

7. Nuclear models

7.1 Fermi gas model

7.2 Shell model

Fermi Gas Model

- Free electron gas: protons and neutrons moving quasi-freely within the nuclear volume
- 2 different potentials wells for protons and neutrons.
- Spherical square well potentials with the same radius

Statistics of the Fermi distribution

Given a volume V the numer of states dn goes like:

$$dn = \frac{4\pi p^2 dp V}{(2\pi\hbar)^3}$$

$$n = \int_0^{p_F} \frac{4\pi p^2 dp V}{(2\pi\hbar)^3} = \frac{V p_F^3}{6\pi^2 \hbar^3}$$

$$N = \underset{\downarrow}{2n} = \frac{V p_{F,n}^3}{3\pi^2 \hbar^3} \quad Z = \frac{V p_{F,p}^3}{3\pi^2 \hbar^3}$$

If $T=0$ the nucleus is in the ground state and p_F (**Fermi Momentum**) is the maximum possible momentum of the ground state.

N = number of neutrons

Z = number of protons

Spin 1/2 particles

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A \quad R_0 = 1.21 \text{ fm}$$

$$Z = N = A/2 \quad \frac{A}{2} = \frac{4}{3} \frac{\pi R_0^3 A p_F^3}{3\pi^2 \hbar^3} \Rightarrow p_F = p_{F,n} = p_{F,p} = \frac{\hbar}{R_0} \left(\frac{9\pi}{8} \right)^{\frac{1}{3}} \approx 250 \text{ MeV}/c$$

The nucleon moves in the nucleus with a large momentum

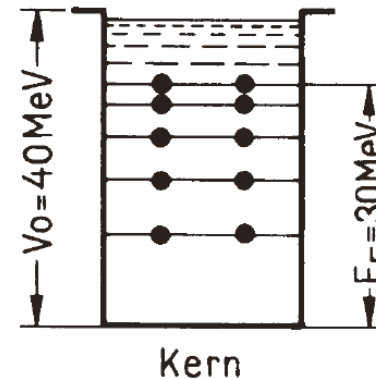
Fermi Energy

$$E_F = \frac{p_F^2}{2m_N} \approx 33 \text{ MeV}$$

Binding Energy: $BE/A = 7-8 \text{ MeV}$

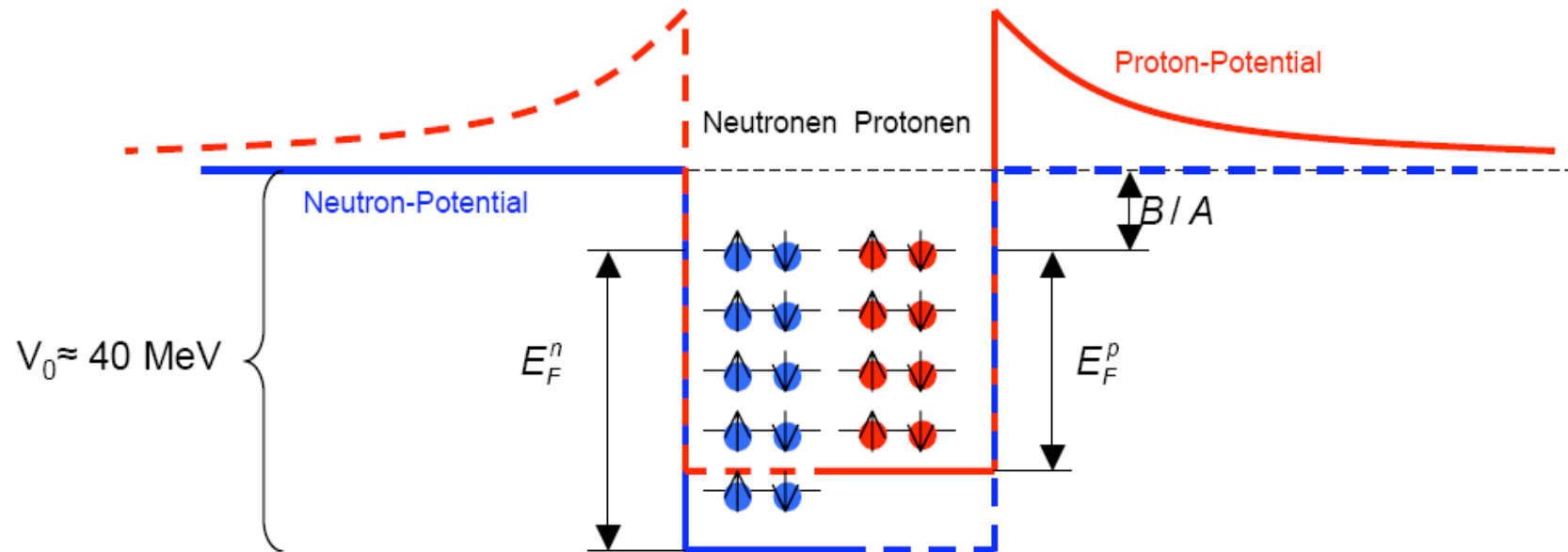
$$V_0 = E_F + B/A \sim 40 \text{ MeV}$$

→ Nucleons are rather weakly bound in the nucleus



Potential well in the Fermi-gas model

The **neutron** potential well is deeper than the **proton** well because of the missing Coulomb repulsion. The Fermi Energy is the same, otherwise the $p \rightarrow n$ decay would happen spontaneously. This implies that there are more neutron states available and hence $N > Z$ the heavier the nuclei become.



$$\langle E_{Kin} \rangle = \frac{\int_0^{p_F} E_{Kin} p^2 dp}{\int_0^{p_F} p^2 dp} = \frac{3}{5} \frac{p_F^2}{2m_N} \approx 20 \text{ MeV}$$

$$E_{Kin}(N, Z) = N \langle E_{Kin, N} \rangle + Z \langle E_{Kin, Z} \rangle = \frac{3}{10m_N} (Np_{F,N}^2 + Zp_{FZ}^2)$$

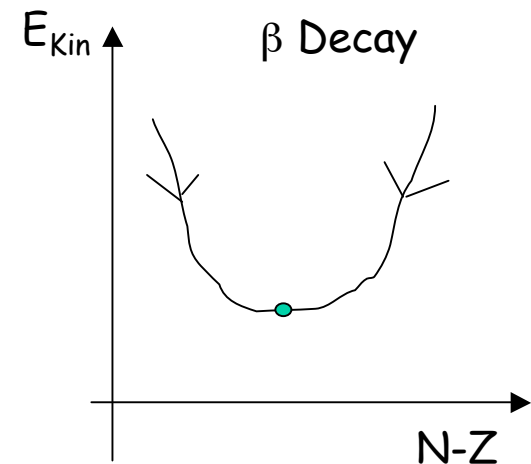
$$p_{FN} = \left(\frac{N \cdot 3\pi^2 \hbar^3}{V} \right)^{\frac{1}{3}} = \left(\frac{N \cdot 9\pi^2 \hbar^3}{4\pi R_0^3 A} \right)^{\frac{1}{3}}$$

$$p_{FZ} = \left(\frac{Z \cdot 3\pi^2 \hbar^3}{V} \right)^{\frac{1}{3}} = \left(\frac{Z \cdot 9\pi^2 \hbar^3}{4\pi R_0^3 A} \right)^{\frac{1}{3}}$$

$$E_{Kin} = \frac{3}{10m_N} \left(\frac{9\pi^2 \hbar^3}{4\pi R_0^3} \right)^{\frac{2}{3}} \left(\frac{Z^{\frac{5}{3}} + N^{\frac{5}{3}}}{A^{\frac{2}{3}}} \right)$$

$$E_{Kin} = \frac{\hbar^2}{R_0^2} \frac{3}{10m_N} \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} \frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}} = \frac{3}{10m_N} \frac{\hbar^2}{R_0^2} \left(\frac{9\pi}{8} \right)^{\frac{2}{3}} \left(A + \frac{5}{9} \frac{(N-Z)^2}{A} + \dots \right)$$

p^2 , se $Z = N$, di ciascun nucleone
F



L'espressione precedente si ottiene da:

$$D = N - Z$$

$$N = \frac{A}{2} + \left(\frac{N - Z}{2} \right) = \frac{1}{2}(A + D)$$

$$Z = \frac{A}{2} - \left(\frac{N - Z}{2} \right) = \frac{1}{2}(A - D)$$

sostituendo in:

$$\left(\frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \right) = \left(\frac{\left(\frac{A + D}{2} \right)^{5/3} + \left(\frac{A - D}{2} \right)^{5/3}}{A^{2/3}} \right)$$

e sviluppando i binomi arrestandosi al secondo ordine:

$$\left(\frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \right) = \left(\frac{1}{2} \right)^{5/3} \left(\frac{A^{5/3} + \frac{5}{3} A^{2/3} D + \frac{10}{18} A^{-1/3} D^2 + A^{5/3} - \frac{5}{3} A^{2/3} D + \frac{10}{18} A^{-1/3} D^2}{A^{2/3}} \right)$$

$$\left(\frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \right) = \left(\frac{1}{2} \right)^{5/3} \left(2A + \frac{10}{9} \frac{D^2}{A} \right) = \left(\frac{1}{2} \right)^{2/3} \left(A + \frac{5}{9} \frac{D^2}{A} \right)$$

NOTA BENE:

A) Il risultato prova che l'energia cinetica totale ha un MINIMO quando $N = Z$, e dato che $E(\text{totale}) = T - V$, con T proporzionale ad A , $E(\text{totale})$, che è negativa ed è anche la BINDING ENERGY totale, è proporzionale ad A (primo termine dell'equazione semiempirica delle masse)

B) Aggiungendo il secondo termine, come nell'equazione semiempirica delle masse, la BINDING ENERGY diminuisce in modulo. A parte i valori dei coefficienti, il modello a gas di Fermi predice correttamente

$$EB = T - V = -c_{Vol} A + c_{Sym} \frac{(N - Z)^2}{A}$$

Calculation of the state density

Considering the approximation that the nuclear potential should have a sharp edge in correspondence of the nuclear radius, one can approximate that to particles trapped in the potential

$$-\frac{\hbar^2}{2m}\Delta\psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2} = E_x X(x) \dots \psi(r) = X(x)Y(y)Z(z)$$

$$YZE_x X + XZE_y Y + X YE_z Z = EXYZ$$

$$\frac{\partial^2 X(x)}{\partial x^2} = -kX \quad k = \frac{1}{\hbar}\sqrt{2mE}$$

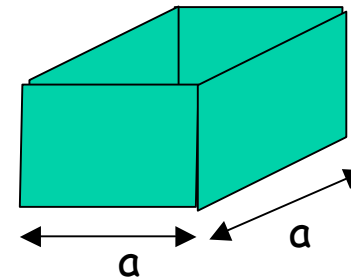
$$X_\lambda = A_\lambda e^{ik_\lambda x} + B_\lambda e^{-ik_\lambda x} \quad \text{Boundary Conditions: at } x = y = z = \frac{a}{2} : X = Y = Z = 0$$

$$X_\lambda^+ = \frac{2}{\sqrt{2a}} \cos k_\lambda^+ x \quad X_\lambda^- = \frac{2i}{\sqrt{2a}} \sin k_\lambda^- x \quad k_\lambda^+ = \frac{\pi\lambda^+}{a}, \lambda = 1, 3, 5, \dots \quad k_\lambda^- = \frac{\pi\lambda^-}{a}, \lambda = 0, 2, 4, \dots$$

$$EX = \frac{\hbar^2}{2m} k_\lambda^2 = \frac{1}{2m} \left(\frac{\pi\lambda\hbar}{a} \right)^2$$

$$E = \frac{1}{2m} \left(\frac{\pi\hbar}{a} \right)^2 (\lambda_x^2 + \lambda_y^2 + \lambda_z^2) \quad p^2 = 2mE = \left(\frac{\pi\hbar}{a} \right)^2 (\lambda_x^2 + \lambda_y^2 + \lambda_z^2)$$

$$\Omega \approx \lambda^3 \approx a^3 p^3$$



Volume in which the state density is calculated

We count how many states we have in a spherical volume of radius between ρ and $\rho+\delta\rho$

$$\rho^2 = \lambda_x^2 + \lambda_y^2 + \lambda_z^2 \quad d\Omega = 4\pi\rho^2\delta\rho$$

$$dn = \frac{1}{8} d\Omega = \frac{\pi}{2} \rho^2 \delta\rho \quad \rho = \frac{a}{\pi\hbar} p \quad d\rho = \frac{a}{\pi\hbar} dp$$

$$dn = \frac{\pi}{2} \frac{a^2}{\pi^2 \hbar^2} p^2 \frac{a}{\pi\hbar} dp = \frac{a^3}{2\pi^2 \hbar^3} p^2 dp = \frac{4\pi p^2 dp v}{(2\pi\hbar)^3}$$

Momentum volume

Space volume

Phase-space occupied by one state

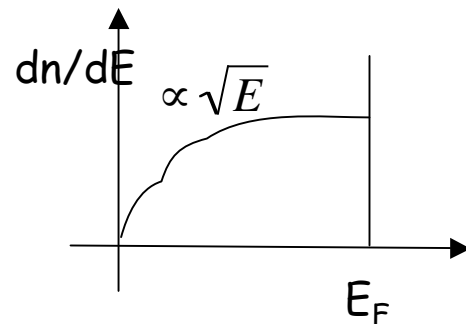
$$p^2 = 2mE \quad p^2 dp = \sqrt{2m^3 E} dE$$

$$dn = m^{\frac{3}{2}} (\sqrt{2\pi^2 \hbar^3})^{-1} v dE \sqrt{E} = C_1 \sqrt{E} dE$$

Number of spin 1/2 particles which sit in the well:

$$n = \int_0^{E_F} 2 \frac{dn}{dE} dE = 2C_1 v \int_0^{E_F} \sqrt{E} dE = (\sqrt{8} m^{\frac{3}{2}}) (3\pi^2 \hbar^3)^{-1} v E_F^{\frac{3}{2}}$$

E_F = maximal energy of the particle in the well



$$E_F = \left(\frac{1}{2m} \right)^{\frac{2}{3}} 3^{\frac{2}{3}} \pi^{\frac{4}{3}} \hbar^2 \left(\frac{n}{v} \right)^{\frac{2}{3}} \approx 30 \text{ MeV}$$

$$\text{in Cu atoms: } \frac{n}{v} = 6 \cdot 10^{23} \cdot \frac{9}{64} = 8 \cdot 10^{22} \text{ cm}^{-3} \Rightarrow E_F \approx 7 \text{ eV}$$

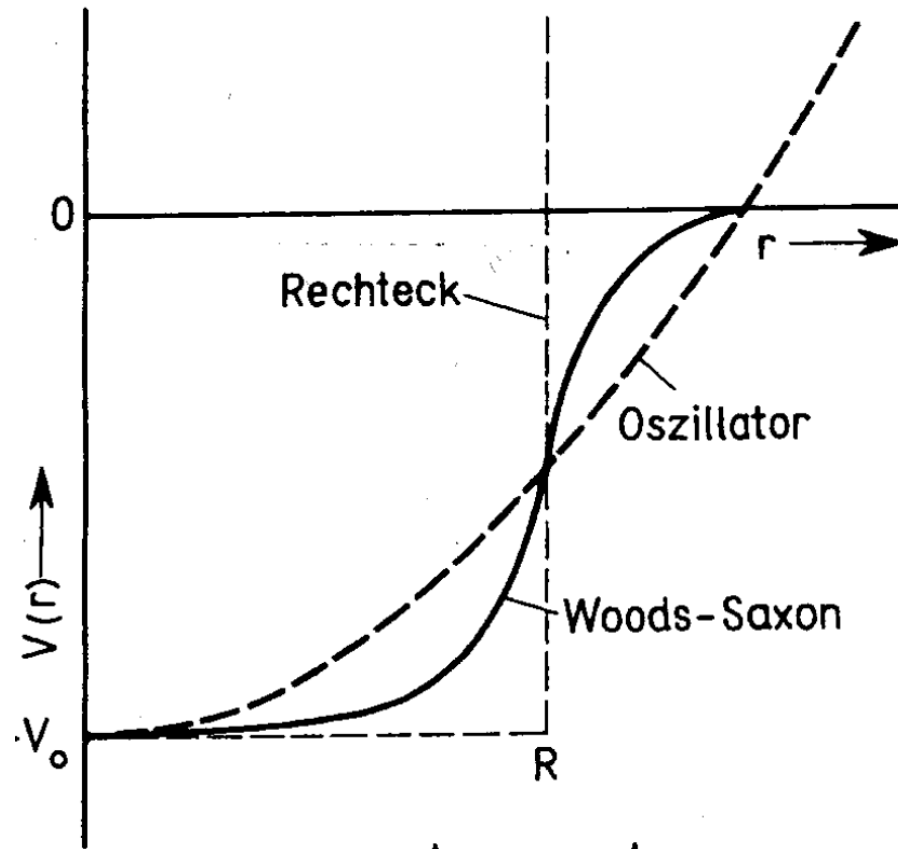
Hierarchy of energy eigenstates of the harmonic oscillator potential

N	0	1	2	2	3	3	4	4	4	...
nl	1s	1p	1d	2s	1f	2p	1g	2d	3s	...
Degeneracy	2	6	10	2	14	6	18	10	2	...
States with $E \leq E_{nl}$	2	8	18	20	34	40	58	68	70	...

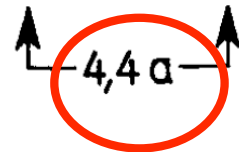
$$N = 2(n-1) + l$$

$$E = h\nu(N + 3/2)$$

Shell Model Potential



Skin thickness



Oszillator Potential		Rechteck Potential
$6 \hbar \omega$ [168]	1 i (26) 3 p (6)	[138] [132]
$5 \hbar \omega$ [112]	2 f (14) 1 h (22)	[106] [92]
$4 \hbar \omega$ [70]	3 s (2) 2 d (10) 1 g (18)	[68] [58] [40]
$3 \hbar \omega$ [40]	2 p (6) 1 f (14)	[34] [20] [18]
$2 \hbar \omega$ [20]	2 s (2) 1 d (10)	[8]
$1 \hbar \omega$ [8]	1 p (6)	[2]
$0 \hbar \omega$ [2]	1 s (2)	[2]
λ	$[\Sigma \nu]$	$[\Sigma \nu]$
	$n \ell(\nu)$ $\nu = 2(2\ell + 1)$	

$E - E_0$

Spin Orbit Coupling

It changes the hierarchy of the energy levels:

$$V(r) = V_{central}(r) + V_{ls}(r) \frac{\langle \vec{l}s \rangle}{\hbar^2}$$

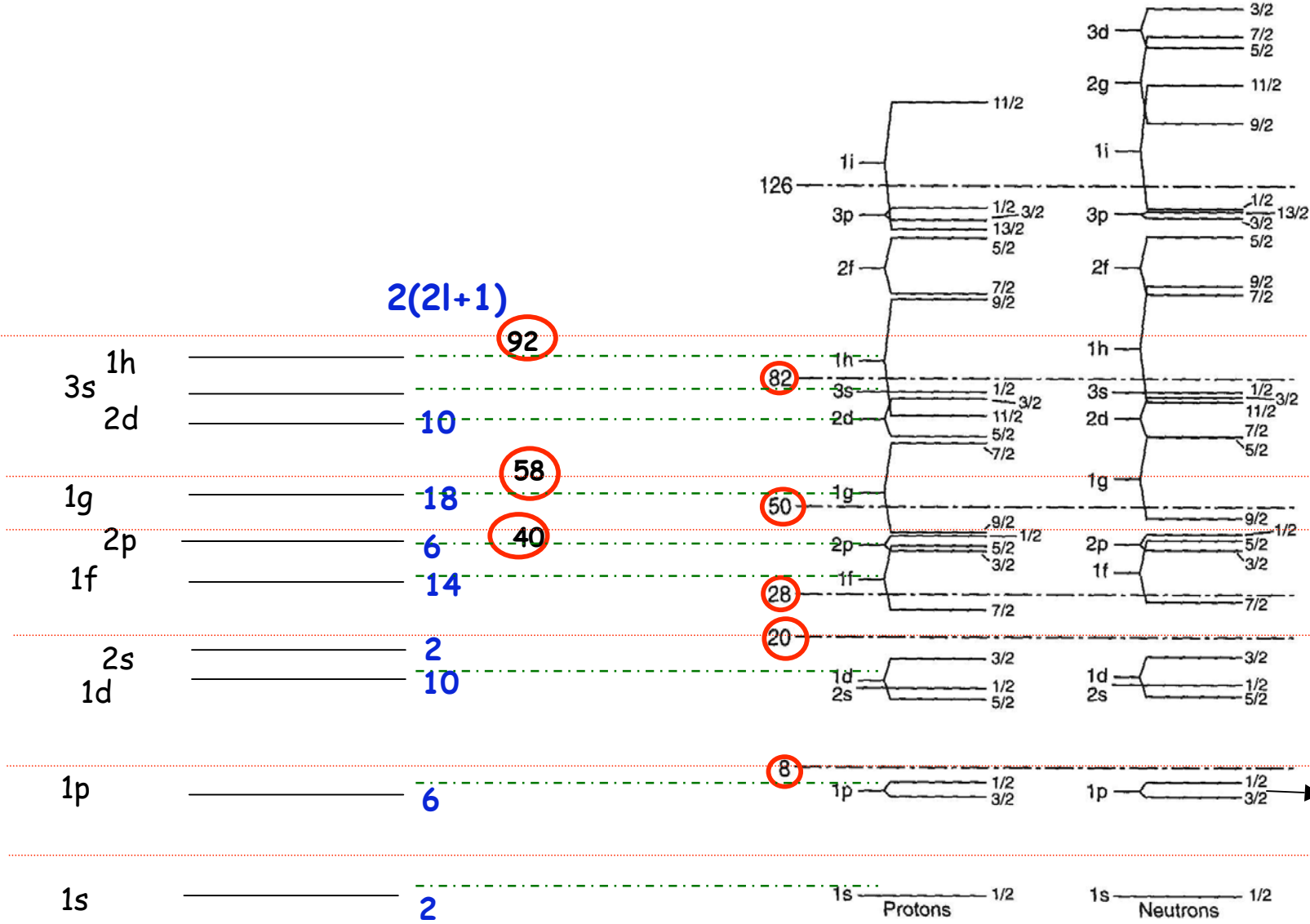
$$\frac{\langle \vec{l}s \rangle}{\hbar^2} = \frac{1}{2}(j(j+1) - l(l+1) - s(s+1)) = \begin{cases} l/2 & j = l + 1/2 \\ -(l+1)/2 & j = l - 1/2 \end{cases} \begin{array}{l} \text{Moving below} \\ \text{Moving above} \end{array}$$

$$\Delta E_{ls} = \frac{2(l+1)}{2} \langle V_{ls}(r) \rangle$$

The energy levels transform into n_{lj} levels.

Saxon-Wood Potential

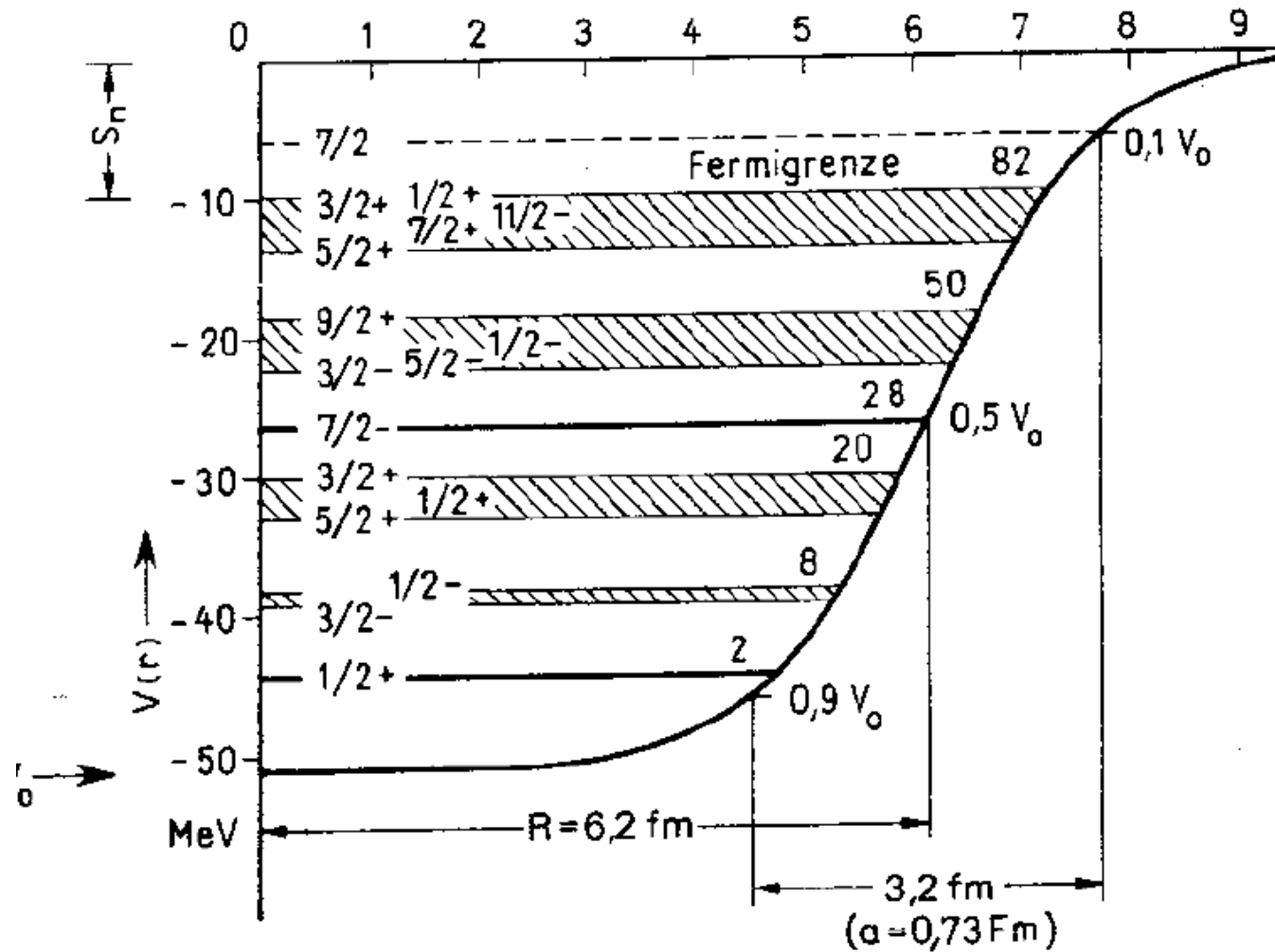
Shell model



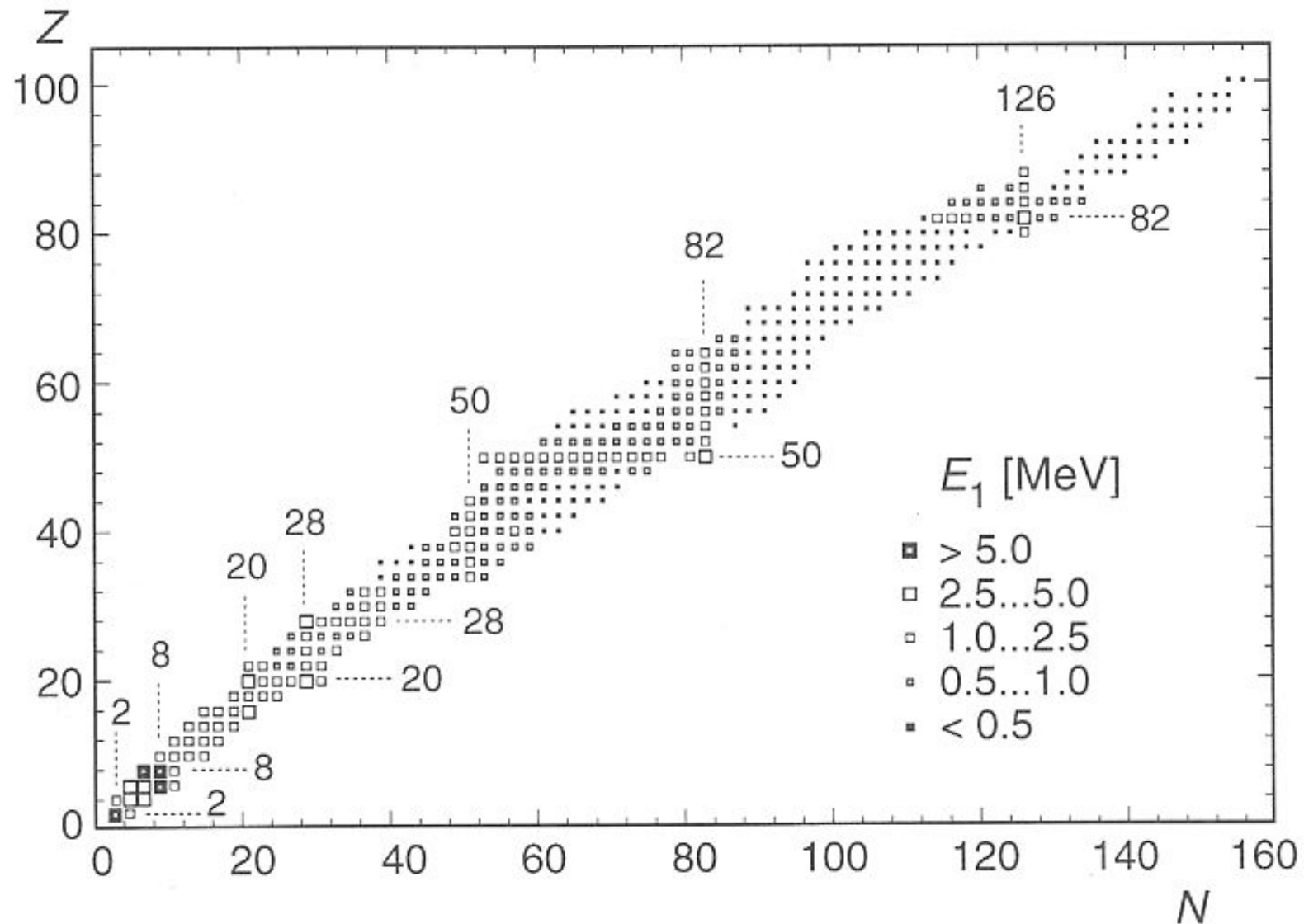
Spin-orbit coupling

Single particle level calculated in the shell model.

Single-Particle-Spectrum of the Wood-Saxon Potential for a heavy nucleus



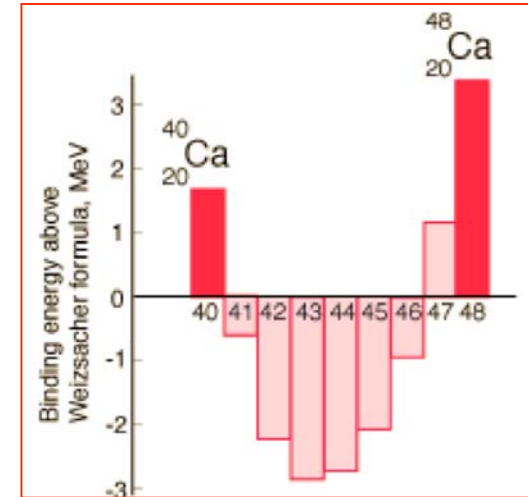
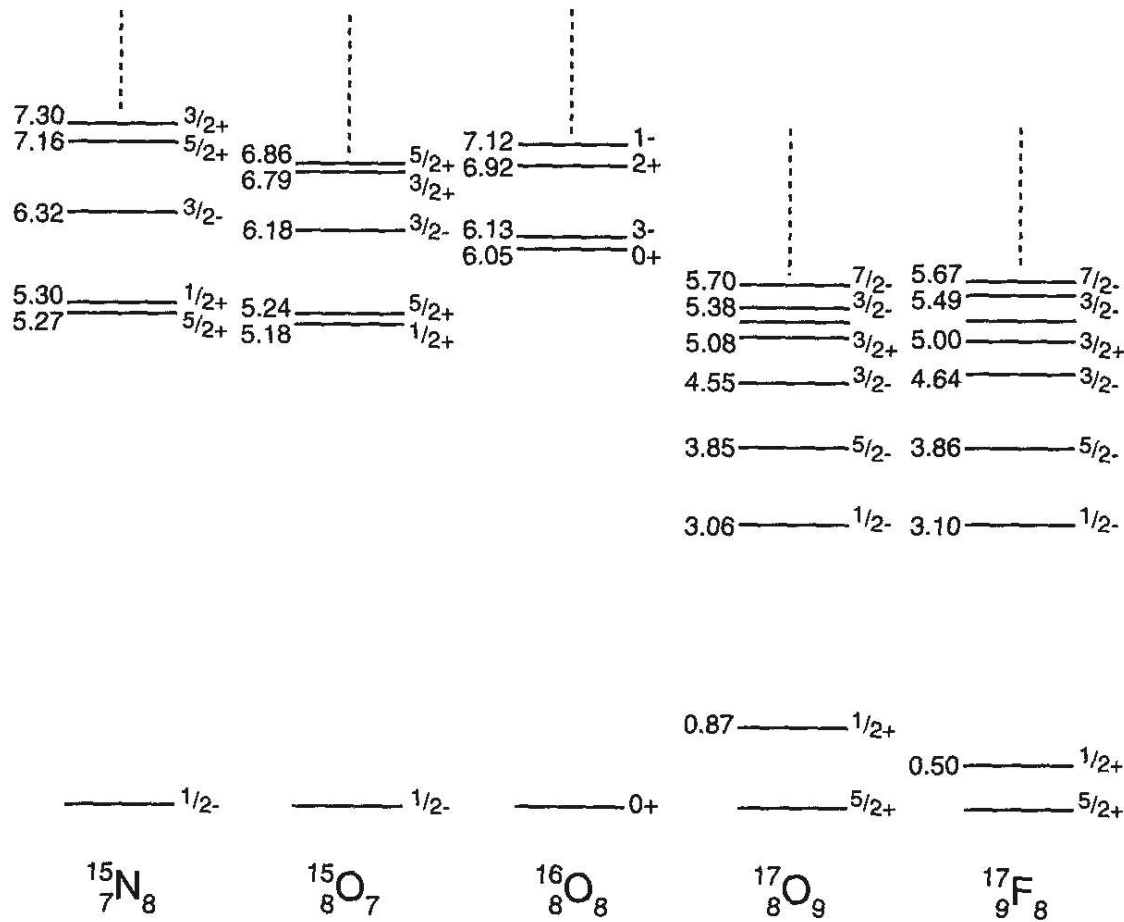
Energy of first excited nuclear level in gg-nuclei



Energy levels of some nuclei

${}^4_2\text{He}$ ${}^{16}_8\text{O}$ ${}^{40}_{20}\text{Ca}$ ${}^{48}_{20}\text{Ca}$ ${}^{208}_{82}\text{Pb}$

Excitation Energy



Nuclear Magnetic Moments in Shell Model

$$\mu_{nucl} = \mu_N \sum_{i=1}^A (g_l \vec{l}_i + g_s \vec{s}_i) / \hbar$$

$$g_l = \begin{cases} 1 & \text{for } p \\ 0 & \text{for } n \end{cases} \quad g_s = \begin{cases} 5.58 & \text{for } p \\ -3.83 & \text{for } n \end{cases}$$

$$\langle \mu_{nucl} \rangle = \mu_N \sum_{i=1}^A \langle \psi_{nucl} | g_l \vec{l}_i + g_s \vec{s}_i | \psi_{nucl} \rangle / \hbar$$

Wigner-Eckart Theorem: The expectation value of any vector operator of a system is equal to the projection onto its angular momentum

$$\langle \mu_{nucl} \rangle = g_{nucl} \mu_N \frac{\langle \vec{J} \rangle}{\hbar} \quad g_{nucl} = \sum_{i=1}^A \frac{\langle JM_J | g_l \vec{l}_i + g_s \vec{s}_i | JM_J \rangle}{\langle JM_J | J^2 | JM_J \rangle}$$

In the case of a single nucleon in addition to the closed shell the angular momentum of the closed shell nuclei couples to 0. The nuclear magnetic momentum of is equal to the nuclear magnetic mom. of the valence nucleon.

$$g_{nucl} = \frac{g_l(j(j+1) + l(l+1) - s(s+1)) + g_s(j(j+1) + s(s+1) - l(l+1))}{2j(j+1)}$$

$$\frac{\mu_{nucl}}{\mu_N} = g_{nucl} J = \left(g_l \pm \frac{g_s - g_l}{2l+1} \right) \quad J = l \pm 1/2$$

Magnetic moments:

Shell model calculations and experimental values

Nucleus	State	J^P	μ/μ_N	
			Model	Expt.
^{15}N	p- $1p_{1/2}^{-1}$	$1/2^-$	-0.264	-0.283
^{15}O	n- $1p_{1/2}^{-1}$	$1/2^-$	+0.638	+0.719
^{17}O	n- $1d_{5/2}$	$5/2^+$	-1.913	-1.894
^{17}F	p- $1d_{5/2}$	$5/2^+$	+4.722	+4.793