LECTURE 1: INTRODUCTION TO LATTICE GAUGE THEORY

1) GLUONIC ACTION IN CONTINUUM

$$S = \frac{1}{2g_0^2}\int d^4x \; \mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right]$$

$T^a = T^{a+}$

$\mathrm{Tr}\left[T^a T^b\right] = \frac{\delta^{ab}}{2}$

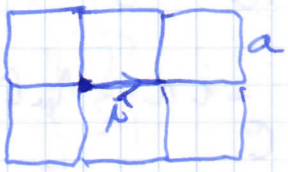
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] \quad ; \quad F_{\mu\nu} = F_{\mu\nu}^a T^a$$

GAUGE TRANSFORMATION

$$G(x) = e^{i\Lambda(x)} \quad , \quad \Lambda(x) = \lambda^a(x) T^a$$

$$A_\mu^G(x) = G(x) A_\mu(x) G^+(x) + i\, G(x)\,\partial_\mu G^+(x)$$

2) WILSON GLUONIC ACTION



- LINKS ARE FOND. GAUGE FIELD

$$U_\mu(x) = e^{-ia A_\mu(x)} \in SU(N)$$

- GAUGE TRANSFORM.

$$U_\mu^G(x) = G(x)\, U_\mu(x)\, G^+(x+\hat{\mu})$$

- PLAQUETTE

$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^+(x+\hat{\nu})\, U_\nu^+(x)$$

$$\Downarrow$$

$$U_{\mu\nu}^G(x) = G(x)\, U_{\mu\nu}(x)\, G^+(x)$$

WILSON ACTION

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{2N_c} \mathrm{Tr} \left\{ U_{\mu\nu}(x) + U^{+}_{\mu\nu}(x) \right\} \right]$$

$$\beta = \frac{2N_c}{g_0^2}$$ (We will be interested in $N_c = 3$)

NOTE: EXACT GAUGE INVARIANCE AT FINITE LATTICE SPACING

3) CLASSICAL CONTINUUM LIMIT

BY USING CAMPBELL-HAUSDORFF FORMULA

$$e^A e^B = e^{\{A + B + \frac{1}{2}[A,B]\} + \dots}$$

$$\Downarrow$$

$$U_{\mu\nu}(x) = e^{-ia\left\{A_\mu(x) + A_\nu(x+\hat{\mu}) - \frac{ia}{2}\left[A_\mu(x), A_\nu(x+\hat{\mu})\right] + \dots\right\}}$$
$$\times e^{ia\left\{A_\mu(x+\nu) + A_\nu(x) + \frac{ia}{2}\left[A_\mu(x+\nu), A_\nu(x)\right] + \dots\right\}}$$

AND USE AGAIN

$$U_{\mu\nu}(x) = e^{-ia\left\{A_\mu(x) + A_\nu(x+\hat{\mu}) - \frac{ia}{2}\left[A_\mu(x), A_\nu(x+\hat{\mu})\right] - A_\mu(x+\nu) - A_\nu(x) - \frac{ia}{2}\left[A_\mu(x+\nu), A_\nu(x)\right] + \frac{ia}{2}\left[A_\mu(x) + A_\nu(x+\hat{\mu}), A_\mu(x+\nu) + A_\nu(x)\right] + \dots\right\}}$$

$$\Downarrow$$

$$U_{\mu\nu}(x) = \exp\left\{-ia^2\left\{D_\mu A_\nu(x) - D_\nu A_\mu(x) - i\left[A_\mu(x), A_\nu(x)\right] + O(a)\right\}\right\}$$

↳ this is an Hermitian algebra thanks to the unitarity of $U_{\mu\nu}$.

WHERE THE FORWARD DERIVATIVE IS

$$\nabla_\mu f(x) = \frac{f(x+\hat{\mu}) - f(x)}{a}$$

BY DEFINING ANALOGOUSLY TO THE CONTINUUM

$$F_{\mu\nu}(x) \equiv \nabla_\mu A_\nu(x) - \nabla_\nu A_\mu(x) - i\left[A_\mu(x), A_\nu(x)\right]$$

THEN

$$U_{\mu\nu}(x) = \exp\left\{-ia^2\left[F_{\mu\nu}(x) + O(a)\right]\right\}$$

$$\Downarrow$$

$$S_G = \frac{\beta}{2}\sum_x \sum_{\mu,\nu}\Big[\cancel{1} - \frac{1}{2N_c}\mathrm{Tr}\Big[\cancel{1} - ia^2\cancel{\left\{F_{\mu\nu} + O(a)\right\}}$$

$$- \frac{1}{2}a^4 F_{\mu\nu}(x)F_{\mu\nu}(x) + \cancel{1} + ia^2\cancel{\left\{F_{\mu\nu} + O(a)\right\}}$$

$$- \frac{1}{2}a^4 F_{\mu\nu}(x)F_{\mu\nu}(x) + O(a^5)\Big]\Big]$$

By using Equations of motion becomes $O(a^6)$

SO AT THE END

$$\boxed{S_G = \frac{1}{2g_0^2}a^4\sum_x\sum_{\mu\nu}\mathrm{Tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + O(a)}$$

$$\Downarrow$$

$$\boxed{Z = \int \delta U\, e^{-S_G}}$$

Ok

4) FERMIONIC ACTION IN EUCLIDEAN:

$$S_F = \int d^4x\, \bar{\psi} \left\{ \gamma_\mu D_\mu + m \right\} \psi$$

$D_\mu \equiv \partial_\mu - i A_\mu \quad ; \quad \not{D} \equiv \gamma_\mu D_\mu$

5) NAIVE FERMION ACTION ON THE LATTICE

- WE CONSIDER THE FREE CASE

$$\nabla_\mu^* f(x) = \frac{f(x) - f(x-\mu)}{a}$$

AND WE USE THE SYMMETRIC DERIVATIVE

$$\frac{1}{2} \left\{ \nabla_\mu + \nabla_\mu^* \right\} \psi(x) = \frac{1}{2a} \left\{ \psi(x+\hat{\mu}) - \psi(x-\hat{\mu}) \right\}$$

THE NAIVE DISCRETIZATION IS GIVEN BY

$$S_F = a^4 \sum_x \frac{1}{2a} \left\{ \bar{\psi}(x) \gamma_\mu \psi(x+\hat{\mu}) - \bar{\psi}(x) \gamma_\mu \psi(x-\hat{\mu}) + \right.$$
$$\left. + m\, \bar{\psi}(x) \psi(x) \right\}$$

BY USING THE FOLLOWING CONVENTION FOR F.T.

$$f(x) = \int_{BZ} \frac{d^4k}{(2\pi)^4} \tilde{f}(k)\, e^{ikx} \qquad BZ^1 = \left[-\frac{\pi}{a}, \frac{\pi}{a} \right]$$

$$\tilde{f}(k) = a^4 \sum_x f(x)\, e^{-ikx}$$

$$a^4 \sum_x e^{-ix(k_1-k_2)} = (2\pi)^4 \delta_P^{(4)}(k_1-k_2)$$

WE INSERT F.T. IN S_F AND OBTAIN

(FROM NOW ON $\tilde{f}(k) \to f(k)$, IT IS CLEAR FROM ARG.)

$\to \bar{\psi}(k_1) \to \bar{\psi}(-k_1)$

$$S_F = \frac{a^4}{2a} \sum_x \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \, e^{-ik_1x} \, e^{ik_2x} \times$$

$$\times \left\{ \bar{\psi}(k_1) \gamma_\mu e^{ik_2\hat{\mu}} \psi(k_2) - \bar{\psi}(k_1) \gamma_\mu e^{-ik_2\hat{\mu}} \psi(k_2) + m\bar{\psi}(k_1)\psi \right.$$

$$\Downarrow$$

$$S_F = \int \frac{d^4q}{(2\pi)^4} \bar{\psi}(q) \left\{ \gamma_\mu \frac{2i}{2a} \sin(q_\mu a) + m \right\} \psi(q)$$

WE DEFINE

$$\boxed{\begin{gathered} \bar{q}_\mu = \frac{1}{a} \sin(q_\mu a) \;\; ; \;\; K(q) \equiv i\,\bar{q}_\mu \gamma_\mu + m \\ S_F = a^4 \int_{BZ} \frac{d^4q}{(2\pi)^4} \bar{\psi}(q) K(q) \psi(q) \end{gathered}}$$

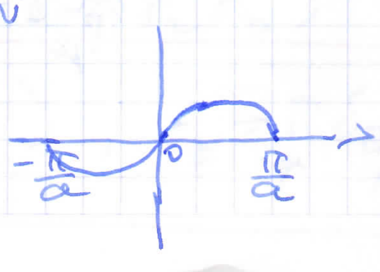
6) DOUBLING PROBLEM.

WE LOOK AT THE PROPAGATOR

$$\boxed{\langle \psi(x) \bar{\psi}(0) \rangle = \int_{BZ} \frac{d^4q}{(2\pi)^4} \frac{-i\gamma_\mu \bar{q}_\mu + m}{\sum_\mu \bar{q}_\mu^2 + m^2} e^{ikx}} \quad (6.1)$$

FOR $m=0$, IN ONE DIRECTION

$\bar{q}_\mu^2 = 0 \iff q_\mu = 0, \pm \frac{\pi}{a}$



THEREFORE IN 4-DIMENSIONS THE ϕ ARE

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ \pm\pi & \pm\pi & \pm\pi & \pm\pi \end{array} \Rightarrow 2^4 = 16 \text{ ZEROS}$$

ALL THESE ZEROS SURVIVE IN CONTINUUM LIMIT !!

7) CHIRALITY OF DOUBLERS :

LET US REWRITE (6.1) AS AN INTEGRAL OVER 16 ZONES, IN EACH DIRECTION

$$|q_\mu| < \frac{\pi}{2a} \quad , \quad \frac{\pi}{2a} \leq |q_\mu| \leq \frac{\pi}{a}$$

WE REWRITE

$$q_\mu = \bar{p}_\mu + \tilde{p}_\mu$$

WHERE $a\tilde{p}_\mu = (0,0,0,0), (\pi,0,0,0) + \ldots, (\pi,\pi,0,0) \ldots$ permutations

$$\boxed{\langle\psi(x)\bar{\psi}(0)\rangle = \sum_{\tilde{p}} e^{i\tilde{p}x} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4 p}{(2\pi)^4} \frac{-i\sum_\mu \delta_{\tilde{p}_\mu}\gamma_\mu \bar{p}_\mu + m}{\sum_\nu \bar{p}_\nu^2 + m^2} e^{i\bar{p}x}}$$

where

$$\delta_{\tilde{p}_\mu} = e^{i\tilde{p}_\mu a}$$

BY NOTICING THAT

$$\tau_{\tilde p}\, \gamma_\mu\, \tau_{\tilde p}^{-1} = \underbrace{\delta_{\tilde p_\mu}}\, \gamma_\mu$$

i.e. ~~THESE MATRICES ARE~~ RELATED TO THE ORIGINAL ON BY A SIMILARITY TRANSFORMATION ($\tilde\tau_\mu = \gamma_\mu \gamma_5$, $\tilde\tau_{\mu\nu} = \gamma_\mu \gamma_\nu$, $\tilde\tau_{\mu\nu\lambda} = \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_5$, $\tilde\tau_{\mu\nu\lambda\rho} = \gamma_5$)

$$\boxed{\langle \psi(x) \bar\psi(0) \rangle = \sum_{\tilde p} e^{i \tilde p x}\, \tau_{\tilde p} \int_{-\frac{\pi}{2a}}^{\pi/2a} \cdots \frac{d^4 p}{(2\pi)^4} \frac{-i \bar p_\mu \gamma_\mu + m}{\sum_\mu \bar p_\mu^2 + m^2}\, e^{ipx}\, \tau_{\tilde p}}$$

ONE NAIVE FERMION ON THE LATTICE REPRESENT 16 CONTINUUM FERMION SPECIES ALL WITH MASS m

- LET US FOCUS ON 1 CHIRAL FERMION IN CONTINUUM → THE NAIVE LATTICE COUNTERPART IS

$$\psi_{R,L} \equiv \frac{1}{2}(1 \pm \gamma_5)\psi \quad ; \quad \bar\psi_{R,L} \equiv \frac{1}{2}\bar\psi(1 \mp \gamma_5)$$

$$= P_\pm \psi \qquad\qquad = \bar\psi P_\mp$$

THE PROP

$$\langle \psi_L(x)\, \bar\psi_L(0) \rangle = \sum_{\tilde p} e^{i\tilde p x}\, P_L \int_{-\frac{\pi}{2a}}^{\pi/2a} \frac{d^4p}{(2\pi)^4}\, \tau_{\tilde p} \left\{ \frac{-i \sum_\mu \gamma_\mu \bar p_\mu}{\sum_p \bar p^2} \right\} \tau_{\tilde p}^{-1}\, e^{ipx}$$

$\tau_{\tilde p}^{-1} \tau_{\tilde p}$

Now it is easy to verify ($\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$)

$$\boxed{\tau_{\tilde p}^{-1} (1 - \gamma_5)\, \tau_{\tilde p} = (1 - \varepsilon_{\tilde p}\, \gamma_5)\,, \quad \varepsilon_{\tilde p} = \prod_\mu \delta_{\tilde p_\mu}}$$

THEREFORE

$$\langle \psi_L(x)\bar{\psi}_L(0)\rangle = \sum_{\tilde{p}} e^{i\tilde{p}x}\, \tau_{\tilde{p}} \left\{ \int_{-\frac{\pi}{2a}}^{\pi/2a} \cdots \frac{d^4p}{(2\pi)^4}\, \frac{1}{2}\left(1-\varepsilon_{\tilde{p}}\gamma_5\right) \times \right.$$
$$\left. \times \left(\frac{-i\,\varepsilon_\mu \gamma_\mu \bar{p}_\mu}{\varepsilon_\mu \bar{p}_\mu^2} \right) e^{ipx} \right\} \tau_{\tilde{p}}^{-1}$$

BY COUNTING WE FIND 8 POLES WITH $\varepsilon_{\tilde{p}}=1$ AND 8 POLES WITH $\varepsilon_{\tilde{p}}=-1$

ONE NAIVE LATTICE CHIRAL FERMION REGOLARIZE 8 LEFT-HANDED FERMIONS AND 8 RIGHT-HANDED FERMIONS !!

IT IS NOT A CHIRAL FERMION !!

8) NIELSEN - NINOMIYA THEOREM

DESIRABLE PROPERTIES OF $K(q)$

1) $D(q)$ SMOOTH FUNCTION OF q_μ WITH PERIOD $\frac{2\pi}{a}$

2) FOR $q_\mu \ll \frac{\pi}{a}$ $\quad D(q) = i\gamma_\mu q_\mu + O(aq^2)$

3) $D(q)$ INVERTIBLE AT ALL $q \neq 0$ (MOD $\frac{2\pi}{a}$)

4) $\{\gamma_5, D(q)\} = 0$

SKETCH OF DEMONSTRATION

LET US ASSUME 1), 2) AND 4) FOR A CHIRAL FERMION. THEN

$$D(q) = i\, P_- \gamma_\mu F_\mu(q) \qquad F_\mu(q) \in \mathbb{R}e$$

- LET US DEVELOP AROUND A GENERIC ϕ OF F_μ

$$F_\mu(q) = M_{\mu\nu}(q_\nu - \tilde{p}_\nu) + \cdots$$

AND WE ASSUME THE GENERIC CASE WHERE $M_{\mu\nu}$ IS NON-SINGULAR (hypothesis 2+3).

- M IS REAL MATRIX THAT CAN BE DECOMPOSED

$$M = O\, S \quad ; \quad O O^T = 1, \quad z^T S z > 0 \quad \|z\| \neq 0$$

- THE MATRIX O CAN BE REABSORBED INTO A REDEFINITION OF FERMION FIELDS

$$P_- \gamma_\mu O_{\mu\nu} = \tau_{\tilde{p}} \frac{(1 - \varepsilon_{\tilde{p}} \gamma_5)}{2} \gamma_\nu \tau_{\tilde{p}}^{-1} \quad ; \quad \varepsilon_{\tilde{p}} = \det O$$

- SO:

$$\boxed{D(q) = i\, \tau_{\tilde{p}} \frac{(1 - \varepsilon_{\tilde{p}} \gamma_5)}{2} \gamma_\alpha \tau_{\tilde{p}}^{-1} S_{\alpha\nu} (q_\nu - \tilde{p}_\nu)}$$

$S_{\alpha\nu}(q_\nu - \tilde{p}_\nu)$ is a positive rescaling and a rotation. **DOES NOT AFFECT MOMENTUM SPACE ORIENTATION** $\Rightarrow$ so the ϕ has chirality $\varepsilon_{\tilde{p}}$

THEOREM POINCARÉ - Hopf $\Rightarrow$ the sum of the ε_p

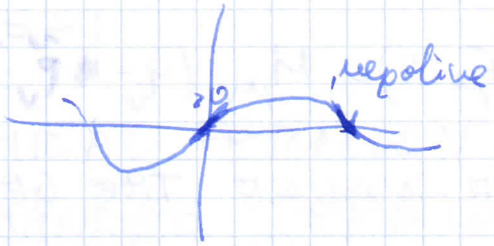
$$\sum_p \varepsilon_p = 0$$

ONE DIMENSIONAL CASE:

continuous periodic function vanishes with positive derivative as many times as with negative

$$\Downarrow$$

$$\sum_p \varepsilon_p = 0$$



IF $\sum_p \varepsilon_p = 0 \Rightarrow$ LATTICE NAIVE CHIRAL FERMION IS ACTUALLY A REGULARIZATION FOR AS MANY LEFT FERMIONS AS RIGHT ONES

HYPOTESYS 3 CANNOT BE SATISFIED!!

9) WILSON ACTION

SOLUTION: ABANDON 4) AND BREAKS CHIRALITY EXPLICITLY

$$S_F = a^4 \sum_x \bar{\psi} \{ D_W + m_0 \} \psi$$

$$D_W = \frac{1}{2} \{ \gamma_\mu (\nabla_\mu^* + \nabla_\mu) - \nabla_\mu^* \nabla_\mu \}$$

$$\nabla_\mu \psi = U_\mu(x) \psi(x+\hat{\mu}) - \psi(x)$$

$$\nabla_\mu^* \psi = \psi(x) - U_\mu^+(x-\hat{\mu}) \psi(x-\hat{\mu})$$

$$K(q) = i \gamma_\mu \bar{q}_\mu + m_0 + \frac{a}{2} \hat{q}^2$$

$$\hat{q} = \frac{2}{a} \sin\left(\frac{q_\mu a}{2}\right)$$

$$\boxed{\langle \psi(x) \bar{\psi}(0) \rangle = \int \frac{d^4 q}{(2\pi)^4} \frac{-i \gamma_\mu \bar{q}_\mu + m_0 + \frac{a}{2} \hat{q}^2}{\sum_\mu \bar{q}_\mu^2 + (m_0 + \frac{a}{2} \hat{q}^2)^2}}$$

For $m=0$ and $q_\mu = \frac{\pi}{a}$, $\hat{q} = \frac{2}{a}$ => DOUBLERS HAVE MASS THAT $\to \infty$ WHEN $a \to 0$. ONLY ONE MASSLESS POLE IN CONTINUUM LIMIT.