Lattice QCD with light quarks compares to chiral perturbation theory

Leonardo Giusti

CERN - Theory Group

In Collaboration with L. Del Debbio (Edinburgh), M. Lüscher (CERN), R. Petronzio and N. Tantalo (Tor Vergata)
The Wilson action for the $SU(3)$ Yang–Mills theory is ($\beta = 6/g^2$)

$$S_{YM} = \beta \sum_{x, \mu < \nu} \left\{ 1 - \frac{1}{6} \text{Tr} \left[ U_{\mu\nu}(x) + U_{\mu\nu}^\dagger(x) \right] \right\}$$

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \mu)U_\mu^\dagger(x + \nu)U_\nu^\dagger(x)$$

Periodic boundary conditions for gauge fields
QCD with two degenerate flavors with the Wilson action

The fermion Wilson action we use is

\[ S_F = \sum_{i=1}^{2} \sum_{x,y} \bar{\psi}_i(x) D_m(x, y) \psi_i(y) \]

\[ \psi \equiv \{ \psi_1, \psi_2 \} \]

\[ D_m = \frac{1}{2} \left\{ \gamma_{\mu} \left( \nabla^*_{\mu} + \nabla_{\mu} \right) - a \nabla^*_{\mu} \nabla_{\mu} \right\} + m_0 \]

where \( am_0 = (1/k - 8)/2 \) and

\[ \nabla_{\mu} \psi_i(x) = \frac{1}{a} \left[ U_{\mu}(x) \psi_i(x + a\hat{\mu}) - \psi_i(x) \right] \]

\[ \nabla^*_{\mu} \psi_i(x) = \frac{1}{a} \left[ \psi_i(x) - U_{\mu}^{\dagger}(x - a\hat{\mu}) \psi_i(x - a\hat{\mu}) \right] \]

Fermion fields with periodic boundary conditions in space and anti-periodic in time
It is possible to define renormalized operators

\[ \hat{A}_\mu^a(x) = Z_A A_\mu^a(x) \quad A_\mu^a(x) = \bar{\psi}(x)\gamma_\mu\gamma_5 \frac{\sigma^a}{2} \psi(x) \]

\[ \hat{P}^a(x) = Z_P P^a(x) \quad P^a(x) = \bar{\psi}(x)\gamma_5 \frac{\sigma^a}{2} \psi(x) \]

that satisfy renormalized axial Ward identities of the form

\[ \partial_\mu \langle \hat{A}_\mu^a(x) \hat{P}^a(0) \rangle = 2 \hat{m} \langle \hat{P}^a(x) \hat{P}^a(0) \rangle + \mathcal{O}(a) \quad x \neq 0 \]

The “on-shell” non-perturbative definition of the quark mass is

\[ m = \frac{1}{2} \frac{\partial_\mu^* \langle A_\mu^a(x) P^a(0) \rangle}{\langle P^a(x) P^a(0) \rangle} \quad \hat{m} = \frac{Z_A}{Z_P} m \]
Non-linear sigma model with two degenerate flavors

- The fundamental fields

\[ U \equiv \exp \left\{ \frac{2i}{F} \Phi \right\}, \quad \Phi = \sum_a \phi^a \sigma^a \]

transforms under chiral symmetry as

\[ U \rightarrow V_R U V_L^\dagger, \quad U^\dagger \rightarrow V_L U^\dagger V_R^\dagger \]

with \( V_L V_L^\dagger = I \) and \( V_R V_R^\dagger = I \)

- The \( \mathcal{O}(p^2) \) Euclidean action which encodes the SSB is

\[ S^{(2)} = \int d^4x \frac{F^2}{4} \left\{ \text{Tr} \left[ \partial_\mu U^\dagger \partial_\mu U \right] - M^2 \text{Tr} \left[ U^\dagger + U \right] \right\} \]

where \( M^2 = 2B\hat{m} \)
Meson mass and decay constant at NLO

The $\mathcal{O}(p^4)$ Euclidean Action is given by

\[
S^{(4)} = \int d^4x \left\{ \frac{M^4(\hat{l}_4 - \hat{l}_3)}{16} \text{Tr}[U^\dagger + U] \text{Tr}[U^\dagger + U] + \right.
\]

\[
\frac{M^2\hat{l}_4}{8} \text{Tr}[\partial_\mu U^\dagger \partial_\mu U] \text{Tr}[U^\dagger + U] + \text{four deriv. terms} \right\}
\]

The meson mass and decay constant at $\mathcal{O}(p^4)$ are given by

\[
M^2_P = M^2 \left\{ 1 + \frac{M^2}{32\pi^2 F^2} \log \left( \frac{M^2}{\mu^2} \right) + \frac{2M^2}{F^2} \hat{l}_3(\mu) \right\}
\]

\[
F_P = F \left\{ 1 - \frac{M^2}{16\pi^2 F^2} \log \left( \frac{M^2}{\mu^2} \right) + \frac{M^2}{F^2} \hat{l}_4(\mu) \right\}
\]
Matching a non-linear sigma model with the experiment: $M_P^2$ Gasser Leutwyler 84

If we define

$$\hat{l}_3(\mu) = \frac{-1}{64\pi^2} \left( l_3 + \log \left( \frac{M^2}{\mu^2} \right) \right) \bigg|_{M=139.6\text{MeV}}$$

$$\bar{l}_3 = \log \left( \frac{\Lambda^2}{M^2} \right) \bigg|_{M=139.6\text{MeV}}$$

then

$$M_P^2 = M^2 \left( 1 + \frac{M^2}{32\pi^2 F^2} \log \left( \frac{M^2}{\Lambda^2} \right) \right)$$

A crude estimate from experimental values of meson masses gives

$$\bar{l}_3 = 2.9 \pm 2.4$$
If we define

\[ \hat{l}_4(\mu) = \frac{1}{16\pi^2} \left( \bar{l}_4 + \log \left( \frac{M^2}{\mu^2} \right) \right) \bigg|_{M=139.6\text{MeV}} \]

\[ \bar{l}_4 = \log \left( \frac{\Lambda_F^2}{M^2} \right) \bigg|_{M=139.6\text{MeV}} \]

then

\[ F_P = F \left\{ 1 - \frac{M^2}{16\pi^2 F^2} \log \left( \frac{M^2}{\Lambda_F^2} \right) \right\} \]

An estimate from the scalar radius of the pion gives

\[ \bar{l}_4 = 4.4 \pm 0.2 \]
Decomposition of the lattice into blocks with Dirichlet b.c. with \( q \geq \pi / L > 1 \) GeV

Asymptotic freedom: quarks are weakly interacting in the blocks \( \Rightarrow \) QCD easy (cheaper) to simulate

Block interactions are weak and are taken into account exactly

\[
S(x, y) \sim \frac{1}{|x - y|^3}
\]
Block decomposition of the Dirac operator

The Wilson–Dirac operator

\[ D_m = \frac{1}{2} \left\{ \gamma_\mu (\nabla^*_\mu + \nabla_\mu) - \nabla^*_\mu \nabla_\mu \right\} + m_0 \]

can be decomposed as

\[ D = D_{\Omega^*} + D_\Omega + D_{\partial \Omega^*} + D_{\partial \Omega} \]

where

\[ D_{\Omega^*} = \sum_{\text{white } \Lambda} D_\Lambda \quad D_\Omega = \sum_{\text{black } \Lambda} D_\Lambda \]

\( \Omega^*, \Omega \) are white and black blocks, \( \partial \Omega, \partial \Omega^* \) are exterior boundaries
The determinant of the Dirac operator written as

$$\det D_W = \prod_{\text{all } \Lambda} \det \hat{D}_\Lambda \det R$$

with the block interaction

$$R = 1 - P_{\partial \Omega^*} D_\Omega^{-1} D_{\partial \Omega} D_{\Omega^*}^{-1} D_{\partial \Omega^*}$$

For two flavors can be written as integral over scalar fields

$$S_{\phi \chi} = \sum_{\text{all } \Lambda} \| \hat{D}_\Lambda^{-1} \phi_\Lambda \|^2 + \| R^{-1} \chi \|^2$$

where $\phi_\Lambda$ defined on $\Lambda$ and $\chi$ on $\partial \Omega^*$.
In molecular dynamics force naturally split

\[
\frac{d}{dt} \Pi(x, \mu) = -F_G(x, \mu) - F_\Lambda(x, \mu) - F_R(x, \mu)
\]

\[
\frac{d}{dt} U(x, \mu) = \Pi(x, \mu) U(x, \mu)
\]

Integration step-sizes chosen such that

\[
\epsilon_G \|F_G\| \sim \epsilon_\Lambda \|F_\Lambda\| \sim \epsilon_R \|F_R\|
\]

i.e. the most expensive force computed less often!

Do not give up first-principles: teach Physics to exact algorithms for being smarter (faster)!

\[
C_{\text{ost}} \propto m_q^{-1}
\]
Collaboration: L. Del Debbio (Edinburgh), L. G. and M. Lüscher (CERN), R. Petronzio and N. Tantalo (Tor Vergata)

**Fermi Institute** PC cluster with 80 nodes (160 Xeon procs)
64 nodes used for this project (≈200 Gflops sustained)

**Bern Physics Institute** PC cluster with 32 nodes (64 Xeon procs)
8 nodes used for this project (≈25 Gflops sustained)

**CERN** PC cluster with 32 nodes (64 Xeon procs)
All nodes used for this project (≈160 Gflops sustained)
Parameters of the runs with the Wilson action

<table>
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<tr>
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<th>$N_{\text{trj}}$</th>
<th>$N_{\text{sep}}$</th>
<th>$N_{\text{conf}}$</th>
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</tr>
<tr>
<td>0.15462</td>
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<td>50</td>
<td>102</td>
</tr>
</tbody>
</table>

Parameter ranges:

1. $m \sim \frac{1}{4} m_s - m_s$
2. $a \sim 0.050 - 0.075$ fm
3. $L \sim 1.75$ fm

All confs archived @ CERN

All following results preliminary!
Pseudoscalar meson mass

\[
\begin{array}{cc}
 k & aM_P \\
 0.15750 & 0.2744(21) \\
 V = 24^3 \times 32 & 0.15800 & 0.1969(16) \\
 \beta = 5.6 & 0.15825 & 0.1554(31) \\
 t_1 - t_2 = 12 - 16 & 0.15835 & 0.1204(44) \\
 24^3 \times 32 & 0.15410 & 0.1965(8) \\
 \beta = 5.8 & 0.15440 & 0.1481(11) \\
 t_1 - t_2 = 18 - 32 & 0.15462 & 0.1040(12) \\
 24^3 \times 64 & 0.15455 & 0.1151(12) \\
 \end{array}
\]

Pseudoscalar meson mass extracted from

\[
C_{PP}(t) = \sum_{\vec{x}} \langle P^a(x) P^a(0) \rangle
\]

by fitting the effective mass to a plateaux
Pseudoscalar meson mass

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Pseudoscalar meson mass extracted from

$$C_{PP}(t) = \sum_{\vec{x}} \langle P^\alpha(\vec{x}) P^\alpha(0) \rangle$$

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Pseudoscalar decay constant

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Pseudoscalar decay constant extracted by combining \(C_{PP}(t)\) with

\[
C_{AP}(t) = \sum_x \langle A_0^a(x) P^a(0) \rangle
\]

and by fitting the effective decay constant to a plateaux
Pseudoscalar decay constant

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Pseudoscalar decay constant extracted by combining $C_{PP}(t)$ with

$$C_{AP}(t) = \sum_{\vec{x}} \langle A_0^a(x) P^a(0) \rangle$$

and by fitting the effective decay constant to a plateaux.
Quark mass

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Quark mass extracted from

\[ 2m(t) = \frac{\partial_l^* C_{AP}(t)}{C_{PP}(t)} \]

by fitting to a plateaux
Quark mass

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Quark mass extracted from

$$2m(t) = \frac{\partial_t^* C_{AP}(t)}{C_{PP}(t)}$$

by fitting to a plateaux
Statistical gain with five sources

Two-point pseudoscalar correlation functions computed for 5 sources

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<tr>
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A general error reduction observed

A clear pattern of error reduction in $F_P$
Finite volume corrections

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Meson masses and decay constants at \(O(p^4)\) in finite volume

\[
M_P^2 = M^2 \left\{ 1 + \frac{M^2}{32\pi^2 F^2} \log \left(\frac{M^2}{\Lambda^2_{\pi}}\right) + \frac{1}{2F^2} g_1^4(M^2) \right\}
\]

\[
F_P = F \left\{ 1 - \frac{M^2}{16\pi^2 F^2} \log \left(\frac{M^2}{\Lambda^2_F}\right) - \frac{1}{F^2} g_1^4(M^2) \right\}
\]

The finite volume corrections in \(M_P^2\) for the various masses are

\(\beta = 5.6\)

\(\{0\%, 0.2\%, 0.7\%, 2.1\%\}\)

\(\beta = 5.8\)

\(\{0\%, 0.6\%, 0.9\%, 1.3\%\}\)
Reference point defined to be

\[ \left( \frac{M_P}{M_V} \right)^2 \bigg|_{m=m_{\text{ref}}} = \left( \frac{M_{K}^{\text{exp}}}{M_{K}^{\text{exp}}} \right)^2 = 0.30657 \]

If we fix \( M_{\text{ref}} = M_{K}^{\text{exp}} \) to fix the lattice spacing

\[
a^{-1} = 2.70(3) \text{ GeV} \quad \beta = 5.6
\]

\[
a^{-1} = 3.77(4) \text{ GeV} \quad \beta = 5.8
\]

If we use \( Z_A \) from RI-MOM D. Bećirević et al 05

\[
F_{\text{ref}} = 111(2) \quad \beta = 5.6
\]

\[
F_{\text{ref}} = 108(2) \quad \beta = 5.8
\]
A remarkable linear behavior is observed
A remarkable linear behavior is observed

...... and results from the two lattices are consistent
In QCD with two light flavors the mass of the light pseudoscalar meson shows a remarkable linearity in the quark mass.
In QCD with two light flavors the mass of the light pseudoscalar meson shows a remarkable linearity in the quark mass.
In QCD with two light flavors the mass of the light pseudoscalar meson shows a remarkable linearity in the quark mass.

The mass dependence is also compatible with the “experimental” curve.
Comparison with quenched data

\[
\frac{(M_p^2/m_q)}{(M_p^{2\text{ref}}/m_q^{\text{ref}})} \text{ vs } \frac{(m_q/m_q^{\text{ref}})}{\text{beta=5.6}}
\]

For quenched data thanks to: P. Hernández, C. Pena, J. Wennekers and H. Wittig
In QCD with two light flavors the decay constant of the light pseudoscalar meson shows a clear dependence on the quark mass.
In QCD with two light flavors the decay constant of the light pseudoscalar meson shows a clear dependence on the quark mass

and results from the two lattices are consistent
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The mass dependence is also compatible with the “experimental” curve.
ChPT fits for the pseudoscalar decay constant

The lightest three points are compatible with a linear behavior.
The lightest three points are compatible with a linear behavior.

. . . . . . and also with the NLO ChPT fit function.

Light and precise points are needed for an accurate determination of $F$. 
First clover run

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N_{trj}$</th>
<th>$N_{sep}$</th>
<th>$N_{conf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.13550</td>
<td>5200</td>
<td>50</td>
<td>104</td>
</tr>
<tr>
<td>$V = 24^3 \times 48$</td>
<td>0.13590</td>
<td>4620</td>
<td>30</td>
</tr>
<tr>
<td>$\beta = 5.3$</td>
<td>0.13610</td>
<td>5070</td>
<td>30</td>
</tr>
<tr>
<td>$c_{sw} = 1.90952$</td>
<td>0.13620</td>
<td>1770</td>
<td>30</td>
</tr>
</tbody>
</table>

TBD
Conclusions

- Our experience for two flavor QCD shows that SAP is very stable in the ranges
  1. \( m \sim \frac{1}{4} m_s - m_s \)
  2. \( a \sim 0.050 - 0.075 \text{ fm} \)
  3. \( L \sim 1.75 \text{ fm} \)

- The production for two Wilson lattices completed. The first clover run is finishing

- Discretization effects in the quark mass dependence of \( M_P^2 \) and \( F_P \) are small

- The mass dependence of \( M_P^2 \) turns out to be very linear for \( M_P = 300 - 600 \text{ MeV} \)
  Data compatible with NLO ChPT + exp.

- \( F_P \) shows a clear quark mass dependence. Data compatible with NLO ChPT + exp.

- Precise points at light quark masses are necessary to extract the LECs reliably