bar{\lambda} \in \mathbb{C}$:

$\bar{a}\bar{\lambda} = 1 - e^{i\alpha} \Rightarrow (\gamma_5 D + D\gamma_5)|u_\lambda\rangle = \bar{a} D\gamma_5 D |u_\lambda\rangle \Rightarrow D\gamma_5 |u_\lambda\rangle = \frac{1}{\bar{a}\bar{\lambda} - 1} \bar{\lambda} \gamma_5 |u_\lambda\rangle$

$\Rightarrow D\gamma_5 |u_\lambda\rangle = \bar{\lambda}^* \gamma_5 |u_\lambda\rangle$

$\Downarrow$

$\bar{\lambda}, \lambda^* \in$ eigenspectrum, $|u_{\lambda^*}\rangle = \gamma_5 |u_\lambda\rangle$, $\langle u_\lambda | \gamma_5 | u_\lambda\rangle = 0$

SINCE $\mathrm{Tr}[\gamma_5]=0$

$$\mathrm{Tr}[\bar{a}\,\gamma_5 D] = \mathrm{Tr}[\gamma_5(\bar{a}D-2)] \Rightarrow (u_+ - u_-) = -(u'_+ - u'_-)$$

AND THEREFORE

$$Q = n_+ - n_- \quad ; \quad \mathrm{Tr}[\hat{\gamma}_5] = 2(u_+ - u_-)$$

$$\boxed{d\bar{\psi}'\,d\psi' = e^{-2i\,\varepsilon_A^0\,Q}\,d\bar{\psi}\,d\psi \quad , \quad Q = u^+ - u^-}$$

4) NEUBERGER OPERATOR

$$D = \frac{1}{\bar{a}}\left\{1 + \gamma_5 \frac{Q}{\sqrt{Q^2}}\right\} \; ; \; Q = \gamma_5(aD_W - 1 - 1)$$

SO WE NEED TO COMPUTE THE SIGN OF HERMITEAN WILSON-DIRAC OPERATOR WITH NEGATIVE MASS $-\frac{(1+1)}{a}$.

<u>NOTE:</u> OPERATOR IS NOT ULTRALOCAL, BUT IS LOCAL WITH EXPONENTIALLY DECAYING TAILS

$$\left\| \frac{1}{\sqrt{Q^2}}(x,y) \right\| \sim e^{-\frac{c}{a}\|x-y\|_1}$$

4) CHIRAL MULTIPLETS AND SYMMETRY

$$\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5) \quad ; \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

$$\psi_{R,L} = \hat{P}_\pm \psi \quad , \quad \bar{\psi}_{R,L} = \bar{\psi} P_\mp$$

TRANSFORMATION OF CHIRAL GROUP $U(1)_L \otimes U(1)_R$

$$\psi_L' = V_L \psi_L \quad ; \quad \psi_R' = V_R \psi_R$$
$$\bar{\psi}_L' = \bar{\psi}_L V_L^+ \quad ; \quad \bar{\psi}_R' = \bar{\psi}_R V_R^+$$

AND BILINEARS WITH CORRECT CHIRAL PROPERTIES

$$O_\Gamma \equiv \bar{\psi} \Gamma \tilde{\psi} \quad , \quad \tilde{\psi} \equiv \left(1 - \frac{\bar{a} D}{2}\right)\psi$$

VECTOR SUBGROUP: $V_L = V_R = e^{i \varepsilon_V^0}$

AXIAL TRANSF. : $V_L^+ = V_R = e^{i \varepsilon_A^0}$

5) ACTION FOR MASSIVE FERMIONS

$$S = a^4 \sum_x \left\{ \bar{\psi} D \psi + \bar{\psi}_R M^+ \psi_L + \bar{\psi}_L M \psi_R \right\}$$

- All terms invariant under $U(1)_V$
- $\bar{\psi} D \psi$ INVARIANT UNDER $U(1)_A$
- $\bar{\psi}_R M^+ \psi_L$ invariant if
- $\bar{\psi}_L M \psi_R$ " "

$$\boxed{\begin{array}{l} M^+ \to M^{+\prime} = V_R M^+ V_L^+ \\ M \to M' = V_L M V_R^+ \end{array}}$$

↳ SPURION TRANSFORMATION

6) ν AND M DEPENDENCE OF THE FREE-ENERGY

$$S = S_G - i\nu Q + S_F$$

$$e^{-F(\nu, M, M^+)} = \frac{\int \delta U\, d\bar\psi d\psi\, e^{-S_G - S_F}\, e^{i\nu Q}}{\int \delta U\, d\bar\psi d\psi\, e^{-S_G - S_F}} = \langle e^{i\nu Q} \rangle = \frac{Z(\theta)}{Z(0)}$$

By the anomalous transformation

$$\varepsilon^0_A = -\frac{\nu}{2} \Rightarrow V_L^+ = V_R = e^{-i\frac{\nu}{2}}$$

then

$$d\bar\psi d\psi = e^{-i\nu Q}\, d\bar\psi' d\psi'$$

$$M' = e^{i\nu} M$$

$$M^{+\prime} = e^{-i\nu} M^+$$

$$\int \delta U\, d\bar\psi d\psi\, e^{-S_G - S_F(M, M^+)}\, e^{i\nu Q} = \int \delta U\, d\bar\psi' d\psi'\, e^{-S_G - S_F(M', M^{+\prime})}$$

$$\Downarrow$$

$$\boxed{F(\nu, M, M^+) = F(0, V_L M V_R^+, V_R M^+ V_L^+) = F(0, e^{i\nu} M, e^{-i\nu} M^+)}$$

(6.1)

7) AWI FOR TOPOLOGICAL SUSCEPTIBILITY

THE CUMULANTS OF CHARGE DISTRIBUTION

$$C_n \equiv (-1)^{n+1} \frac{1}{V} \frac{d^{2n}}{d\theta^{2n}} F(\theta)\Big|_{\theta=0} = \frac{a^{8n}}{V} \sum_{x_1 \dots x_{2n}} \langle q(x_1) \dots q(x_{2n}) \rangle_c$$

ARE INTEGRATED CORREL. FUNCTIONS OF $q(x)$

FOR $n=1$

$$\chi \equiv \frac{1}{V} \frac{d^2}{d\theta^2} F(\theta)\Big|_{\theta=0}$$

By using (6.1) e.h.s.

$$\chi = a^4 \sum_x \langle q(x) q(0) \rangle \leftarrow$$ $\langle q(x) q(0) \rangle \underset{x \to 0}{\sim} \frac{1}{(x^2)^4}$ SO HAS SHORT DISTANCE SINGULARITIES

By using (6.1) r.h.s

$$\frac{d^2}{d\theta^2} F(\theta)\Big|_{\substack{\theta=0 \\ M=M^+}} = -a^4 \sum_x \langle \bar\psi_L M \psi_R + \bar\psi_R M^+ \psi_L \rangle +$$

$$+ a^8 \sum_{x,y} \langle (\bar\psi_L M \psi_R - \bar\psi_R M^+ \psi_L)(x) (\bar\psi_L M \psi_R - \bar\psi_R M^+ \psi_L)(y) \rangle$$

- By USING TRANSL. INVARIANCE, DIVIDING BY V, AND BY DEFINING

$$M = M^+ = m, \quad S^0 = \bar\psi_L \psi_R + \bar\psi_R \psi_L, \quad P^0 = \bar\psi_L \psi_R - \bar\psi_R \psi_L$$

Eq. (6.1) BECOMES

$$\boxed{a^4 \sum_x \langle q(x) q(0) \rangle = m^2 a^4 \sum_x \langle P^0(x) P^0(0) \rangle - m \langle S^0 \rangle}$$

NOTE: NOT VERY USEFUL AS IT STANDS SINCE ON R.H.S. THERE ARE ILL DEFINED QUANTITIES!

- $\langle S^0 \rangle$ U.V. DIVERGENT FOR $m \neq 0$
- $\sum_x \langle P^0(x) P^0(0) \rangle$ INTEGRATED CORR. WITH POTENTIAL CONTACT TERMS NON INTEGRABLE
- $\sum_x \langle q(x) q(0) \rangle$ SAME AS BEFORE

NOTE: χ HAS A CHANCE TO BE DEFINED ONLY IF ON R.H.S. THERE ARE CANCELLATIONS

8) CHIRAL MULTIPLETS AND SYMMETRY WITH MANY FLAVOURS

$$\psi = (\psi_1, \dots, \psi_{N_f}) \quad , \quad \psi_{R,L} = \hat{P}_\pm \psi \quad , \quad \bar{\psi}_{R,L} = \bar{\psi} \hat{P}_\mp$$

$$\bar{\psi} = - - -$$

TRANSFORMATION UNDER CHIRAL GROUP $U(N_f)_L \otimes U(N_f)_R$

$$\psi'_L = V_L \psi_L \quad ; \quad \psi'_R = V_R \psi_R$$

$$\bar{\psi}'_L = \bar{\psi}_L V_L^\dagger \quad ; \quad \bar{\psi}'_R = \bar{\psi}_R V_R^\dagger$$

WITH

$$V_L = e^{i\varepsilon_L^0} e^{i\varepsilon_L^a T^a} \quad ; \quad V_R = e^{i\varepsilon_R^0} e^{i\varepsilon_R^a T^a}$$

$T^a = \frac{\tau^a}{2}$ $N_f = 2$, $T^a = \frac{\lambda^a}{2}$ $N_f = 3$

VECTOR SUBGROUP:

$$V_L = V_R = e^{i\varepsilon_V^0} e^{i\varepsilon_V^a T^a} \Rightarrow \varepsilon_L^0 = \varepsilon_R^0 = \varepsilon_V^0$$

$$\varepsilon_L^a = \varepsilon_R^a = \varepsilon_V^a$$

AXIAL TRANSFORM:

$$V_L^+ = V_R = e^{i\varepsilon_A^0} e^{i\varepsilon_A^a T^a} \Rightarrow -\varepsilon_L^0 = \varepsilon_R^0 = \varepsilon_A^0$$

$$-\varepsilon_L^a = \varepsilon_R^0 = \varepsilon_A^0$$

MASS TERM:

$$M^{+\prime} = V_R M^+ V_L^+ \quad ; \quad M' = V_L M V_R^+$$

9) θ AND M DEPENDENCE OF $F(\theta)$ WITH MANY FLAVOU

$$S = S_G - i\theta Q + S_F$$

$$S_F = a^4 \sum_x \left\{ \bar\psi D \psi + \bar\psi_R M^+ \psi_L + \bar\psi_L M \psi_R \right\}$$

AGAIN WE DEFINE

$$e^{-F(\theta, M, M^+)} \equiv \langle e^{i\theta Q} \rangle$$

AS BEFORE WE TAKE THE TRANSF.

$$\varepsilon_A^0 = -\frac{\theta}{2N_f} \Rightarrow V_L^+ = V_R = e^{-i\frac{\theta}{N_f}}$$

AND ~~THEREFORE~~:

$$\boxed{F(\theta, M, M^+) = F\left(0, e^{i\frac{\theta}{N_f}} M, e^{-i\frac{\theta}{N_f}} M^+\right)} \quad (9.$$

10) SINGLET AWI WITH SEVERAL FLAVOURS

FOLLOWING EXACTLY THE VERY SAME STEPS AS FOR $N_f = 1$

$$\boxed{a^4 \sum_x \langle q(x)\, q(0) \rangle = -\frac{m}{N_f^2} \langle S^0 \rangle + \left(\frac{m}{N_f}\right)^2 a^4 \sum_x \langle P^0(x)\, P^0(0) \rangle} \tag{10.1}$$

WHERE

$$M = M^+ = m I \quad , \quad S^0 = \bar\psi_L \psi_R + \bar\psi_R \psi_L \quad , \quad P^0 = \bar\psi_L \psi_R - \bar\psi_R \psi_L$$

<u>NOTE</u>: AS FOR $N_f = 1$ R.H.S. AND L.H.S. **ARE** ILL DEFINED ~~AS THEY STAND~~ UNLESS INTERESTING CANCELLATION HAPPEN ON R.H.S

11) NON-SINGLET AWIs :

FROM POINT 8

$$V_L^{+} = V_R = 1 + i\, \varepsilon_A^a T^a + \dots$$

$$\psi_L' = (1 - i\, \varepsilon_A^a T^a + \dots)\psi_L \quad , \quad \psi_R' = (1 + i\, \varepsilon_A^a T^a)\psi_R$$

$$\Downarrow$$

$$\psi' = \psi + i\, \varepsilon_A^a T^a \hat{\gamma}_5 \psi \quad , \quad \bar{\psi}' = \bar{\psi} + i\, \bar{\psi} \gamma_5 T^a \varepsilon_A^a$$

ANALOGOUSLY TO THE SINGLET CASE

$$\delta S_F = -a^4\, i\, \varepsilon_A^a \sum_x \bar{\psi}(x) T^a \Big\{ \cancel{\left[\gamma_5 D + D \hat{\gamma}_5 \right]}^{\,GW} \psi \Big\}(x) +$$

$$+ a^4 (+i\, \varepsilon_A^a) \sum_x \bar{\psi}_R \{ T^a, M^+ \} \psi_L +$$

$$+ a^4 (-i\, \varepsilon_A^a) \sum_x \bar{\psi}_L \{ T^a, M \} \psi_R$$

$$\Downarrow$$

$$\delta S_F = -a^4\, i\, \varepsilon_A^a \sum_x \left[-\bar{\psi}_R \{ T^a, M^+ \} \psi_L + \bar{\psi}_L \{ T^a, M \} \psi_R \right]$$

FOR A GENERIC OPERATOR ($\nu = 0$ IN THIS CASE)

$$\langle O \rangle = \frac{1}{Z} \int \delta U\, d\bar{\psi}\, d\psi\, e^{-S_G - S_F}\, O$$

BY REMEMBERING THAT

$$d\bar{\psi}' d\psi' = d\bar{\psi}\, d\psi$$

$$\langle O \rangle = \frac{1}{Z} \int \delta U\, d\bar{\psi}' d\psi'\, e^{-S_G}\, e^{-S_F - \delta S_F} (O + \delta O)$$

$$\Downarrow$$

$$\langle \delta S_F\, O \rangle = \langle \delta O \rangle$$

LET US TAKE $O = P^b$

$$P^b \equiv \bar{\psi}_L T^b \psi_R - \bar{\psi}_R T^b \psi_L$$

IT IS EASY TO SHOW

$$\delta P^b = -i\, \varepsilon_A^a \left[\bar{\psi}_L \{T^a, T^b\} \psi_R + \bar{\psi}_R \{T^a, T^b\} \psi_L \right]$$

IF WE NOW TAKE

$$M = M^+ = m\,\mathbb{1}$$

THE WIS READS

$$\boxed{2m\, a^4 \sum_x \langle P^a(x)\, P^b(0) \rangle = \frac{\delta^{ab}}{N_f} \langle S^0 \rangle} \qquad (11.1)$$

12) COMBINING SINGLET AND NON SINGLET AWI's!

BY COMPUTING THE WICK CONTRACTIONS

$$\langle P^0(x)\, P^0(0)\rangle = -N_f \langle \bigcirc \rangle + N_f \langle \mathrm{D}\,\mathrm{D} \rangle$$

$$\langle P^a(x)\, P^a(0)\rangle = -\frac{1}{2} \langle \bigcirc \rangle$$

THEREFORE IN EQ. (10.1) WE CAN REPLACE THE CONDENSATE BY $\langle P^a P^a \rangle$ AND OBTAIN

$$\boxed{a^4 \sum_x \langle q(x)\, q(0)\rangle = m^2 a^4 \sum_x \langle P_{\ell\ell}(x)\, P_{ss}(0)\rangle \qquad \ell \neq s,\ \text{NO SUMMATION OVER REPEATED INDICES} \qquad P_{ii}(x) = \bar{\psi}_{iL}\psi_{iR} - \bar{\psi}_{iR}\psi_{iL}}$$

(13.1)

<u>NOTE:</u> R.H.S. STILL ILL DEFINED UNLESS THERE ARE CANCELLATIONS