Very slowly decaying auto-correlations: Application to volatility in financial markets

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Abstract

We discuss empirical data associated to the Dow-Jones Industrial Average stock market index over a daily basis during the period 1928-2007. We study the statistical properties of the logarithmic daily variations of the index, such as the probability distribution function and autocorrelation functions, for both log-returns and their absolute values. We consider in particular the effect of detrending the empirical data with polynomial functions of different degrees and find that stationary results are obtained for degrees larger than about four. A model based on an autoregressive scheme with long-time memory is briefly discussed.

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1 Introduction

Stock price variations generally behave independently to past changes on not too short time scales, behavior believed to be due to the so-called noise traders. Roughly speaking, this fact is traditionally interpreted by affirming that once a transaction has taken place all the information governing the price change is already contained in the price itself (the so-called efficient market hypothesis, EMH, see for instance [1]).

There are other intriguing issues related to the behavior of stock markets which make them so fascinating and challenging, that is, their non-stationarity and the long-time autocorrelation functions for absolute price returns. In this paper we address these two issues and study for simplicity a single time series, corresponding to a large and liquid market, the Dow Jones Industrial Average (DJIA) (see e.g. [2]).

We consider the logarithm of the index as variable and study the statistical behavior of the log-returns and its absolute values. We show that a trend present in the daily log-returns can be described by a polynomial of degree L, with $0 \leq L \leq 5$ giving the most relevant results for our purposes. The probability distribution function (PDF) of log-returns is calculated for the raw data and for each set of detrended values depending on the degree L. We find that stationary results are obtained already for $L \simeq 4$ and discuss results for the case $L = 5$. We find that the autocorrelation function for absolute log-returns displays a power-law decay with an exponent $\simeq -0.2$ for the raw data, decreasing down to about -0.4 for the detrended case with $L = 5$. A model based on an autoregressive process with long-range memory is discussed to generate surrogate data with similar statistical properties.

2 Daily log-returns for the DJIA (1928-2007)

In this section, we consider logarithmic Index returns and their PDF. We discuss the issues of non-stationarity, detrending of the data and the asociated auto-correlation functions of log- and absolute-log returns.

2.1 Log-returns and the PDF

The quantity of interest to us is the logarithmic price return (or simply log-return),

$$\Delta S_t^{(n)} = \log P_n(t) - \log P_n(t-1), \tag{1}$$

where $P_n(t)$ is the closing price of market Index n at trading day t. Here, n stands for the DJIA and $1 \leq t \leq T$, with $T = 20000$ trading days corresponding to the period (1928-2007). In order to study the PDF of log-returns, we consider the scaled variable,

$$y_t^{(n)} = \frac{1}{\Sigma_n}\big[\Delta S_t^{(n)} - \big\langle \Delta S_t^{(n)} \big\rangle_T\big], \tag{2}$$

where Σ_n is the standard deviation of log-returns over the T trading days for the index n, and $\big\langle \Delta S_t^{(n)} \big\rangle_T$ is the corresponding mean value. The PDF associated to $y_t^{(n)}$, $P_n(y)$, has zero mean and unit variance. In the following, we discard the label n from the resulting expressions and denote the PDF simply as $P(y)$.

Log-returns for the DJIA are shown in Fig. 1 as a function of the trading day. The data starts at year 1928 and therefore the first few data points correspond to the 1929 famous market crash. Despite it, the largest one day

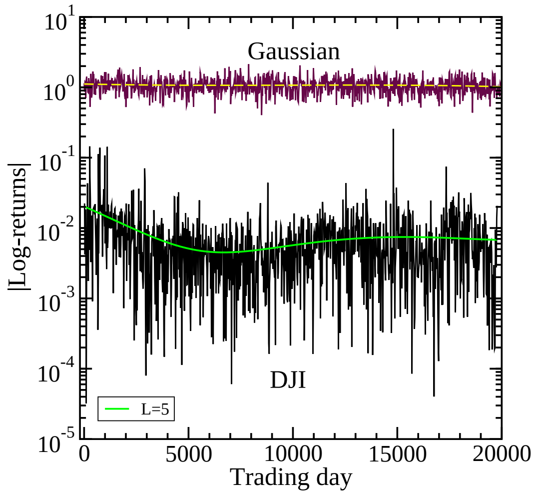

Figure 1: The absolute values of log-returns for the DJIA as a function of the trading day in semi-log scale. Raw data taken from [2]. The peak around day 15000 corresponds to black monday crash (19 October 1987). The continuous line is a polynomial fit of fifth degree ($L = 5$). For comparison we show results drawn from a Gaussian distribution with the same standard deviation as the empirical data (data shifted upwards for convenience). The dashed line is a polynomial fit of third degree, indicating a constant standard deviation.

variation occurred in October 1987, during the so-called black monday. As one can see, the amplitude of log-returns are not constant over the time, but fluctuate somehow wildly over the years. This behavior can be compared with the standard one, i.e. a constant 'volatility' process, of a Gaussian variable displaying the same Σ_n (see upper curve in Fig. 1). Regarding the empirical data, we show in the same plot the fifth-degree polynomial fit (continuous line), denoted as $P_5(t)$, yielding the detrended log-returns $\Delta S_t^{(5)} = \Delta S_t - P_t^{(5)}$.

The PDF's are shown in Fig. 2 in the cases of the raw data (upper panel) and detrended ones $\Delta S_t^{(5)}$ (lower panel). As one can see, the main effect of the 5th-degree polynomial trend on the PDF is the narrowing of its shape, which in our power-law representation of the distribution leads to a larger power-law exponent of about -3.5 compared to the raw-data one of -3.1.

Due to the limited data presently available for the DJIA, we can not draw a definitive conclusion about the shape of the PDFs in neither case, but the present results suggest in fact a 'lesser-fat-tailed' distribution for the detrended log-returns. We consider next the issue of autocorrelations and discuss the effects of trends on them.

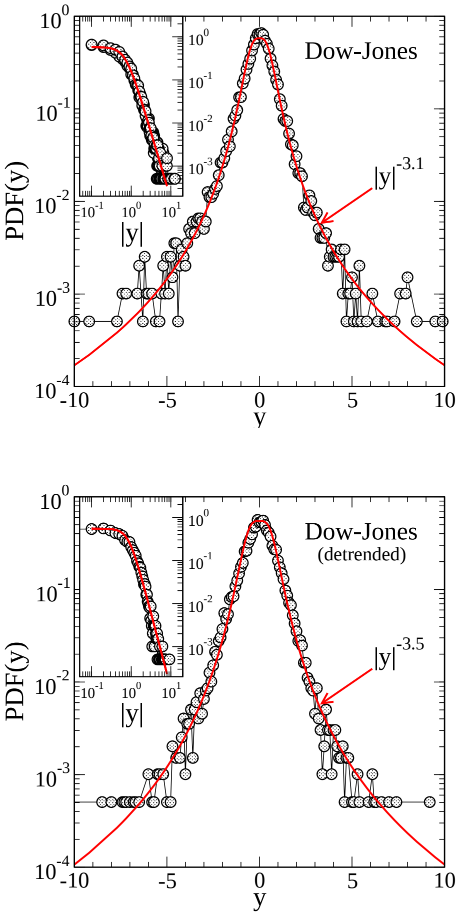

Figure 2: The PDF of log-returns for the DJIA (1928-2007). (*Upper panel*) Raw data: The line is a fit with the form $F = 0.58/(1 + |y/0.73|^{3.1})$. The inset shows the PDF of absolute returns, $P(|y|)$ (scale to the right side), and for the fit, in a double-logarithmic plot. (*Lower panel*) Detrended data using a fifth degree polynomial. The line is a fit with the form $F = 0.55/(1 + |y/0.87|^{3.5})$ and the inset shows the absolute log-return counterparts in double-log scale.

2.2 Autocorrelations

The EMH implies that the autocorrelation function of log-returns, defined as $\mathcal{C}_y(\tau) = \left\langle y_{t+\tau}\ y_t \right\rangle_T$, vanishes for all time scales $|\tau| > 0$, i.e. $\mathcal{C}_y(\tau) = \delta_{\tau,0}$. This has been verified numerically for the present data, as shown in Fig. 3. More intriguing is the behavior of the autocorrelation function of absolute log-returns, $\mathcal{C}_{|y|}(\tau) = \left(\left\langle |y_{t+\tau}|\ |y_t| \right\rangle_T - \left\langle |y_t| \right\rangle_T^2 \right) / \Sigma^2$, where $\Sigma^2 = \left\langle |y_t|^2 \right\rangle_T - \left\langle |y_t| \right\rangle_T^2$,

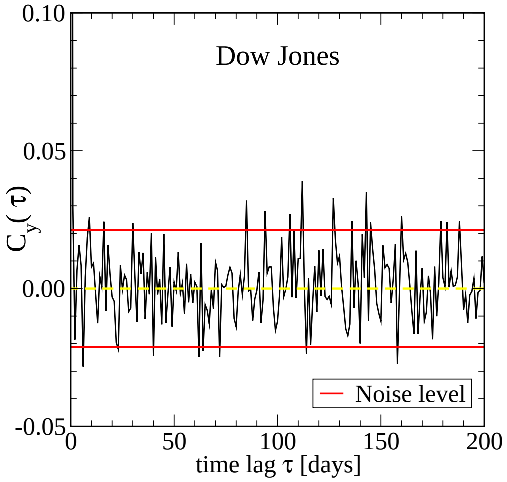

Figure 3: The autocorrelation function of log-returns for the DJIA (1928-2007). The horizontal continuous lines represent the noise level, within which the function is considered to vanish.

which is shown in Fig. 4, suggesting a very slow power-law decay, $\mathcal{C}_{|y|}(\tau) \sim \tau^{-\gamma}$, with $\gamma \simeq 0.2$, at least up to about 200 days. Remarkable is the effect of trends on $\mathcal{C}_{|y|}(\tau)$, which we have calculated for $\Delta S_t^{(5)}$ and displayed by the square symbols in Fig. 4. Now, the autocorrelations are much weaker, with a decaying exponent $\gamma \simeq 0.4$.

3 Long-range memory and stochastic volatility

In this section, we consider the modeling of log-returns using an autoregressive model with conditional heteroskedasticity (ARCH) supplemented by the presence of a long-range memory for the standard deviation or volatility. The asset log-return at time step t, here denoted as ΔX_t, is modeled according to,

$$\Delta X_t = \sigma_t \ \eta_t, \quad \text{with} \quad \sigma_t^2 = a + b \ [\Delta X_{\mathrm{AB}}(t-1)]^2, \tag{3}$$

where η_t are uncorrelated random numbers drawn from a normal distribution (zero mean and unit variance), σ_t obeys an ARCH-type [3] recursion relation with a and b positive constants, and $\Delta X_{\mathrm{AB}}(t)$ is a long-time correlated variable, based on a fractional Brownian motion [4], defined as [5],

$$\Delta X_{\mathrm{AB}}(t) = \frac{\Delta X_t}{C_{\mathrm{H}}(t)} + \sum_{j=1}^{t-1} \left[\frac{(2j+1)^\beta}{C_{\mathrm{H}}(t)} - \frac{(2j-1)^\beta}{C_{\mathrm{H}}(t-1)} \right] \Delta X_{t-j}, \quad t \geq 1, \tag{4}$$

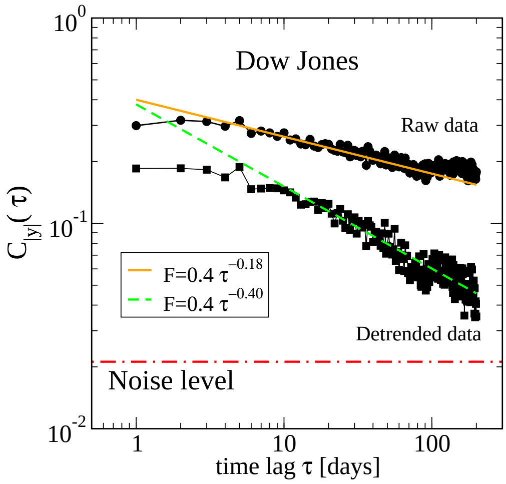

Figure 4: The autocorrelation function of absolute log-returns for the DJIA (1928-2007) for the: Raw data (full circles), and detrended data (full squares) using a polynomial of fifth degree (see Fig. 1). The lines are power-law forms with slopes: $\gamma = -0.18$ (continuous line) and -0.40 (dashed line), respectively. The dashed-dotted line represents the noise level.

where $C_{\rm H}^2(t) = 1 + \sum_{i=1}^{t-1} \kappa_j^2(\beta)$, $\beta = (1-\gamma)/2$ and $\Delta X_0 = 0$. The above model is found to yield results consistent with the empirical data, and is able to describe accurately the so-called 'leverage effect' observed in log-returns [6]. More details on this interesting issue will be presented elsewhere.

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