ARCH process: A simple model of stochastic variance - H.E. Roman

Consider a random variable $\Delta x_t = x_t - x_{t-1}$, representing the variation of the stochastic quantity x_t at the discrete time step t. Within an ARCH scheme, Δx_t is defined as,

$$\Delta x_t = \sigma_t \; \eta_t, \qquad t \geq 1, \tag{1}$$

where σ_t is a $t-$dependent standard deviation of the process and η_t is a random number drawn from a normal distribution (zero mean and unit variance). For a simple ARCH process (Engle, 1982), the variance σ_t^2 in Eq. (1) obeys the recursion relation

$$\sigma_t^2 = a + b \; (\Delta x_{t-1})^2, \tag{2}$$

where a and b are positive constants. To start the recursion relation one can take $\Delta x_0 = 0$. It is known that Δx_t has a probability distribution function (PDF) which decays as a power law at the tails,

$$P(\Delta x) \sim |\Delta x|^{-(1+\alpha)}, \qquad |\Delta x|/\sigma_{\rm A} \gg 1, \tag{3}$$

where $\sigma_{\rm A} \equiv \langle \sigma_{\rm A}^2 \rangle^{1/2}$ (see below). The power-law exponent α is a function of the parameter b alone obeying [**P79**],

$$b^{-\alpha/2} = \frac{2^{\alpha/2}}{\sqrt{\pi}} \Gamma\left(\frac{1+\alpha}{2}\right), \tag{4}$$

where $\Gamma(x)$ is the Gamma function. From the latter relation one sees that $P(\Delta x)$ enters the Lévy regime ($0 < \alpha < 2$) when $b > 1$. Indeed, $\alpha = 2$ when $b = 1$, as can be easily seen from Eq. (4). Note that the ARCH variable Δx_t has vanishing autocorrelation for finite time lags, i.e.

$$\langle \Delta x_\ell \Delta x_m \rangle = \langle \sigma_{\rm A}^2 \rangle \; \delta_{\ell,m}, \tag{5}$$

where the ARCH mean variance $\langle \sigma_{\rm A}^2 \rangle$ can be obtained directly from Eq. (2) yielding

$$\langle \sigma_{\rm A}^2 \rangle = \frac{a}{1-b}. \tag{6}$$

Note that $\langle \sigma_{\rm A}^2 \rangle \to \infty$ when $b \to 1^-$.
The generalization of Eq. (2) to an arbitrary exponent $q > 0$ of the form,

$$\sigma_t^2 = (a + b \; |\Delta x_{t-1}|^q)^{2/q}, \tag{7}$$

has been discussed in [**P79**].

[**P79**] H.E. Roman and M. Porto, Phys. Rev. E63, 36128 (2001)